Stephen Lerman
*Editor*

# Encyclopedia of Mathematics Education

Springer Reference

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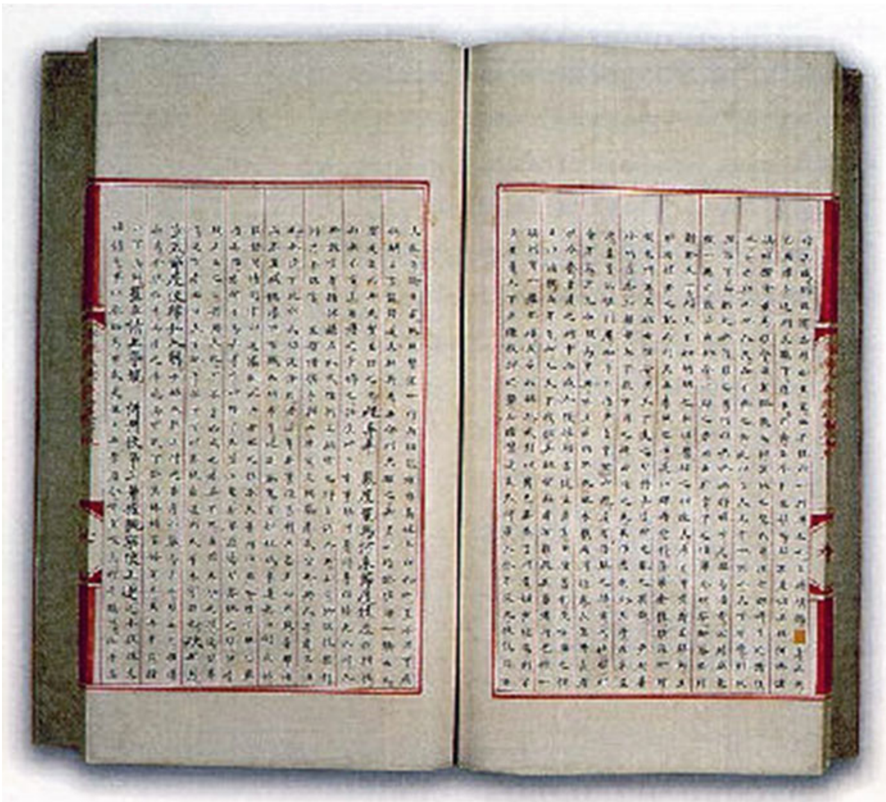

Yongle Encyclopedia (1403) (taken from: http://en.wikipedia.org/wiki/Yongle_Encyclopedia, public domain)

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# Encyclopedia of Mathematics Education

With 67 Figures and 9 Tables

Springer Reference

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# Foreword

Two millennia ago, encyclopedias, beginning with that of Pliny the Elder, were the work of one person. Single authors remained the rule for almost 18 centuries until the Enlightenment, when Denis Diderot and Jean d'Alembert assembled dozens of writers to produce their encyclopedia. These days, encyclopedias must rely on hundreds of contributors if they are to provide a reasonably full treatment – even if they are restricted to a single field. No one individual could possibly construct an all-inclusive encyclopedia today. If some group had attempted to produce an encyclopedia of mathematics education a century ago, the tome would necessarily have been fairly meager. Mathematics education was just getting started as a scholarly field and, in most countries, was not present in the academy. Over the following decades, however, the field has continued to grow rapidly, and its literature has become substantial. A search of the scholarly literature on the Web using the term *mathematics education* yielded 129,000 hits in 2008 and 287,000 in 2013 – more than doubling in only 5 years.

The present encyclopedia offers an up-to-date, wide-ranging reference source spanning a field that is growing and in continuing flux. The ambition of the encyclopedia is to deal with every topic in mathematics education, delineating theoretical positions, describing research findings, and citing relevant literature. The length of an entry is tailored to its importance in the field as determined by the editor in chief and his distinguished international editorial board. The publication has three formats: a printed work in one volume, an e-book and an online work that is searchable and will be updated. The printed format will be usable everywhere, including locations without online access; the online work will make the reference comparable to other online encyclopedias, offering opportunities not simply for readers to search the text but also for contributors to add new entries and revise old ones.

Novice mathematics educators will find that the encyclopedia provides a panoramic view of the field, introducing them to whole realms of work they may never have encountered. Old-timers will find entries by giants in the field as well as by contributors from outside the usual circles. Whatever the topic, every reader will find valuable information, including citations of prominent publications. Researchers undertaking a study in mathematics education will want to check first with this reference source to get not only pertinent theoretical analyses of the topic and relevant research but also a sense of recent controversies and open questions. This encyclopedia represents a major step forward in the field of mathematics education, bringing to everyone with a professional interest in mathematics education access to the latest and best thinking in the field. It is the most timely, comprehensive, and useful reference we have.

Jeremy Kilpatrick

# Preface

The encyclopedia is intended to be a comprehensive reference text, covering every topic in the field of mathematics education research with entries ranging from short descriptions to much longer ones where the topic warrants more elaboration. The entries have been written by leaders in the field as a whole, and in most cases they are originators and innovators in the specific entry topic.

The entries provide access to theories and to research in the area and refer to some of the key publications for further reading, including the core texts as well as cutting-edge research, and point also to future developments. We have tried to be comprehensive in terms of drawing on work from around the world, particularly through the knowledge and experience of the section editors. The vast majority of the hard work of soliciting, encouraging, and editing has been carried out by these editors. The list of entries was mapped out at an intensive seminar of the editors, in sections of common theme. Each editor took on responsibility for a theme according to their interests and expertise. They then worked with all the authors to develop and edit the entries in their section. As things progressed, while some editors were overloaded with work, others took on part of their tasks. They have been exemplary in their roles, and an enormous debt of gratitude is owed to them.

Michèle Artigue took responsibility for the section on research on mathematics curriculum topics and Information and Communication in education; Ruhama Even, for research on teaching; Mellony Graven, for research on teacher education; Eva Jablonka, for research on mathematics in out-of-school contexts and for research methods, paradigms, and sociological perspectives; Robyn Jorgensen, for research on curriculum, assessment, and evaluation; Yoshinori Shimizu, for research on learning; and Bharath Sriraman, for research on the nature of mathematics and mathematical thinking and theories of learning.

We have been supported by the excellent team at Springer, including Michael Hermann, Daniela Graf, Clifford Nwaeburu, and Jutta Jaeger-Hamers. The SpringerReference system has been modified and developed in part by the suggestions and needs of the section editors, and credit must go to the developers for making those modifications. In the years to come, I am sure further changes will be needed.

The encyclopedia should be informative for graduate students, researchers, curriculum developers, policy makers, and others with interests in the field of mathematics education. It can be used to support students in their review of literature and in finding the sources of knowledge in the field. It is our hope, too, that it will enable researchers to connect their research with what has gone before. Too frequently, we see research that either has largely been done before or does not take the opportunity to build on prior work and develop it, but repeats it. Furthermore, we hope that it will support researchers in making links between theoretical approaches and frameworks and the ways they carry out their research, their methodology, and methods. As experts in the field, the entry authors exemplify how these connections should be made, in their descriptions and in the references they provide.

In this first iteration of the encyclopedia, we have not succeeded fully in our goal of being comprehensive. Some entries were not completed in time, potential authors withdrew at the last minute, but on a more positive note colleagues around the world have already indicated topics that should be included in the future. This is not an open access encyclopedia. We welcome and encourage comments, suggestions, critique, and further ideas, which can be made on the particular entry pages. They will be reviewed and considered by the entry authors, and we will periodically invite the authors to make changes in their entry as they see fit, in communication with the editors.

We look forward, also, to reactions to me, editor-in-chief, about what works and what does not, in more general terms, and we will do our best to respond. Recently, we have celebrated 100 years of the international mathematics education community, and we have seen a proliferation of research orientations, journals, and conferences and the growth of research communities around the world. If this venture contributes in substantial ways to these developments, we will be very satisfied that the work has been worthwhile.

Stephen Lerman

# About the Editors

**Stephen Lerman** was a secondary teacher of mathematics in the United Kingdom and Israel for many years and then became Head of mathematics in a London comprehensive school before completing a PhD and moving into mathematics teacher education and research. He was temporary lecturer at the Institute of Education, University of London, and at the University of North London before taking up a permanent position at what is now called London South Bank University. He is a former President of the International Group for the Psychology of Mathematics Education (PME) and Chair of the British Society for Research into Learning Mathematics (BSRLM). He is now Emeritus Professor at London South Bank University, part-time Professor at Loughborough University, Visiting Professor at the University of the Witwatersrand, and Adjunct Professor at Griffith University. His research interests are in sociocultural and sociological theories in mathematics education research and in the use of theories in general in the field.

**Bharath Sriraman** is Professor of mathematics at the University of Montana and on the Faculty and Advisory Board of Central/SW Asian Studies, where he occasionally offers courses on Indo-Iranian studies/languages. He holds degrees from Alaska (BS in mathematics, University of Alaska–Fairbanks) and Northern Illinois (MS and PhD in mathematics, minor in mathematics education). He maintains active interests in mathematics education, philosophy, history of mathematics, gifted education, and creativity. He has published over 300 journal articles, commentaries, book chapters, edited books, and reviews in his areas of interest and presented more than 200 papers at international conferences, symposia, and invited colloquia. Bharath is the founding editor of *The Mathematics Enthusiast* and the founding co-series editor of *Advances in Mathematics Education* (Springer Science) and of four other book series. He serves on the editorial panel of a dozen or so journals, including *Roeper Review*, *Gifted Child Quarterly*, and *High Ability Studies*. Bharath is fairly fluent in seven to nine languages (English, German, Farsi, Hindi, Tamil, Urdu, Kannada, basic French, and Danish) and travels/holds active ties with researchers all over the world. He received the 2002 NAGC Outstanding Brief of the Year Award; was nominated for the 2006, 2007 NAGC Early Career Scholar Award; and was named the 2007 Outstanding Early Scholar by the School Science and Mathematics Association, and in 2009, Northern Illinois University named him as one of 50 "golden alumni" in the last 50 years for his significant contributions to research in mathematics education, gifted education, and interdisciplinary research at the intersection of mathematics–science–arts.

**Eva Jablonka** holds a Chair in mathematics education at King's College, London, UK. Before joining King's, she held a range of academic positions in different countries and contexts, including Sweden, Germany, and Australia. Her research includes the study of school mathematics curricula at macro and micro levels (in particular, mathematical modeling and mathematical literacy), the sociology of mathematics, the role of theorizing in mathematics education, cross-cultural comparative studies of mathematics education, and students in transition between different sectors of mathematics education with a focus on the emerging achievement disparities related to these transitions.

**Yoshinori Shimizu** is a Professor of mathematics education at the Graduate School of Comprehensive Human Sciences, University of Tsukuba, in Japan. His primary research interests include international comparative study on mathematics classrooms and student assessment. He was a member of the Mathematics Expert Group for OECD/PISA 2003, 2006, and 2009. He is one of the founders of Learner's Perspective Study (LPS), a 16 countries' comparative study on mathematics classrooms, and has been the Japanese team leader of the project. He serves as a member of editorial boards on international research journals, such as *International Journal of Science and Mathematics Education* and *ZDM-International Journal of Mathematics Education*.

**Michèle Artigue** is Emeritus Professor at the Paris Diderot University – Paris 7. After completing a PhD in mathematical logic, she progressively moved to the field of mathematics education. In that field, beyond theoretical contributions on the relationships between epistemology and didactics, didactical engineering, the reproducibility of didactic situations, the instrumental approach, and more recently the networking of theoretical frameworks, her main research areas have been the teaching and learning of mathematics at the university level, especially the didactics of calculus and elementary analysis, and the use of digital technologies in mathematics education. She has many editorial and scientific responsibilities at national and international levels, and after being from 1998 to 2006 Vice President of the International Commission on Mathematical Instruction (ICMI), she was its President from 2007 to 2010.

**Ruhama Even** is Full Professor at the Weizmann Institute of Science and holds the Rudy Bruner Chair of Science Teaching. Her main research and development work is structured around three main interrelated foci: (a) the professional education and development of mathematics teachers, (b) mathematics curriculum development and analysis, and (c) the interplay of factors involved in shaping students' opportunities to learn mathematics. She has been member of the International Committee of PME and Cochair of ICMI Study 15 on the professional education and development of teachers of mathematics and serves as an editorial board member of the *Journal of*

*Mathematics Teacher Education* (JMTE) and *Mathematics Education Research Journal* (MERJ).

**Robyn Jorgensen** has been working in the area of mathematics education since undertaking her honors and doctoral work at Deakin University. She has focused her work in the area of equity, particularly focusing on the social practices that contribute to the patterns of success (or not) of social, cultural, and linguistic groups. Her strong interest in equity has been in the areas of social class, Indigenous, and issues around language and culture. She has international recognition for her work in this area as evidenced by numerous invitations for keynote addresses; state, national, and international panels; and invited publications and submissions. In 2008 she was invited co-convenor of the ICMI Centenary Conference for the social context working group; in 2008–2009, she was a member of the Ministerial Advisory Committee for Science, Technology, Engineering and Mathematics (STEM) as well as Chair of the Queensland Studies Authority Mathematics Advisory Committee. From 2009, she has been serving as the eminent mathematics education Professor on the national project (Turn the Page) of the Australian Association of Mathematics Teachers for enhancing mathematics learning for Indigenous Australians. She has worked in an advisory capacity for state projects and innovations in various states including Queensland, South Australia, and the Northern Territory. In 2009–2010, she took leave from the university to work as CEO/Principal of an Aboriginal corporation in Central Australia.

Robyn has secured numerous competitive grants including eight Australian Research Grants since 2001. Each grant has had a strong equity dimension to it. Collectively these have spanned the range of learning contexts from early childhood through to workplace learning. She has a critical edge to her work where she seeks to identify and redress issues of inequality in participation, access, and success in mathematics learning and teaching. Her work focuses strongly on practice – whether in formal school settings or settings beyond the school. The work seeks to challenge the status quo that has been implicated in the construction of unequal outcomes for particular groups of people. Her most recent ARC grants indicate the culmination of her challenge to contemporary practices in mathematics education. The work in the Kimberley region is an example of reforming teaching so as to enable Indigenous students greater access to mathematics learning. The newest ARC grant seeks to draw on the impact of digital technologies on young people's mathematical thinking. This project may provide explanations for new numeracies that have been observed in other ARC projects where older adolescents were found to have different dispositions to using and undertaking numeracy than their employers and teachers. These two projects will offer considerable challenges to current practices in school mathematics that are known to have profound (and negative) implications for many disadvantaged groups in Australia and internationally. Robyn is currently working on a 4 year longitudinal study to investigate the effects of early years swimming in under-5s on their development.

Robyn's work seeks to impact on the practices of the various sectors within which she works – whether schools, workplaces, or policy. This can be seen in the ways in which her research is undertaken with a range of industry partners for whom the research is most relevant. In most cases, the industries are actively involved in the studies and use the outcomes to inform their own practice. She is frequently sought by various stakeholders – schools, community groups, industry, policy, state authorities – to provide input into their activities including reports, professional development work, and advice on reform. The work that Robyn has undertaken has been recognized internationally and nationally. She is active in reviewing for a wide range of mathematics education and general education journals, being as well an Oz reviewer for the ARC and a reviewer for national research council grants including for Israel and South Africa. She is currently chief editor of the *Mathematics Education Research Journal* and serves on the editorial board of the *International Journal for Science and Mathematics Education*.

**Mellony Graven** is the South African Chair of Numeracy Education, Rhodes University. Her work as Chair involves the creation of a hub of mathematical activity, passion, and innovation that blends teacher and learner numeracy development with research focused on searching for sustainable ways forward to the challenges of mathematics education. She is the President of the Southern African Association for Research in Mathematics, Science and Technology Education and past editor of the journal *Learning and Teaching Mathematics*.

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A

## Ability Grouping in Mathematics Classrooms

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### Definitions

Ability means a certain amount of intelligence that individuals are thought to possess. Ability grouping is a generic term to encompass any grouping, whether it be within class or between classes, flexible or inflexible, that involves students being separated according to perceptions of their ability.

The term setting is used for grouping organization used in England whereby students attend different classes according to ideas of their ability. In primary schools there are typically 2–4 sets for mathematics; secondary schools may have as many as 10 sets. The varying sets move at a different pace and cover different mathematics content. Streaming is an older practice used in England whereby students were grouped by ability for all of their subjects together. This was used in secondary schools with students frequently being placed into streamed groups as soon as they started the schools. Tracking is an organizational practice used in the USA whereby different classes are offered, with different content, such as algebra, geometry, advanced algebra, and calculus. Tracking usually begins in middle school; the track students who are placed in middle school determine the high school courses they are able to take.

### Introduction

Whether or not to group students according to their current mathematics achievement is one of the most contentious issues in education. Research studies that have provided evidence to inform this question have been conducted in different subject areas and in various countries throughout the world. This chapter will review the ability grouping practices that are prevalent and summarize the results of some of the research studies conducted on the impact of ability grouping on students' mathematics experiences and understandings.

### Ability Grouping Practices in Different Countries

Beliefs about the purposes of education, the potential of students, and the nature of learning are deeply cultural (Altendorff 2012; Stigler and Hiebert 1999) and result in different ability grouping practices in countries across the world. In some countries, such as England, there is widespread belief that students have a certain "ability," and the role of teachers is to determine what that is, as early as possible, and teach different levels of content to different

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groups of students. This has resulted in extensive ability grouping with children as young as 5, but more typically 7 or 8, being placed into different classes (sets) for mathematics according to perceptions of their potential. By the time students reach secondary school, they are likely to have been placed in one of many (6–10) different sets. In England students and parents often do not know the implications of the set they are in, until they are entered for final examinations.

Other countries in Europe have moved away from ability grouping because it is judged to work against the pursuit of equity (Sahlberg 2011; Boaler 2008). In Scandinavian countries in particular, ability grouping is rare or nonexistent. Finland is one of the most successful countries in the world in terms of international achievement and chooses to group student heterogeneously for the majority of their school career (Sahlberg 2011).

Many Asian countries, particularly those in the Pacific Rim, have education systems that are based upon the idea that learning is a process determined by effort rather than fixed notions of ability (Stigler and Hiebert 1999). The idea of separating students into different levels is thought to be undesirable or even acceptable, as reflected in the following commentary about education in Japan:

> In Japan there is strong consensus that children should not be subjected to measuring of capabilities or aptitudes and subsequent remediation or acceleration during the nine years of compulsory education. In addition to seeing the practice as inherently unequal, Japanese parents and teachers worried that ability grouping would have a strong negative impact on children's self-image, socialization patterns, and academic competition. (Bracey 2003)

Mathematics classes in the USA are often organized through a form of ability grouping called "tracking." Students usually learn together in elementary school, but in middle school different courses are on offer – usually algebra, pre-algebra, and advanced algebra. The placement in middle school determines the courses available to students in high school, with only those who have completed algebra in middle school typically reaching calculus by the time they finish high school. Tracking is a more "open" practice than setting as it is clear to everyone which course students are placed into.

These different countrywide practices in ways of grouping students have been analyzed using the results of both the Second and Third International Mathematics and Science Studies (SIMSS and TIMSS, respectively), with researchers finding that countries that group by ability, the least and the latest, are the most successful countries in the world (Boaler 2008). In recent international achievement tests (TIMSS and PISA), Finland, Japan, and Korea, all countries that reject rigid ability grouping, have taken up the highest places in the world rankings. Studies of ability grouping have also been conducted within countries and these will be summarized below.

## Ability Grouping, Achievement, and Equity

A number of studies have taken place within the countries that divide students into ability groups – this chapter will include examples from England, Australia, Israel, and the USA – comparing the achievement of those who are taught in ability groups with those who are taught heterogeneously.

In the USA, Burris et al. (2006) compared 6 annual cohorts of students attending a middle school in the district of New York. For the first 3 years of the study, students were taught in tracked classes with only high-track students being taught an advanced curriculum, as is typical for schools in the USA. In the next 3 years, all students in grades 7–9 were taught the advanced curriculum in mixed-ability classes and all of the 9th graders were taught an accelerated algebra course. The researchers looked at the impact of these different middle school experiences upon the students' achievement and their completion of high school courses, using four achievement measures, including scores on the advanced placement calculus examinations. They found that the students from de-tracked classes took more advanced classes, pass rates were significantly higher, and students passed exams a year earlier than the average in New York State. The increased success from de-tracking applied to students across the achievement range – from the lowest to the highest achievers.

In England researchers followed 14,000 children through years 4 and 6 comparing those taught in sets with those grouped heterogeneously over the period of a year. They found that setting hindered the progress of students and that those taught heterogeneously performed significantly better on tests of mathematical reasoning (Nunes et al. 2009). The Primary Review, a government report in the UK, considered the impact of ability grouping and concluded that "The adoption of structured ability groupings has no positive effects on attainment but has detrimental affects on the social and personal outcomes for some children." The researchers conducting the review realized that primary teachers choose to group children according to notions of "ability" because they think that they can offer more appropriate work for children when they are in such groups. However, the review found that "the allocation of pupils to groups is a somewhat arbitrary affair and often depends on factors not related to attainment" and also that although teachers think they are giving children in low groups more appropriate work, "the evidence suggests that many pupils find the work they are given is inappropriate; often it is too easy" (Blatchford et al. 2008, pp. 27–28).

In addition to studies that track large cohorts of students through classes with different groupings, more detailed studies of students attending schools in sets and heterogeneous groups have found that ability grouping reduces achievement for students overall. This takes place through two processes – limiting opportunities for success by teaching high-level content to only some students (Porter and associates 1994) and discouraging students through communication of the idea that only some students are high achievers (Boaler et al. 2005). Boaler conducted longitudinal studies of students progressing through schools with contrasting grouping arrangements, in both the UK (Boaler 2012, 2005, 2002, 1997a, b) and the USA (Boaler and Staples 2008; Boaler 2008). She followed 500 students through 3 years of two schools in England and 700 students through 4 years of three schools in California. In both studies the students who worked in schools in mixed-ability groups performed at higher levels overall than those who worked in set or tracked groups. The schools teaching to mixed-ability groups also achieved more equitable outcomes. In a follow-up study of the students who had attended the different schools in England, some 8 years later, the adults who have been in the school employing ability grouping were in less professional jobs, and the adults interviewed linked the limits in their job prospects to the ability grouping used in school (Boaler 2012, 2005).

In Australia, Zevenbergen (2005) conducted research on the beliefs of students in low- and high-achievement groups and found that those in low groups had a fundamentally different experience of mathematics and constructed a different sense of self around their placement in groups. Those in high groups reported high-quality teaching and a sense of empowerment, whereas those in low groups reported low-quality teaching and a sense of disempowerment. Zevenbergen/Jorgensen's finding that students in low groups reported work being too easy with their achievement being limited by not being taught content that would be assessed in examinations was also reported from students in low groups in England (Boaler et al. 2000).

Linchevski and Kutscher (1998) conducted two different studies in Israel, investigating the impact of grouping upon student achievement. They found that students of average and below average attainment achieved at higher levels when taught in mixed-ability classes and high attainers achieved at the same level as those taught in same-ability classes. This finding – of high students achieving at similar levels in same- or mixed-ability classes and low and average students achieving at higher levels in mixed-ability classes – is one that has been reported in different studies (Slavin 1990; Hallam and Toutounji 1996).

In addition to the lower overall achievement of students taught in ability groups, reported in studies of setting and tracking, ability grouping has also been found to result in severe inequities as lower-ability classes are disproportionately populated by students of lower socioeconomic status and ethnic minority students and are usually taught by less well-qualified teachers and teachers who often have low expectations for their students (Oakes 1985). Mixed-ability approaches to teaching have consistently demonstrated more equitable outcomes (Boaler 2008, 2005; Cohen and Lotan 1997; Linchevski and Kutscher 1998).

## Conclusion

The weight of evidence from countries across the world indicates that ability grouping harms the achievement of students in low and middle groups and does not affect the achievement of high attaining students. Despite this evidence, ability grouping continues to be widespread in some countries – particularly English-speaking countries in the West, probably reflecting a common Western belief that students have a certain ability that is relatively unchangeable. Where countries recognize that high achievement is possible for all students (Dweck 2006) or hold equity as a central principle (Sahlberg 2011), ability grouping is less prevalent and not used with young children. Deeply held cultural beliefs about learning and about what it means to be "smart" are difficult to change, which may be the reason for the persistence of ability grouping in some countries, a practice that appears to benefit some students at the expense of others.

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## Abstraction in Mathematics Education

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### Keywords

Processes of abstraction; Concretion; Empirical abstraction; Reflecting abstraction; Objectification; Reification; Procept; Shift of attention; APOS; Webbing; Situated abstraction; Abstraction in context

### Definition

An abstraction, to most mathematicians, is an object, such as a vector space, which incorporates a structure – elements and relationships between them – common to many instances appearing in diverse contexts. For the example of vector space, these instances include Euclidean 3-space, the complex plane, the set of solutions to a system of linear equations with real coefficients, and the space of states of a quantum mechanical system. The nature of the elements that serve as vectors in different contexts may be different: an element of Euclidean 3-space is a point, an element of the complex plane is a complex number, a solution of a system of linear equations is an $n$-tuple of (real) numbers, and a state of a quantum mechanical system is represented by a function. Nevertheless, if we ignore (abstract from) these contextual differences, in each case the vectors can be added and multiplied by scalars (numbers) according to exactly the same rules, and each of the spaces is closed under these two operations. Focusing on operations with and relationships between vectors while ignoring the specific nature and properties of the vectors in each context, the mathematician obtains the abstract vector space. Hence, to mathematicians, abstraction is closely linked to decontextualization.

### Characteristics

Mathematics educators, on the other hand, are more interested in the processes that lead learners to grasp a structure than in the structure itself. Hence, for mathematics educators, abstraction is a process rather than an object. Mathematics educators investigate the processes by which learners attempt, succeed, or fail to reach an understanding of the structure of a concept or a strategy or a procedure, where structure refers to the elements and the relationships and connections between them. Mathematics educators also study conditions, situations, and tasks that facilitate or constrain such processes. Since mathematical notions generally have some structure, these processes are relevant not only to unquestionably abstract structures like vector space or group but also to most of the notions usually learned in schools including addition, the algorithm for multiplying multi-digit numbers, negative number, ratio, rate of change, sample space, and the integral. Moreover, since learners are usually led to approach these notions in a restricted context, context is an important factor to be taken into consideration when investigating processes of abstraction.

Wilensky (1991) has remarked that "concretion" might be a more appropriate term than abstraction for what mathematics educators intend to achieve: attaining an understanding of structure means establishing connections and "the more connections we make between an object and other objects, the more concrete it becomes for us" (p. 198). Hence, the goal is to make notions that are considered abstract by mathematicians more concrete for learners.

Piaget may have been the first to attend to the issue of abstraction as a cognitive process in mathematics and science learning, in particular young children's learning; his distinction between empirical and reflecting abstraction and his work on reflecting abstraction have been enormously influential. Empirical abstraction is the process of a learner recognizing properties common to objects in the environment. Reflecting abstraction is a two-stage process of (i) projecting properties of a learner's actions onto a higher level where they become conscious and (ii) reorganizing them at the higher level so they can be connected to or integrated with already existing structures. As Campbell, the editor, remarks, projecting refers to the optical meaning of reflecting, whereas reorganizing refers to its cognitive meaning, and the term "reflecting" is more accurate than the usually used "reflective."

Mitchelmore and White (1995) focus on and further develop Piaget's notion of empirical abstraction. They build on Skemp's elaboration of empirical abstraction as lasting change allowing the learner to recognize similarities between new experiences and an already formed class and propose a theory of teaching for abstraction that links the lasting nature of the change to the learner's connections between different contexts.

A number of researchers have further developed Piaget's thinking about reflecting abstraction and applied it to school age learners. Among them, Thompson (1985) has proposed a theoretical framework in which mathematical knowledge is characterized in terms of processes and objects. The central issue is how a learner can conceptualize a process such as counting, multiplication, or integration as a mathematical object such as number, product, or integral. The learner usually first meets a notion as a process and is later asked to act on the object corresponding to this process. The transition from process to object has been called objectification. The notion of reification, proposed by Sfard (1991), is closely related to objectification; the relationship has been discussed in the literature by Thompson and Sfard.

Gray and Tall (1994) have pointed out that mathematical understanding and problem solving requires the learner to be able to flexibly access both, the process and the object. They proposed the term procept to refer to the amalgam of three components: a process, which produces a mathematical object, and a symbol, which is used to represent either process or object.

The notions of process and object are central to learning mathematics, and it is very important for mathematics educators to gain insight into learners' processes of objectification and into how such processes can be encouraged and supported. Mason (1989) proposed to consider abstraction as a delicate shift of attention and the essence of the process of abstraction as coming to look at something differently than before. The shift from a static to a dynamic view of a function graph may be an example; the shift from seeing an (algebraic) expression as an expression of generality to seeing it as an object is another one. Researchers investigating processes of abstraction only gradually took Mason's perspective seriously, possibly because of the heavy investment in time and effort it implies. Indeed, in order to gain insight into learners' shifts of attention and hence processes of abstraction, microanalytic analyses of learning processes are required. Such analyses have been carried out by several teams of researchers and are at the focus of the remainder of this article.

Dubinsky and his collaborators (2002) observed undergraduate students' learning process by means of the theoretical lens of schemas composed of processes and objects; they did this for case of mathematical induction, predicate calculus, and several other topics. For each topic, the analysis led to a genetic decomposition of the topic and to conclusion on the design for instruction supporting conceptual thinking.

In parallel, Simon et al. (2004), using the fraction concept as illustrative example, elaborated a mechanism for conceptual learning. The elements of that mechanism include the learners' goals, their activities in attempting to attain these goals, and their observation of the effect of each activity. The researchers then link the relationship between activity and effect as perceived by the learner to reflecting abstraction.

Taking up Wilensky's (1991) theme, Noss and Hoyles (1996) stress the gain of new meanings (rather than a loss of meaning) in the process of abstraction and hence consider this process as experiential, situated, activity-based, and building on layers of intuition, often in a technology-rich learning environment. They introduce the metaphor of webbing, where local connections become accessible to learners, even if the global picture escapes them. Recognizing that in each instance such webbing is situated in a particular setting, they coin the term *situated abstraction*. Pratt and Noss have recently discussed design heuristics implied by this view of abstraction.

Another characteristic of situated abstraction, possibly because of the authors' focus on the use of computers, is a strong role of visualization in processes of abstraction. This has led Bakker and Hoffmann (2005) to propose a semiotic theory approach according to which learners proceed by forming "hypostatic abstractions," that is, by forming new mathematical objects which can be used as means for communication and further reasoning.

Whereas Noss and Hoyles' approach to abstraction is fully situated, the other approaches discussed above are cognitive in nature. An approach that bridges cognition and situatedness by analyzing cognitive processes with respect to the context in which they occur has been proposed and developed by Schwarz et al. (2009) under the name *Abstraction in Context*. Schwarz et al.'s view of abstraction has its roots in the notion of vertical mathematization in the sense of the Freudenthal school and in Davydov's ascent to the concrete (which recalls Wilensky's concretion). Their methodology is determined by a model of three nested epistemic actions that have been identified as relevant to processes of abstraction: recognizing, building with, and constructing. This model provides the tools for analyzing learners' progress through processes of abstracting in a manner that is strongly linked to the learning environment as well as to the mathematical, the social, the historical, and the physical context. This notion of context is, of course, far wider than the one mentioned above, in the first paragraph. It allows the researcher, for example, to follow the flow of ideas in small groups of students during processes of abstraction or to analyze the influence of technological and other tools on processes of abstraction. The analytical power of the model has been demonstrated by its power to determine learners' partially correct constructs and by identifying patterns of interaction of constructing actions that are indicative for the learner's enlightenment in justification processes (see Dreyfus 2012 for references).

## Cross-References


- ▶ Actions, Processes, Objects, Schemas (APOS) in Mathematics Education
- ▶ Algebra Teaching and Learning
- ▶ Concept Development in Mathematics Education
- ▶ Design Research in Mathematics Education
- ▶ Mathematical Representations
- ▶ Realistic Mathematics Education
- ▶ Situated Cognition in Mathematics Education
- ▶ Visualization and Learning in Mathematics Education


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## Actions, Processes, Objects, Schemas (APOS) in Mathematics Education

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### Keywords

Encapsulation; Interiorization; Mental mechanisms; Mental structures; Totality

### The Mental Structures and Mechanisms of APOS Theory

APOS Theory is a theory of mathematical understanding, its nature, and its development. It is an outgrowth of Piaget's theory of reflective abstraction (Piaget 1971) and, although originally created to apply Piaget's ideas about children's learning to postsecondary mathematics, it has been applied to elementary school and high school mathematics as well. The basic tenet of APOS Theory, a constructivist theory, is that an individual's understanding of a mathematical topic develops through reflecting on problems and their solutions in a social context and constructing or reconstructing certain mental structures and organizing these in schemas to use in dealing with problem situations. The main ideas in APOS Theory were introduced in Dubinsky (1984). The acronym APOS was first used in Cottrill et al. (1996).

The mental structures proposed by APOS Theory are actions, processes, objects, and schemas (and thus the acronym APOS). The structures are constructed by means of certain mental mechanisms including interiorization, encapsulation, de-encapsulation, coordination, reversal, generalization, and thematization. Following are brief descriptions and examples of these mental structures and mechanisms.

According to APOS Theory, a mathematical concept is first understood as an *action*, a set of step-by-step instructions performed explicitly to transform physical or mental objects. For example, with the function concept, an action would consist of plugging a value into an expression and calculating the result. As an individual repeats and reflects on the action, it may be *interiorized* into a mental *process*. A process is a mental structure that performs the same operation as the action but wholly in the mind of the individual. For the concept of function, this means that the individual can imagine any element in the domain being transformed into an element of the range by an expression or by any other means. As the individual becomes aware of the total process, realizes that transformations can act on it, and/or actually constructs such transformations, the process is *encapsulated* into a mental *object*. With an object conception of function, an individual is able, for instance, to perform arithmetic operations on functions to obtain new functions. In developing an understanding of a mathematical topic, an individual may construct many actions, processes, and objects.

When these are organized and linked into a coherent framework, the individual has constructed a *schema* for the topic. The coherence of a schema is what allows one to decide if it can be used in a particular mathematical situation. For example, the coherence of an individual's function schema might consist of an abstract definition of function: a domain set, a range set, and a means of going from an element of the domain to an element of the range. This would allow the individual to see functions in situations where "function" is not explicitly mentioned, and use functions to solve problems.

The mental mechanism of *coordination* is used to construct a new process from two or more processes that already exist in the mind of the individual. This coordination can take the form of connecting processes in series, in parallel, or in any other manner. One example of coordination is the composition of two functions in which the process is constructed by taking outputs of one function and using them as inputs to the other function. The mechanism of *reversal* creates a new process by reversing the operation of an existing process, as in forming the inverse of a function, and *generalization* changes a process by applying it to objects in a context more general than previously considered, as, for example, in extending the domain of a function represented by an expression from real numbers to complex numbers. Finally, *thematization* constructs an object by applying actions and/or processes to existing schemas. For instance, comparing the concept of function with the concept of relation, which may result in the implication "every function is a relation," is an example of an action on two schemas resulting in an object, the implication. For a more detailed description of APOS Theory, see Arnon et al. (in press).

## Studies That Use APOS Theory

APOS Theory is an analytic tool that can be, and has been, used for investigating individuals' understanding of a mathematical concept and for describing the development of that understanding in the individual's mind. Some studies use APOS Theory for evaluating understanding, some use it for describing development, and some use it for both. The description of development of understanding a concept obtained from a developmental study is called a *genetic decomposition*. It is often used to design an APOS-based instructional treatment of the concept which is implemented and studied in a subsequent evaluative study. Such evaluative studies may involve comparison of results with a control group that received instruction not based on an APOS analysis.

For example, studies of the process conception of functions, the definite integral and area, calculus graphing schemas, the set $P(N)$ of all subsets of the natural numbers, infinite iterative processes, and the relation between the infinite repeating decimal 0.999... and 1 investigated how various mathematical concepts might be constructed in the mind of an individual. Studies of prospective elementary school teachers' understanding of the multiplicative structure of the set of natural numbers, including least common multiple; college students' understanding of the set of natural numbers; and preservice teachers' understanding of the relation between an infinite repeating decimal and the corresponding rational number evaluated the effectiveness of APOS-based instructional treatment. Finally, studies of college students' understanding of permutations and combinations, calculus students' graphical understanding of the derivative, comparative APOS Theory analyses of linear transformations on vector spaces, and grade 5 students' understanding of fractions were all investigations of both the mental constructions of mathematical concepts and evaluations of the effectiveness of APOS-based instructional treatments.

## APOS-Based Instruction

As we indicated above, APOS-based research often involves APOS-based instruction, and thus, evaluation applies not only to the understanding that students have but also to the extent to which that understanding is due to the instruction that was used. APOS-based instruction applies the basic tenet of APOS Theory mentioned in the first paragraph of this entry. According to this hypothesis, students deal with mathematical problem situations by constructing and reconstructing mental structures. One result of these constructions, according to APOS Theory, is that learning can then take place. Therefore, the first goal of APOS-based instruction for a particular concept is to help students make the mental constructions called for in the genetic decomposition for that concept. If this happens, APOS Theory hypothesizes, understanding the concept will not be difficult for the student and can be achieved through a number of pedagogical strategies, both traditional and nontraditional.

The question then arises of what pedagogical strategies might help students perform actions, interiorize actions into processes, encapsulate processes into objects, and gather everything into coherent schemas. Although work in APOS Theory has gone on for several decades and has had considerable success, there is still much that needs to be done in the area of pedagogical strategies. Several approaches have been used involving cooperative learning, role-playing, and writing essays. So far, the most effective pedagogy involves students writing computer programs. For example, if students can express an action as a computer procedure and run it with various inputs, then they will tend to interiorize this action into a process. If the software used has the capability of treating such a procedure as data and performing operations on it (e.g., ISETL), then using this feature to solve problems helps students encapsulate the process underlying the computer procedure into an object. It has also been shown that it is possible to foster encapsulation by having students use specially designed software to perform operations on processes, even infinite processes. In another approach, APOS has been used as a grounding learning theory for the development of textbooks for college-level courses in calculus, discrete mathematics, abstract algebra, linear algebra, and Euclidean and non-Euclidean geometries. The textbooks rely on the use of various software such as ISETL, Maple, and Geometer's Sketchpad. Working with such software can help students make the mental constructions that lead to learning mathematical concepts and provide a dynamic interactive environment for students to explore the properties of geometric and other mathematical objects and their relationships.

## Results

There has been considerable research and curriculum development based on APOS Theory since its inception in the early 1980s. Most of this work demonstrates the efficacy of this theory for theoretical analysis, evaluation of learning, and describing the development of concepts in the mind of an individual. A survey by Weller et al. (2003) summarizes the results of student learning through the use of APOS-based instruction during the first two decades of its existence. Mathematical topics included in this summary are the derivative, the chain rule, and the definite integral in calculus; binary operations, groups, subgroups, cosets, normality, and quotient groups in abstract algebra; the concept of function; existential and universal quantification; and mathematical induction. In the decade since that summary appeared, work with APOS Theory has expanded. Many research reports and doctoral theses have been written and additional topics have been investigated. These include fractions; permutations and combinations; vector spaces, bases, spanning sets, systems of linear equations, and linear transformations in linear algebra; and divisibility properties of integers, functions of several variables, differential equations, countable and uncountable infinity, equivalence structures on sets, statistics, and logic. It would not be easy to survey this growing body of work.

## Ongoing and Future Work

APOS Theory is a developing set of ideas as new studies and new researchers appear. For example, several researchers are providing new insights into the development of understanding many topics in linear algebra. And a recent study has produced data suggesting the need for a new stage in APOS Theory, tentatively called totality, which would lie between process and object and refers to seeing a process as a whole, with all steps present at once. One area in which it may be time to begin investigations would be the relationship between APOS Theory and related theoretical frameworks such as the duality theory of Sfard (1991) and the procepts of Gray and Tall (1994). Another possible area of study would be to investigate whether there is a connection between thinking at one of the APOS stages and brain function. Finally, it would be interesting and possibly important to study the relation between APOS Theory and the sociology of mathematical knowledge as it develops in the classroom. With a successful foundation developed over the past 30 years, it is reasonable to expect continued development of APOS Theory and its use in helping students understand mathematical concepts.

## Cross-References


- ▶ Abstraction in Mathematics Education
- ▶ Concept Development in Mathematics Education
- ▶ Constructivism in Mathematics Education
- ▶ Discursive Approaches to Learning Mathematics
- ▶ Epistemological Obstacles in Mathematics Education
- ▶ Theories of Learning Mathematics


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# Activity Theory in Mathematics Education


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## Keywords

Vygotsky; Leont'ev; Dialectics; Consciousness; Personality; Change


## Definition

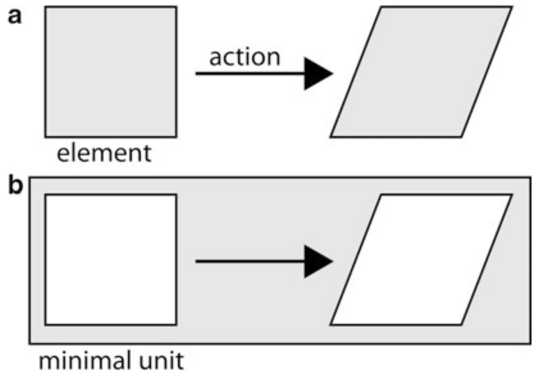
Activity theory is the result of an attempt to construct a psychology that draws on and concretely implements epistemological principles of materialist dialectics as K. Marx presented them (Leont'ev 1978; Vygotsky 1997). Like Marx's *Das Kapital*, activity theory is intended to explain change, learning, and development as an *immanent* feature of a system rather than in terms of externally produced cause-effect relations.

## History of Activity Theory

L. S. Vygotsky generally is recognized as the founding father of activity theory because he introduced the idea of tool-mediated activity as a way of overcoming on-going psychological ideas consistent with stimulus–response or disembodied thinking approaches to cognition. Responding to the crisis of psychology, he explicitly stated the need for developing a Marxist psychology. Expanding on Vygotsky's work, A. N. Leont'ev articulated what is now known as second-generation cultural-historical activity theory in his *Activity, Consciousness, and Personality* (Leont'ev 1978), in which the first two chapters are devoted to establish the Marxist foundation of the theory. The third-generation activity theory was formulated in two different lineages. The Helsinki version originally established by Y. Engeström (1987) focuses on the structural-systemic aspects of activity, whereas the Berlin version, developed by K. Holzkamp (1993) and his colleagues, is a subject-oriented psychology that focuses on the person and consciousness. Fourth-generation cultural-historical activity theory builds on both third-generation versions and also includes emotions (affect) and ethics as irreducible fundamental moments of human activity (Roth and Lee 2007).

## Minimum Unit

In all other psychologies, individuals and objects are the minimal units of analysis. Thus, for example, the transformation of a square into a parallelogram by means of shearing would involve a human agent, who, by acting on the square, would turn it into the result (Fig. 1a). The action is external to the object. The human subject and his/her actions are the causes for the transformation. Activity theory, on the other hand, conceives the situation in a radically different way. In this theory, the entire production of some outcome from the beginning conception to its material realization is the minimum unit (Fig. 1b). This unit bears an inner contradiction, because depending on how and when we look at it, we would see a square, a person, a parallelogram, none of which exists independent of the entire unit. Because of this inner contradiction, the unit is referred to as a dialectical unit; it sublates – simultaneously integrates and




**Activity Theory in Mathematics Education, Fig. 1** (**a**) In traditional theories, people and objects are the minimum units; change is the result of outside actions on objects (elements). (**b**) In activity theory, the minimum unit encompasses the entire change process; it is impossible to speak of causes and effects

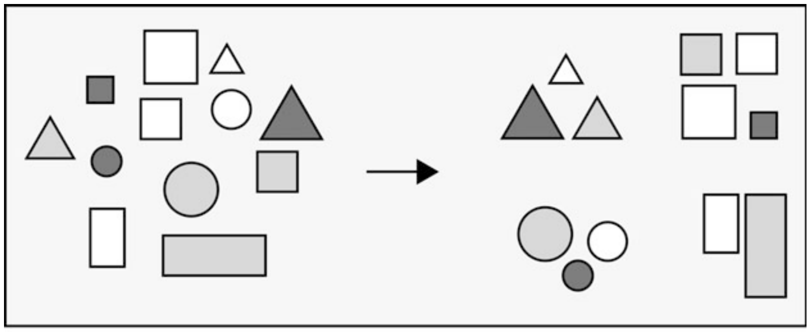
overcomes – what manifests itself in irreconcilable differences. If we were to look at school mathematics, then prior knowledge, post-unit knowledge, grades, teacher, and students would all be part of this minimal unit and could not be understood independent of it. By using this unit, change is *immanent* to the minimum category and does not require external agents. To understand the key principles, consider the following two scenarios.

Scenario 1: Connor and his peers in a second-grade mathematics class sort objects into groups, which become constitutive of geometrical relations within and between the objects.

Scenario 2: Erica, a fish culturist, talks about the production of coho salmon smolt to be released into the river to increase natural stocks; she monitors and controls the production process using a spreadsheet-based database and mathematical functions such as graphs, histograms, and mathematical functions (e.g., to calculate amount of food).

In the first scenario, the minimal unit would be schooling; as part of doing schooling, the second-grade students complete tasks. That is, not their grouping task, where a collection of objects is sorted is the activity, but the before and after in the context of schooling belongs into the minimal unit as well (Fig. 2). This is so because the ultimate productions that really count are grades and

**Activity Theory in Mathematics Education, Fig. 2** The entire process by means of which a collection of objects into a groups of like-objects, together with institution, tools, and people constitutes the minimum unit

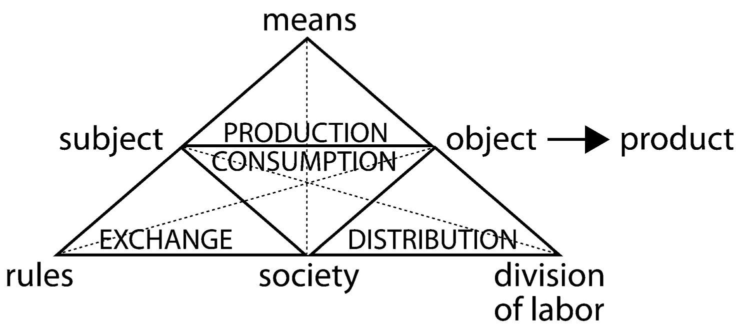



grade reports. Activity theory explains learning as a by-product in the production of grades. It does not account for mathematical activity as if it could occur outside and independent of the schooling context. In the second scenario, the ultimate product is a population of young salmon released into the river. The computer and the mathematics that Erica draws on are means employed in the production. What she can be observed to do is subsumed into the one category of salmon production – which contributes to increased opportunities for commercial fishing (generalized dietary needs), native sustenance fishing (specific dietary needs), and tourism focusing fishing (leisure) (Roth et al. 2008).

## Structure of Activity

The structural approach, as embodied in the mediational triangle, is perhaps the most well-known and used version of cultural-historical activity. As Fig. 3 shows, it makes thematic 7 moments that constitute the parts of the irreducible unit of productive activity: subject, object, means (of production), product, rules, society, and division of labor. All production is oriented towards ultimate consumption, which meets some generalized basic (e.g., food, shelter) or extended need (e.g., leisure). Schooling, in the course of which the second-graders complete the sorting, involves teachers and students who have different roles (division of labor), school buildings, (school) rules of engagement, and society. It is society that ultimately comes to be reproduced in the activity of schooling, both in terms of certain practices as in the hierarchical relationships between those who go to university and those who end up doing menial labor or drop out and never finish school. Society also benefits from the activity in which Erica is a part, because the salmon she contributes to producing ultimately lead to the generalized satisfaction of needs. Thus, dietary needs may be satisfied directly or in exchange of a salary for working on a fishing vessel or in the tourism industry (as fishing guide or maid in a hotel or lodge). What Erica does with the mathematical tools and with mathematics, for example, graphs, numbers, equations, and histograms, cannot be understood outside of the system as a whole. Activity theorists are not interested in understanding the structure at a given point; rather, the entire transformation of goods into products is an integral part of the same unit. A better representation would be similar to Fig. 1, with two different triangles within the same unit. Nothing within the unit makes sense on its own. That is why activity theorists speak of the mediation of actions by the activity as a whole.

Central to activity theory is the transformation of the *object* into a *product*, which initially only consists ideally. The intended transformation is the motive of activity. Activity theorists therefore speak of the *object/motive*. By definition, this category includes both material (the materials started with) and ideal dimensions (e.g., the envisioned product). For Erica, the intended outcome is clear. She wants to end the work cycle with a healthy brood of about one million coho




**Activity Theory in Mathematics Education, Fig. 3** Productive activity may be analyzed in terms of the seven moments that constitute a system; the products are exchanged, coming to be distributed in society, and ultimately are consumed (or used up)

salmon, with an average weight per specimen of about 20 g. In contrast, the second-grade students do not know the intended outcome of their task; and they are not likely aware of the ultimate motive of schooling (Roth and Radford 2011). As a result, they have to engage in the activity constituting task without knowing its object/motive. With respect to the task, they can become conscious of the reasons for doing what they do – that is, they can become aware of the goal – only when they have completed their task. It is when they realize the grouping in Fig. 2 that they can come to understand why the teacher, for example, asked Connor to rethink his actions when he placed one of the squares with the rectangles rather than with the other squares.

## Subjectification and Personality

Cultural-historical activity theory allows us to understand two developmental processes. On the one hand, when a person participates in an activity, such as Connor in schooling or Erica in producing young coho salmon, they undergo subjectification. This concept names the process by means the person, together with everything else that makes the activity system, undergoes change. This change can be noted as the emergence of new capacities for actions of a body together with new forms of talk, neither of which has been identifiable previously. Together, these changes in objects, bodies, and forms of talk reconfigure the field of experience. Thus, for example, as Erica uses the spreadsheet to track and model the coho salmon population, she becomes more proficient with spreadsheets, mathematical models, calculating feed needs, graphs, histograms, and calculations. With these changes, her entire field of experience is reconfigured. Most importantly, activity theory does not allow us to speak of her development independent of everything else at her worksite; her transformation also means transformation of the entire field.

For Erica, working in the hatchery is only part of her everyday life, just as for Connor going to school is only part of his everyday life. Both participate in many other activities: as family members, shoppers, participants in leisure activities, or as members in urban traffic systems. That is, in the course of their everyday lives, both contribute to realizing other object/motives other than producing a population of young coho salmon and doing schooling. Leont'ev introduced the category personality to integrate all these object/motives. Thus, personality is understood as a network of societal object/motives. That is, personality is made up of an ensemble of collective object/motives. However, each network is highly individual. Personality, therefore, is utterly singular while being entirely constituted by societal/collective moments.

## A Holistic Psychology

Cultural-historical activity theory is a holistic approach to psychology. It does not reduce the individual to its thoughts (mental constructions). It in fact integrates body and mind, on the one hand, and individual and collective, on the other hand (Vygotsky 1989). It focuses on change as inherent in life and society and, therefore, inherent in individual life and understanding. That is, everything we do has to be understood in terms of this intersection of dimensions, which manifest themselves in mutually excluding ways: mind versus body, individual versus society, and the natural world versus the social world. The developments at the different time-scales also are irreducible and therefore mutually constitute each other. The moment-to-moment changes – e.g., the transformation of a collection of objects in the second-grade mathematics classroom or the entering of fish size and weight into the database – are related to the development of system and people (i.e., subjectification) over time, and because people and systems are part of society, the cultural-historical changes of the human life form.

## Cross-References


▶ Affect in Mathematics Education
▶ Anthropological Approaches in Mathematics Education, French Perspectives
▶ Mathematical Modelling and Applications in Education
▶ Mathematical Representations
▶ Mathematization as Social Process
▶ Situated Cognition in Mathematics Education
▶ Theories of Learning Mathematics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# Adults Learning Mathematics


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## Keywords

Adults learning mathematics; Adult numeracy; Numeracies; Quantitative literacy; Mathematical literacy; Techno-mathematical literacy; Mathematics education


## Definitions: Contested Meanings in an Emerging Field

Adults learning mathematics is a young field of study and research, emerging towards the end of the twentieth century in what Wedege (2010) has called the borderland between mathematics education and adult education.

Like some other borderlands, this one is disputed, with the area variously termed as follows: "numeracy" (or "numeracies," after Street (2005) and others), "quantitative literacy," "mathematical literacy," "techno-mathematical literacy," and "mathematics education"; the latter sometimes linked with the other STEM subjects (science, technology, and engineering) and often prefaced by "adult."

Numeracy is a particularly contested term. Since its first appearance in 1959 denoting what might now be called scientific literacy (Ministry of Education 1959, p. 270), it has been used variously to denote computational and functional concepts, as well as ideas of numeracy as social practice.

In current usage, there is often a link with literacy and usually a focus on the use of mathematics in adult life; the mathematics involved (sometimes number skills only) is nowadays usually at a basic level (Coben et al. 2003, p. 9). The evolution of concepts of numeracy has been analyzed by Maguire and O'Donoghue (2003) as a continuum with three phases of increasing sophistication: Formative, Mathematical, and Integrative. The latter phase is conceptualized as a complex, multifaceted construct incorporating the mathematics, communication, cultural, social, emotional, and personal aspects of numeracy for each individual in a particular context.

The link with literacy persists. Numeracy was included in the United Nations' Declaration of Education For All, ratified at Jomtien, Thailand, in 1990, as an "essential learning tool," encompassed within literacy and covering the ability "to make simple arithmetical calculations (numeracy)" (Haddad et al. 1990, p. ix). Seven years later, numeracy first appeared in the International Standard Classification of Education as "Literacy and numeracy: Simple and functional literacy, numeracy" (UNESCO 1997).

Numeracy is still sometimes explicitly subsumed within literacy, as in an Australian definition which states that literacy "incorporates numeracy" (Campbell 2009, p. 11). Often the subsumation is implicit, making it difficult to tell whether numeracy is included in statements about literacy and in literacy programs, including those listed on UNESCO's "Effective Literacy Practice" website (UNESCO 2009-2014).

The latest international survey, the Programme for the International Assessment of Adult Competencies (PIAAC), and its immediate precursor, the Adult Literacy and Life Skills Survey (ALLS), both include numeracy, with PIAAC defining numeracy as:

> The ability to use, interpret and communicate mathematical information and ideas in order to engage in and manage the mathematical demands of a range of situations in adult life. (OECD 2012, p. 34)

An alternative term, quantitative literacy, is defined in the first such survey, the International Adult Literacy Survey (IALS), as:

> the knowledge and skills required to apply arithmetic operations, either alone or sequentially, to numbers embedded in printed materials, such as balancing a checkbook, figuring out a tip, completing an order form, or determining the amount of interest on a loan from an advertisement. (Murray et al. 1998, p. 17)

The Programme for International Student Assessment (PISA), which assesses 15-year-olds on the competencies required in adulthood, defines mathematical literacy as follows:

> Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (OECD 2006, p. 12)

The notion of techno-mathematical literacies builds on the PISA definition as "a *specific* characterization of mathematical literacy [. . .] that takes account of the character of the *workplace* in which IT is pervasive" (Kent et al. 2007).

Programs in which adults learn mathematics are also variously termed as follows: "adult numeracy education," "bridging mathematics" (denoting programs preparing students for university study involving mathematics), "mathematics learning support" or "academic numeracy" (denoting programs in which mathematics is a service subject, supporting students to cope with the mathematical demands of their post-school courses), and "basic mathematics." Within and outside formal educational provision, such programs may be geared to the mathematics involved in vocational areas, in work with families, and communities or be more abstract in focus. They may include financial literacy (Atkinson and Messy 2012) or statistical literacy (Gal 2002). The mathematics involved may be at a range of levels, including in the so-called numerate disciplines where an ability to analyze or extrapolate data is required. Mathematics also features to varying degrees in programs termed as the following: "basic education," "basic skills," "foundation learning," "functional skills," "functional mathematics," "vocational skills," "workplace learning," "essential skills," "development education," "*educación popular*" (in Latin America), and "*bildung*" (in parts of Europe).

This diversity of forms and nomenclature reflects the emergent nature and borderland location of the field and the diversity of purposes for which adults learn mathematics, the contexts in which they do so, the forms of educational provision and organization involved, the areas and levels of mathematics covered, and the practice, policy, and research trajectories of the work.

Here we consider adults learning mathematics to be the focus of a field of practice, research, and policy development encompassing the formulations outlined above. At the heart of the field is the learning and use of mathematics by those who are regarded as adults in the society in which they live.

In the following sections, we outline the emergence of the field of adults learning mathematics through an overview of key events, organizations, research, publications, international surveys, and national strategies.

## Characteristics of an Emerging Field

A series of events led to the emergence (or recognition) of the field in the early 1990s. As noted above, in 1990 the Jomtien Declaration put numeracy on the world's educational map as a complement to literacy. At the same time, concern about adults' skill deficits led to a series of international OECD (Organization for Economic Cooperation and Development) surveys which assessed, with respect to mathematics, first, quantitative literacy (in three waves of IALS: 1994; 1996; 1998) and then numeracy, assessed in ALLS in two waves in 2002 and 2006, and in PIAAC, which is due to report from October 2013.

In response to poor results in IALS and ALLS, together with growing evidence of the negative impact of poor numeracy (and literacy) on adults' lives (Bynner 2004) and concerns about the impact of low skills on productivity, national strategies with an adult focus were established in several OECD countries. For some, numeracy was linked to literacy, as in the *Skills for Life* strategy in England, launched in 2001 (DfEE 2001). National centers of various kinds were established to support these strategies, including, for example, the National Research and Development Centre for Adult Literacy and Numeracy (NRDC, www.nrdc.org.uk) in England. In New Zealand, also, a plan to improve adult literacy and numeracy was launched in 2001 (Walker et al. 2001), supported since 2009 by the National Centre of Literacy and Numeracy for Adults (http://literacyandnumeracyforadults.com/). In Australia, the Adult Literacy and Numeracy Australian Research Consortium (ALNARC, www.staff.vu.edu.au/alnarc/index.html) is a national collaboration between university-based research centers funded from 1999 to 2002 under Adult Literacy National Project funds. In the US research in adult language, literacy and numeracy was spearheaded from 1991 to 2007 by the National Center for Adult Literacy (NCAL, later the National Center for the Study of Adult Learning and Literacy: NCSALL), which published a call for research on adult numeracy in 1993 (Gal 1993). A parallel concern with mathematics as a STEM subject led to the establishment in Ireland of the National Centre for Excellence in Mathematics and Science Teaching and Learning (NCE-MSTL, www.nce-mstl.ie/) in 2009. In England the National Centre for Excellence in the Teaching of Mathematics (NCETM, https://www.ncetm.org.uk/) was set up in 2006 to enable access to continuing professional development for teachers of mathematics to learners of all ages.

Meanwhile, in the poorer nations of the South, Torres' comment on the "continued reduction of adult basic education, and even adult education in general, to *literacy*, and continued narrow perceptions of literacy as a simple, elementary skill" applies equally to numeracy, subsumed, as it often is, within literacy (Torres 2003, p. 16). Adults learning mathematics are largely invisible in the literature of "developing education" despite their presence in, for example, microfinance initiatives aimed at alleviating poverty and education geared to adults' livelihoods, as highlighted in a review of *Adult Numeracy: Policy and practice in global contexts of* lifelong learning (Johnston and Maguire 2005). The REFLECT (*Regenerated Freirean Literacy Through Empowering Community Techniques*) program is a notable exception: "distinguished from almost all other literacy programs by its inclusion of numeracy, not just as an add-on, but as a core element in the process of empowerment" (Johnston and Maguire 2005, p. 36).

In 1992, the worldwide response to an article by Coben (1992) stressing the need to bridge the gulf between the cultures of academic researchers and practitioners led to the founding of the international research forum Adults Learning Mathematics (ALM) (www.alm-online.net), an event described by Wedege (2010, p. 13) as "decisive for the growth into a field of what were until then sparse research activities." ALM's focus on research has helped to propel the shift from a field of practice to a field of *research and practice* in which ideas, evidence, and experience are shared across time and across continents. In 1998, the first Mathematics Education and Society (MES) international conference provided a forum for discussing the social, political, cultural, and ethical dimensions of mathematics education, geared to all ages and including a focus on adult learning that is often absent from mathematics education conferences.

Recognition of the emerging field by the wider mathematics education community came with the inclusion of the first working group on "Adults Returning to Study Mathematics" at the 8th International Congress on Mathematics Education (ICME-8) in 1996. Adult-focused topic study groups subsequently met at ICMEs 9, 10, and 11, and edited proceedings were published (Coben and O'Donoghue 2011; FitzSimons et al. 2001); in the case of ICME-10, selected papers were published in the *Adults Learning Mathematics International Journal* (*ALMIJ (1*)2; www.alm-online.net), and a chapter was included in the ICME-10 Proceedings (Wedege et al. 2008). At ICME-11 in 2012, adult educators were catered for in the topic study group "Mathematics education in and for work."

The publications of centers such as NRDC, NCSALL, and NCE-MSTL, the Proceedings of the ICME groups, and ALM's conference proceedings and online journal (*Adults Learning Mathematics International Journal* www.alm-online.net) together constitute a major contribution to the literature of the field. ALM members have also contributed to a growing reference literature, with chapters in the first and second *International Handbooks of Mathematics Education* (FitzSimons et al. 2003, 1996). FitzSimons and Coben also contributed to the *UNESCO-UNEVOC International Handbook of Technical and Vocational Education and Training* (FitzSimons and Coben 2009) and, with O'Donoghue, edited the first book to review the field, *Perspectives on Adults Learning Mathematics: Research and practice* (Coben et al. 2000). The publication of *The Adult Numeracy Handbook: Reframing Adult Numeracy in Australia* (Kelly et al. 2003) added to Australia's strong reputation in adult numeracy education.

The emerging research field (dubbed adult numeracy) was mapped in the first comprehensive review of research in 2003 by NRDC (Coben et al. 2003). This was followed by Condelli et al.'s (2006) review for the US Department of Education and then by Carpentieri et al.'s (2009) review for the BBC. Most recently, NRDC's *Review of Research and Evaluation on Improving Adult Literacy and Numeracy Skills* concluded that continuing investment is needed, but based on stronger evidence of which skills are required than currently exists. The authors recommend "better quality interventions, and large, well designed and more sophisticated studies, that allow for the time and complex causality that connects learning interventions to their outcomes" and recommend that lessons are learned from the US Longitudinal Study of Adult Learning (Reder 2012; Vorhaus et al. 2011, p. 14).

We endorse these comments while noting that despite these limitations, a wealth of research on adults learning mathematics has emerged over a short period. Researchers draw on diverse disciplines and theoretical and methodological frameworks to investigate a wide range of themes. Overall there is a strong orientation towards improving professional practice and outcomes for adult learners and a strong international focus. An international comparative project, "Policies and pedagogies for lifelong numeracy," a collaboration between ALM members in Australia and Ireland, provides a snapshot of the field of practice from the perspective of practitioners in various countries, including several seldom represented in the literature (Maguire et al. 2003). The picture of an emerging field, with much hitherto unrecorded practice, is borne out.

Here we give a brief selective overview of this work, focusing on adults as workers, critical citizens, and thinking beings with emotions and experiences and on teaching, learning, and professional development.

### Adults as Workers

Work on adults learning mathematics in and for the workplace predates the growing interest in the field generally and continues to develop strongly. For example, research for the Cockcroft Report (DES/WO 1982) investigated the mathematics used in work (FitzGerald and Rich 1981), Howson and McClone (1983) reviewed ways in which mathematics is used in adults' working lives, and Sticht and Mikulecky's (1984) examined job-related basic skills in the USA.

Studies of specific work practices include Lave's study of Liberian tailors (Lave 1977), Zevenbergen's (1996) work on the "situated numeracy of pool builders" in Australia, Llorente's (1997) Piagetian analysis of the work-related activities of building workers with little schooling in Argentina, and Smit and Mji's (2012) research on the assessment of numeracy levels of workers in South African chrome mines. Numeracy for nursing has emerged against the background of concern about patient safety and problems with the assessment of numeracy for nursing (Coben et al. 2009). Research has been undertaken in several countries, including Australia (Galligan et al. 2012; Galligan and Pigozzo 2002), the UK (Coben et al. 2010; Hoyles et al. 2001; Pirie 1987) and Finland (Grandell-Niemi et al. 2006).

The impact of technological change in the workplace is another strong theme. For example, Hoyles and colleagues (2010) describe the emerging need to go beyond mere procedural competence with calculations, to interpret and communicate fluently in the language of mathematical inputs and outputs to technologies.

The focus in these studies is on understanding the mathematical demands and affordances of workplaces in order to increase knowledge in and of the field among educators, employers, policy makers, and workers themselves and make evidence-based recommendations for good practice to equip learners to meet those demands.

### Adults as Critical Citizens

A concern with social justice and critical citizenship is evident in the work of many researchers, for example, in Civil's (2002) work with Hispanic parents and communities in the USA, Knijnik's (1997) work with the Landless People's Movement in Brazil and in Benn's (1997) book *Adults Count Too*. Harris' (1997) book, *Common Threads*, celebrates the often unrecognized mathematics in work traditionally done by women, while FitzSimons (2006) explores "numeracy for empowerment in the workplace." The first edited book on ethnomathematics (the study of the relationship between mathematics and culture) was published in 1997 (Powell and Frankenstein 1997) with a worldwide scope and a strong focus on issues of power. In the same period, the REFLECT program was piloted in Uganda, Bangladesh, and El Salvador. REFLECT uses Freirean and ethnographic participatory rural appraisal techniques whereby groups of learners work with a facilitator to produce learning materials such as maps, matrices, calendars, and diagrams that "represent local reality, systematize the existing knowledge of learners and promote the detailed analysis of local issues" (Archer and Cottingham 1996, p. i). Ethnography was also at the heart of work "to bring about change and broaden horizons" (Lide 2007, p. 5) of rural women in South Asia. Like REFLECT, the project is unusual in focusing strongly on numeracy.

### Adults as Thinking Beings with Emotions and Experiences

Biographical research by Hauk (2005) and Coben (2000) has found mathematics to be often the focus of strong feelings, positive and negative, stemming from adults' life experience.

Wedege (1999) has examined adults' personal relationships with mathematics in school, informal settings, and workplaces, using inter alia Bourdieu's concept of habitus and Lave's concept of situated learning to explore the apparent contradiction between many adults' experience of being blocked in relation to mathematics in formal settings and yet competent in their everyday life. Evans' (2000) work on "adults' numerate practices" is noteworthy here. He investigated the ways in which numerate thinking and performance are context-related; the inseparability of thinking and emotion in mathematical activity; the understanding of mathematics anxiety in psychological, psychoanalytical, and feminist theories; and the social differences in mathematics performance, anxiety, and confidence and developed a set of guidelines for teaching and learning. Other authors (e.g., Buxton 1981; Peskoff 2001) have also investigated mathematics anxiety in adults, a factor which can inhibit mathematics learning. Meanwhile, the OECD's Brain and Learning project (2002–2006) focused on literacy, numeracy, and lifelong learning within three transdisciplinary and international networks, in which cognitive neuroscientists were challenged to tackle questions of direct educational relevance (OECD 2007).

### Teaching, Learning, and Professional Development

Studies of teaching, learning, and professional development in the field include, for example, NRDC's studies of effective practice in adult numeracy (Coben et al. 2007), teaching and learning measurement (Baxter et al. 2006), and making numeracy teaching meaningful to adults (Swain et al. 2005). The largest of these, the "Maths4Life: Thinking Through Mathematics" project, aimed to help teachers to develop more connected and challenging evidence-based and learner-centered teaching methods to encourage active learning of mathematics (Swain and Swan 2007). Earlier, in Australia, an 84-h training course for adult numeracy teachers, based on the idea that numeracy is "not less than maths but more" used a critical constructivist approach grounded in learners' and local communities' experiences and perspectives (Johnston et al. 1997, p. 168). The US EMPower project (www.terc.edu/work/644.html) is a mathematics curriculum development project for adult and out-of-school young people (Schmitt et al. 2000). Also in the USA, an environmental scan of adult numeracy professional development initiatives and practices was undertaken in 2007, laying the groundwork for future research in adult numeracy education and professional development and for testing potential models of adult numeracy professional development (Sherman et al. 2007).

## Concluding Remarks

This brief overview aims to give an indication of the scope and vitality of the emerging field of adults learning mathematics. Much remains to be done.

## Cross-References

- ▶ Affect in Mathematics Education
- ▶ Critical Mathematics Education
- ▶ Ethnomathematics
- ▶ Mathematical Literacy
- ▶ Situated Cognition in Mathematics Education

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# Affect in Mathematics Education

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## Keywords

Emotion; Affect; Anxiety

## Definition

There are two different uses for the word "affect" in behavioral sciences. Often it is used as an overarching umbrella concept that covers attitudes, beliefs, motivation, emotions, and all other noncognitive aspects of human mind. In this article, however, the word affect is used in a more narrow sense, referring to emotional states and traits. A more technical definition of emotions, states, and traits will follow later.

## From Anxiety and Problem Solving to Affective Systems

Mathematics is typically considered as the most objective and logical of academic disciplines. Yet, it has been widely acknowledged that mathematical thinking is not purely logical reasoning, but influenced much by affective features. The first systematic research agenda to study mathematics-related affect was initiated within social psychology in the 1970s, focusing on mathematics anxiety as a specific branch of anxiety research. Anxiety is an unpleasant emotion of fear, which is directed towards an expected outcome in the future and it is often out of proportion to the actual threat. Based on a meta-analysis of 151 studies (Hembree 1990), it has been concluded that mathematics anxiety is related to general anxiety, test anxiety, and low mathematics attainment. Also, gender differences have been found. Female students have been found to be more prone to be anxious than male students, although they also seem to cope with their anxiety more efficiently than male students. Moreover, out of different treatments for mathematics anxiety, systematic desensitization was found to be the most effective.

An important distinction in anxiety research is that made between state and trait type of anxiety, which is a specific case of the distinction between emotional (affective) state and trait. The emotional state refers to the emotion that arises in a certain situation, i.e., it is contextual and may change rapidly. On the other hand, more stable personal characteristics are called emotional traits, which refer to a person's tendency to experience certain emotional states across a variety of situations (Hannula 2012).

Within mathematics education, the research on problem solving notified the role of affect early on; already Polya in his classical work (1957) mentioned hope, determination, and emotions.

More explicitly the role of affect in mathematical problem solving was elaborated in several works published in the 1980s (e.g., Cobb et al. 1989; Schoenfeld 1985; for details, see McLeod 1992 or Hannula 2012). This literature on problem solving typically addressed the rapidly changing affective states in the dynamics of problem solving. Somewhat surprisingly, it was found out that in non-routine problem solving both experienced and novice problem solvers experience positive and negative emotions and that these emotions serve an important function in a successful solution process (e.g., Schoenfeld 1985; McLeod 1992).

Most of the research on mathematics-related affect by that time was summarized by McLeod (1992). He also suggested a theoretical framework that has been influential in mathematics education research. He identified emotions as one of the three major domains in the research of mathematics-related affect. Emotions were seen to be less stable, more intense, and to involve less cognition than attitude or beliefs. He also explicated the relationships between these categories in a theoretical framework: beliefs were seen as an element that influenced the initiation of emotions and repeated emotional reactions were seen as the origin of attitudes.

Also more recent research on affect in mathematics education emphasizes the relations between emotions and other affective variables (Hannula 2012). These include not only attitudes and beliefs but also values, motivations, social norms, and identity. The general trend is that a student who has a positive disposition towards mathematics tends to experience positive emotions more frequently and negative emotions less frequently than a student with a negative disposition. On the other hand, different theories (e.g., McLeod 1992) suggest that emotional experiences play a significant role in the formation of attitudes, beliefs, and motivation. Positive emotional experiences are seen as an important ingredient in the formation and development of a positive disposition. However, details are more complex than that. Some of the complexity is analyzed in a recent study (Goldin et al. 2011) that identified a number of behavioral patterns that integrate students' affective and social interactions in mathematics classes.

### Defining Emotions

In the literature, there are several definitions for emotions stemming from three distinct traditions: Darwinian, Freudian, and cognitive tradition (cf. Hannula 2012). Yet, there is a general agreement that emotions consist of three processes: physiological processes that regulate the body, subjective experience that regulates behavior, and expressive processes that regulate social coordination. Moreover, most emotion theories agree that emotions are closely related to personal goals and that they have an important role in human coping and adaptation. Negative emotion (e.g., frustration) is experienced when progress towards a goal (e.g., solving a task) is prevented, and they may suggest approaches (e.g., give up that task and try another) to overcome the perceived causes. Positive emotions, on the other hand, are experienced when progress is smooth. Emotions are an important part of memories and they will influence the choice of strategies in the future.

Emotion theories vary in the number of emotions they identify, the degree of consciousness they attribute to emotions, and the relation they perceive to be between emotion and cognition (ct. Hannula 2011). Some emotion theories identify a large number of different emotions based on the different social scenarios and cognitive appraisals related to the emotion, while some other emotion theories identify a small number of basic emotions (e.g., happiness, sadness, fear, anger, disgust, shame, surprise, and attachment) that differ in their physiology, and the different cognitive appraisals and social scenarios are seen as external (though closely related) to the emotion.

### The Role of Emotional States in Self-Regulation

Emotions function on three different levels of self-regulation: physiological, psychological, and social (Power and Dalgleish 1997). The most clear example of the physiological adaptation is the "fight-or-flight" response to surprising threatening stimulus. Such physiological functions of emotions may have side effects that are relevant for learning. For example, the effects of high anxiety (fear) are detrimental for optimal cognitive functioning.

The psychological self-regulation of cognitive processing is an important function of emotions in any learning context, especially if we acknowledge the learner's agency in the construction of knowledge. This function of emotions is deeply intertwined with metacognition. Empirical research has identified curiosity, puzzlement, bewilderment, frustration, pleasure, elation, satisfaction, anxiety, and despair to be significant in the self-regulation of mathematical problem solving (DeBellis and Goldin 2006). It is now well established that emotions direct attention and bias cognitive processing. For example, fear (anxiety) directs attention towards threatening information and sadness (depression) biases memory towards a less optimistic view of the past (Power and Dalgleish 1997; Linnenbrink and Pintrich 2004). Although there is not yet sufficient evidence to conclude it for all emotions, it seems that positive emotions facilitate creative processes, while the negative emotions facilitate reliable memory retrieval and performance of routines (Pekrun and Stephens 2010). Emotions also seem to play an important role as a memory "fixative." For example, neuropsychological research has identified that activity in the amygdala during an "Aha!" experience predicts which solutions will be remembered (Ludmer et al. 2011).

As learning typically takes place in a social setting of the classroom, the function of emotions in the social coordination of a group is inevitably present. Most emotions have a characteristic facial expression, typically identifiable by movements in the brow region and lip corners. In addition, some emotions have specifically behavioral (e.g., slouching, clenched fists) or physiological (e.g., blushing, tears) expressions. Humans learn to interpret such expressions automatically and they form an important part of intrapersonal communication. Moreover, such visible expressions are more reliable than self-reported thoughts and feelings, which make emotions important observable indicators for related variables, such as goals, attitudes, or values. Perhaps the first to recognize the social emotions in mathematics education were Cobb et al. (1989) who identified students' emotions to be related to two types of problems in collaborative problem solving: mathematical problems and cooperation problems.

## The Spectrum of Emotional Traits

While the research on emotional states in mathematical problem solving and in social coordination of the class has recognized the variety of different emotions, emotional traits have often been explored along a simple positive–negative dimension with anxiety (e.g., Hembree 1990) and enjoyment (e.g., OECD 2004) being measured most frequently. Only recently has the variety of emotional traits in the learning context been addressed quantitatively. Pekrun and his colleagues (2007) have developed a survey instrument to measure a number of achievement emotions, defined as emotions tied directly to achievement activities or achievement outcomes. Achievement-related activities are the origin of activity emotions (enjoyment, boredom, and anger). Outcome emotions include anticipatory emotions (hopelessness and anxiety) as well as emotions based on feedback (anger, pride, and shame).

Emerging empirical research indicates that classrooms are often emotionally flat, and boredom is one of the most frequently experienced emotions (Vogel-Walcutt et al. 2012).

## Regulation of Emotions

Although emotions are functional for the human species, not all emotional reactions are functional in a learning context. For example, expert problem solvers seem to be controlling their emotions better than novices (e.g., Schoenfeld 1985). Emotion regulation refers to "the ways individuals influence which emotions they have, when they have them, and how they experience and express these emotions" (Gross 1998, p. 275). Few studies have addressed how students regulate their emotions in a mathematics class. A study by De Corte et al. (2011) suggests that active coping (i.e., effort), joking and acceptance, and social-emotional coping (i.e., seeking social support) as well as abandoning and negation are important strategies to reduce negative emotions or their effects.

The teacher can help students' emotion regulation through modeling emotion regulation strategies or provide more direct support through controlling student emotions. Perhaps more effective than direct focusing on students' emotion regulation is to develop the classroom climate. Feeling of community, an autonomy supportive teaching style, and an expressive environment have been found to support development of student emotion regulation strategies (Fried 2012).

### Creating an Emotionally Supportive Learning Environment

Although few studies have explored the individual strategies of emotion regulation, there is significant amount of general educational research on characteristics of a classroom that promote optimal emotional climate. Teacher enthusiasm, emphasis on mastery goals, positive feedback, optimal level of challenge, student autonomy and feeling of control, and meeting students' needs can enhance positive student emotions (Pekrun and Stephens 2010).

Several schools have implemented programs to enhance students' social and emotional learning in order to promote a healthy learning environment. Specific goals for these programs include competencies to recognize and manage emotions. According to a meta-analysis, such programs have beneficial effects on positive social behavior, problem behaviors, and academic performance (Durlak et al. 2011).

Cobb et al. (1989) also emphasized the relationship between social norms and emotions. In their experimental classroom, engagement in mathematical activity was the goal, and therefore, even weaker students experienced and expressed positive emotions as they participated in group activities and whole class discussions.

## Cross-References

▶ Creativity in Mathematics Education
▶ Gender in Mathematics Education
▶ Metacognition
▶ Motivation in Mathematics Learning
▶ Problem Solving in Mathematics Education
▶ Students' Attitude in Mathematics Education

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## Algebra Teaching and Learning

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### Keywords

School algebra; Research on the teaching and learning of algebra; Widening perspectives on school algebra; Relational thinking in algebra; Technological tools in algebra learning; Functions and multiple representations; Algebraic reasoning; Algebraic meaning; Algebraic activity; The technical-conceptual interface in algebra; Teaching approaches; Changing nature of research on algebra teaching and learning; Role of teacher questioning in algebra learning; Role of tasks in algebra teaching; Teacher as key stakeholder in algebra research

### Definition

The learning and teaching of the area of mathematics known as school algebra – and the research base accompanying this branch of mathematics education – has typically involved the secondary school student and has focused on forming and operating on polynomial and rational expressions using properties and the field axioms, as well as representing word problems with algebraic expressions containing variables and unknowns. However, over the past several decades, changes in perspective as to what constitutes school algebra have occurred, in addition to its extension in various forms to the elementary school level. Thus, current definitions of school algebra can differ widely, all the more so because what one takes to be algebra depends on factors that vary across communities (see, e.g., Stacey et al. 2004). Decades ago, Freudenthal (1977) characterized school algebra as including not only the solving of linear and quadratic equations but also algebraic thinking, which includes the ability to describe relations and solving procedures in a general way. His characterization remains timely today because it captures not only the symbolic aspects of algebraic activity but also the kinds of relational thinking that underlie algebraic reasoning and that distinguish it from arithmetic activity, which is typically computational in nature.

### Changing Views on School Algebra Over the Years

Up until the second half of the twentieth century, algebra was viewed as the science of equation solving – as per its invention by Al-Khwârizmî in the ninth century. This perspective on algebra, as a tool for manipulating symbols, was reflected in school curricula as they emerged and took shape through the 1800s and into the 1900s. Accordingly, the research conducted during the first half of the twentieth century on the learning of school algebra – scant though it was – tended to focus on the relative difficulty in solving various types of equations, on the role of practice, and on students' errors in applying equation-solving algorithms. During the 1960s, the research took a psychological turn when cognitive behaviorists used the subject area as a vehicle for studying more general questions related to skill development and the structure of memory. In the late 1970s, when algebra education researchers began to increase in number and to coalesce as a community (Wagner and Kieran 1989), research began to center on the ways in which students construct meaning for algebra, on the nature of the algebraic concepts and procedures used by students during their initial attempts at algebra, and on various novel approaches for teaching algebra (e.g., Bednarz et al. 1996). While the study of students' learning of algebra favored a cognitive orientation for some time, sociocultural considerations have added another dimension to the research on school algebra since the end of the 1990s (Lerman 2000).

The years since the late 1980s have also witnessed a broadening of the content of school algebra. While functions had been considered a separate domain of mathematical study during the decades prior, the two began to merge at this time in school algebra curricula and research. Functions, with their graphical, tabular, and symbolic representations, gradually came to be seen as legitimate algebraic objects (Schwartz and Yerushalmy 1992). Concomitant with this evolution was the arrival of computing technology, which began to be integrated in varying degrees into the content and emphases of school algebra. A further change in perspective on school algebra was its encompassing in an explicit way what has come to be called algebraic reasoning: that is, a consideration of the thinking processes that precede – and eventually accompany – activity with algebraic symbols, such as the expression of general rules with words, actions, and gestures. This widening of perspective on algebraic activity in schools reflected a double concern aimed at engaging primary school students in the early study of algebra and at making algebra more accessible to all students.

## A Focus on Algebraic Meaning

As the vision of school algebra widened considerably over the decades – moving from a letter-symbolic and symbol-manipulation view to one that included multiple representations, realistic problem settings, and the use of technological tools – so too did the vision of how algebra is learned. The once-held notion that students learn algebra by memorizing rules for symbol manipulation, and by practicing equation solving and expression simplification, has largely been replaced by perspectives that take into account a multitude of factors and sources by which students derive meaning for algebraic objects and processes.

Several researchers have studied the specific question of meaning making in school algebra (e.g., Kaput 1989; Kirshner 2001). More recently, the various ways of thinking about meaning making in algebra have been expanded (see, e.g., Kieran 2007) to suggest a triplet of sources: (a) meaning from within mathematics, which includes meaning from the algebraic structure itself, involving the letter-symbolic form, and meaning from other mathematical representations, including multiple representations; (b) meaning from the problem context; and (c) meaning derived from that which is exterior to the mathematics/problem context (e.g., linguistic activity, gestures and body language, metaphors, lived experience, and image building). Further theoretical development of this area has been carried out by Radford (2006) with his conceptualization of a semiotic-cultural framework of mathematical learning, which has been applied to the learning of algebra. Through words, artifacts, and mathematical signs, which are referred to as semiotic means of objectification, the cultural objects of algebra are made apparent to the student in a process by which subjective meanings are refined.

## Characterizing Algebraic Activity

Some research studies have used the nature of algebraic activity as a lens for investigating various constituents of students' learning experiences in algebra. Several models have been proposed for describing algebra and its activities (see, e.g., Bell 1996; Mason et al. 2005; Sfard 2008). For example, a model developed by Kieran (1996) characterizes school algebra according to three types of activity: generational, transformational, and global/meta-level.

The generational activity of algebra is typically where a great deal of meaning building occurs and where situations, properties, patterns, and relationships are interpreted and represented algebraically. Examples include equations containing an unknown that represent problem situations, expressions of generality arising from geometric patterns or numerical sequences, expressions of the rules governing numerical relationships, as well as representations of functions by means of graphs, tables, or literal symbols. This activity also includes building meaning for notions such as equality, equivalence, variable, unknown, and terms such as "equation solution."

The transformational activity of algebra, which involves all of the various types of symbol manipulation, is considered by some to be exclusively skill based; however, this interpretation would not reflect current thinking in the field. In line with a broader view, mathematical technique is seen as having both pragmatic and epistemic value, with its epistemic value being most prominent during the period when a technique is being learned (see Artigue 2002). In other words, the transformational activity of algebra is not just skill-based work; it includes conceptual/theoretical elements, as for example, in coming to see that if the integer exponent $n$ in $x^n-1$ has several divisors, then the expression can be factored several ways and thus can be seen structurally in more than one way. However, for such conceptual aspects to develop, technical learning cannot be neglected.

Lastly, there are the global/meta-level activities, for which algebra may be used as a tool but which are not exclusive to algebra. They encompass more general mathematical processes and activities that relate to the purpose and context for using algebra and provide a motivation for engaging in the generational and transformational activities of algebra. They include problem solving, modeling, working with generalizable patterns, justifying and proving, making predictions and conjectures, studying change in functional situations, and looking for relationships or structure – activities that could indeed be engaged in without using any letter-symbolic algebra at all.

### What Does Research Tell Us About the Learning of School Algebra?

There is a considerable body of research on the learning of algebra that has been accumulating since the late 1970s. What does this research have to say?

Many students beginning the study of algebra come equipped with an arithmetical frame of mind that predisposes them to think in terms of calculating an answer when faced with a mathematical problem. A considerable amount of time is required in order to shift their thinking toward a perspective where relations, ways of representing relations, and operations involving these representations are the central focus. Teaching experiments have been designed to explore various approaches to developing in students an algebraic frame of mind. Approaches that have generally been found to be successful include those that (a) emphasize generalizing and expressing generality by using patterns, functions, and variables; (b) focus on thinking about equality in a relational way, starting with number sentences with multiple terms on both sides and moving toward more complex examples involving the "hiding" of the same number on both sides by a box and then by a letter, so as to generate a literal-symbolic equation with an unknown on each side; and (c) use problem situations that are amenable to more than one equation representation and engage pupils in comparing the two (or three) resulting equation representations to determine which one is better in that it is more generalizable.

Research also tells us that students have difficulty with conceptualizing certain aspects of school algebra, for example, (a) accepting unclosed expressions such as $x + 3$ or $4x + y$ as valid responses, thinking that they should be able to do something with them, such as solving for $x$; (b) counteracting well-established natural-language-based habits in representing certain problem situations such as the well-known students-and-professors problem; (c) moving from the solving of word problems by a series of undoing operations toward the representing and solving of these problems with transformations that are applied to both sides of the equation; and (d) failing to see the power of algebra as a tool for representing the general structure of a situation.

These findings illustrate only a very small portion of the information to be gleaned from the large corpus of research literature that is available (see also Kieran 1992, 2006, 2007) – literature that has shifted in focus over the years, due to theoretical developments regarding the learning of algebra, as well as curricular change and the growing use of technological tools. While the research of the 1970s and 1980s was oriented primarily toward issues related to the transition from arithmetic to algebra, later work illustrates the interest in patterning, students' generalizing and use of multiple representations, as well as the ways in which technology environments (e.g., spreadsheets, graphing calculators, calculator-based rangers, computer algebra systems, cell phone technology, and specially designed software environments) can support algebra learning. The studies on the role of technology have reported that students receive the most benefit when the technological tools they use have a pedagogical role in the classroom and are available not just for drill and practice or for checking work; however, exploiting their potential pedagogical role requires special curricular materials that are designed for such tool use. Another shift that has been witnessed concerns what is meant by *algebra problem solving*. In the past, this phrase tended to refer almost exclusively to word problems – an area of algebra learning that continues to challenge many algebra students. However, a much broader interpretation of the phrase exists today that includes many types of nonroutine algebraic tasks, even those in purely symbolic form with no connection at all to so-called "real-world" problems.

With respect to the body of research literature on algebra learning, it is noted that the 12–15-year-old student has received the bulk of the attention of algebra researchers; however, since the turn of the millennium, there has been a significant interest in the development of algebraic reasoning in younger students of elementary and middle school age (see, e.g., Kaput et al. 2007, as well as the Early Algebra Teaching and Learning entry in this volume). Nevertheless, the student older than 15 years of age has not been entirely neglected. Research with this age group has investigated the learning of more advanced algebraic topics, including the study of structure and equivalences, conceptual and technical work involving quadratics and higher-degree expressions, and proof and proving of number-theoretic relations. It has been found, for example, that while many students experience difficulty in "seeing structure," they have shown improvements over their younger counterparts in representing word problems with equations. Older students have also been found to prefer to work with literal symbolic representations than with the graphical. Computer algebra system (CAS) technology has figured in several studies with the older student. Much of this research, which has been grounded in the instrumental approach to tool use (Artigue 2002), has been able to provide evidence for the role played by the CAS tool in the co-emergence of students' technical and conceptual knowledge in algebra.

### Research Involving the Teacher of Algebra

With the main focus in most of the research in school algebra during the 1970s and 1980s on the learner, as well as on teaching approaches aimed at improving student learning, little was revealed about the teacher of algebra. From the few reports available, one could only discern that, just as with teachers of other mathematical subjects, algebra teachers viewed themselves primarily as providers of mathematical information and tended to follow the textbook in their teaching. However, a research interest in the teacher of algebra and the nature of algebra teaching practice took shape in the early 1990s and has continued to this day – research that has begun to deepen our knowledge of this domain. Doerr (2004) has stated that this research tends to fall into three areas: teachers' subject matter knowledge and pedagogical content knowledge, teachers' conceptualizations of algebra, and teachers learning to become teachers of algebra. However, according to Doerr, progress in teacher-oriented research has been hampered by the lack of development of new methodological and theoretical approaches to effectively investigate the practices of teachers of algebra.

Although the research on teachers' practice with respect to promoting algebraic reasoning may still be relatively sparse, it has nevertheless been able to point to two aspects in particular as being critical to student learning. These are the roles played by task design and teacher questioning in encouraging algebraic thinking. Many of the recent studies on innovative approaches to teachers' professional development in the area of school algebra have focused directly on the issue of tasks and their design. These studies have been able to show that the content-related design and careful sequencing of tasks have a clear impact upon the ways in which students come to conceptualize a variety of algebraic ideas and operations, including the following: relational views of equality; meaningful interpretations of algebraic symbols; awareness of the theoretical-technical interface in algebraic work; perception of form, structure, and generality; and the pedagogically effective use of technological tools in algebraic activity.

The role of teacher questioning in developing students' algebraic reasoning has been found to be no less important than that of good task design. For example, data drawn from the eighth-grade TIMSS Video Study (Stigler et al. 1999) illustrate the ways in which teachers' well-conceived questions during whole-class discussions can encourage students to make explicit their problem-solving approaches and to generalize them into literal-symbolic form. However, this research also shows that despite the use of tasks designed to help students engage in classroom discussions that focus on making conjectures and reasoning mathematically, simply using such tasks will not spontaneously promote the desired discussions. Skillful teacher guidance is needed in order to help students engage in the algebraic reasoning that is intended by the tasks.

### For Further Research

The following closing remarks return to a central issue regarding the practice of teaching algebra. The research that was synthesized just above emphasized the need for a certain kind of support by the teacher in order to promote students' algebraic reasoning – support involving both task design and whole-class teacher questioning. However, as has also been noted, research involving the development of such support in teachers has been hindered, at least up to the early 2000s, by a lack of appropriate methodological and theoretical tools. While some advances have clearly been made in this area, including teachers' sharing of their effective approaches with other teachers, further work is needed. A general framework for thinking about models for developing teaching practice that can support students' algebraic reasoning, and which offers a perspective for moving forward in this area, has been described in a recent article on connecting research with practice (Kieran et al. in press). In that article, the researchers attempt to close the distinctive gap between research and practice that exists in much of the mathematics education research literature by viewing teachers as key stakeholders in research – stakeholders who coproduce professional and scientific knowledge – instead of as "recipients of research" and sometimes even as "means" to generate or disseminate knowledge. Their elaboration of the notion of *the teacher as key stakeholder* with examples drawn from five international projects, all of them involving teachers researching their own or their colleagues' practice, offers several viable models and a useful lens for considering how teachers and researchers might collaborate in further developing the teaching and learning of school algebra.

## Cross-References


- ▶ Early Algebra Teaching and Learning
- ▶ Functions Learning and Teaching
- ▶ Information and Communication Technology (ICT) Affordances in Mathematics Education
- ▶ Mathematical Representations
- ▶ Teaching Practices in Digital Environments
- ▶ Technology and Curricula in Mathematics Education
- ▶ Theories of Learning Mathematics

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# Algorithmics

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## Keywords

Algorithmics; Algorithms; Programming; Computer sciences; Programming language

## Definition

"Algorithmics" can be defined as the design and analysis of algorithms (Knuth 2000). As a mathematical domain, algorithmics is not principally concerned by human execution of algorithms, for instance, for arithmetic computation (see 2010/index/chapterdbid/313187 for a discussion), but rather by a reflection on how algorithms are built and how they perform. Algorithms exist and have been studied since the beginning of mathematics. However, the emergence of algorithmics as a mathematical domain is contemporary to digital computers, the work on computability by Church (1936), Turing (1937), and other mathematicians being often considered as seminal. Computer science, also emerging at the same time, is concerned with methods and techniques for machine implementation, whereas algorithmics focuses on the properties of algorithms.

Typical questions addressed by algorithmics are the effectiveness of an algorithm (whether or not it returns the expected result after a finite number of steps), the efficiency (or complexity) of an algorithm (an order of the number of steps for a given set of data), and the equivalence of algorithms (e.g., iterative and recursive equivalent forms). Djiskra (1976, p. 7) notes that "as long as an algorithm is only given informally, it is not a proper object for a formal treatment" and therefore that "some suitable formal notation" is needed "to study algorithms as mathematical objects." This formal notation for algorithms or "language" is a vehicle for abstraction rather than for execution on a computer.

## Algorithms in Mathematics Education Research

Research in mathematics education and computers most often concentrates on the use of technological environments as pedagogical aids. Authors like Papert and Harel (1991), Dubinsky (1999), or Wilensky and Resnick (1999) proposed computer programming as an important field of activity to approach mathematical notions and understanding. This strand of research does not consider the design and analysis of algorithms as a goal in itself. The hypothesis is that building algorithms operating on mathematical objects and implementing these in a dedicated programming language (LOGO or ISTL) is able to promote a "constructive" approach to scientific concepts. The language's features (recursivity, functions, etc.) are chosen in order to support this approach. Students' access to a formal algorithmic language is generally not an issue because the tasks proposed for students generally imply short programs with a simple structure.

In a few countries and regions, curricula for algorithmics have been implemented and, in parallel, research studies have been conducted. For instance, at the end of the year 1980, a curriculum has been written and tested for 7th- and 8th-grade students in a region of Germany (Cohors-Fresenborg 1993). Concepts of algorithmics were taught by making students solve calculation problems using a concrete "register machine."

These research studies are few and do not really tackle questions at the core of algorithmics like effectiveness and complexity, reflecting the fact that at school levels investigated by research studies, students' consideration of algorithmics is still limited by the difficult access to a symbolic language.

## Students' Understanding of Algorithmic Structures and Languages

In France, programming algorithms has been proposed as a task for secondary students in various curricula. Because the time devoted for these tasks was short, students' understanding of algorithmic structures and languages appeared to be the real challenge, algorithmics in the sense of Knuth (2010) being inaccessible to beginners without this prerequisite. Didactic research studies were developed focusing on this understanding.

Samurcay (1985) was interested by 10th-grade students' cognitive problems relatively to variables in iteration. The method was to ask students to complete iterative programs in which instructions were missing. Missing instructions were of three types: the initialization of the iterative variable, an assignment of the iterative variable in the loop body, and the condition for exiting the loop. Important misunderstandings of the semantics of variables were identified. For instance, regarding the initialization, some students think that the initial value has necessarily to be entered by a reading instruction; others systematically initialize variables to zero. They are clearly influenced both by preconception of how a computer works and by previous examples of algorithms that did not challenge these preconceptions. The author concludes that more research studies are essential in order to understand how students conceptualize the notions associated to iteration and to design adequate didactical situations.

Samurcay, Rouchier (1990) studied students' understanding of recursive procedures distinguishing between two aspects: self-reference (relational aspect) and nesting (procedural aspect). They designed teaching sessions with the aim to help pupils to construct a relational model of recursion, challenging students' already existing procedural model. After sessions of introducing the students to the LOGO graphic language without recursion, they designed ten lessons: first introducing the students to graphic recursive procedures, making them distinguish between initial, central, and final recursion and then helping them to generalize recursive structures by transferring recursive procedure to numerical objects for tasks of generating sequences. Observing students, they conclude that introducing recursion is a nonobvious "detour" from already existing procedural model of iteration and a promising field for research.

Lagrange (1995) considered the way 10th- and 11th-grade students understand representations of basic objects (strings, Booleans) in a programming language. Analyzing students' errors in tasks involving simple algorithmic treatments on these objects, he found that misunderstandings result from assimilation to "ordinary" objects and treatments. For instance, when programming the extraction of a substring inside a string, students often forgot to assign the result to a variable; the reason is that they were not conscious of the functional nature of the substring instruction, being influenced by the "ordinary" action oriented language. Another example is that students generally did not consider the assignment to a Boolean value, not understanding that in an algorithmic language, "conditions" are computable entities. Similar difficulties found in this study were analyzed in relationship with analogous obstacles in accessing the algebraic symbolism at middle school level. Programming simple algorithms involving these nonnumerical objects seemed promising for overcoming such obstacles.

Nguyen (2005) questioned the introduction of elements of algorithmics and programming in the secondary mathematical teaching, showing that on one hand, there is a fundamental solidarity between mathematics and computer science based on the history and the current practice of these two disciplines and that on the other hand, the ecology of algorithmics and programming in secondary teaching is not obvious. Focusing on the teaching/learning of loop and of computer variable notions in France and in Vietnam, he proposed an experimental teaching unit in order that 10th-grade students learn the iterative structure. He chose to make students build suitable representations of this structure by solving tasks of tabulating values of polynomial using a dedicated calculator, emulated on the computer, and based on the model of calculator existing in the secondary teaching of the two countries with the additional capacity to record the history of the keys pressed.

The experimental teaching was designed as a genesis of the machine of Von Neumann: the students had to conceive new capabilities for the calculator especially erasable memories and controlled repetition in order to perform iterative calculations and programming through the writing of the successive messages (programs) to machines endowed with different characteristics. This allowed for the emergence of the notion of iterative variables and treatments. In the framework of the Theory of Didactical Situations, a milieu and a fundamental situation are then offered for the construction of the iterative structure.

## Algorithmics and Programming Competencies

In parallel to mathematics education research, studies have been carried out in the field of psychology of programming. Most studies in the field address professional programming and discuss opportunities and constraints of programming languages and design strategies for experts (e.g., see Petre and Blackwell 1997). Some studies focused on programming problem solving by beginners with tasks very close to students' activity in early algorithmics courses. For instance, Rogalski and Samurçay (1990) focused on the acquisition of programming knowledge "as testified by students' ability to solve programming problem", that is to say, to pass from "real" world objects and situations to an effective program implementation. Rogalski and Samurçay (1990) insist on "the variety of cognitive activities and mental representations related to program design, program understanding, modifying, debugging (and documenting)." They stress the necessity for beginners of adequate mental models of data representation and processing.

These models include static schemas and plans. Schemas are defined as sets of organized knowledge used in data processing that help to achieve small-scale goals. Plans are organized sets of dynamic procedures related to the schemas. For instance, when programming the sum of numbers in a list of arbitrary length, schemas are related to different sub-tasks like entering the list and computing iteratively partial sums, and the plans help to define a strategy, separating the two sub-tasks or merging these in a single iteration. More generally, research in the field of psychology of programming by beginners usefully complements math education research because it introduces theoretical models of human thinking to give account of competencies required to build or understand programs or algorithms.

## Perspectives

In spite of nearly 30 years of existence, mathematics education research in algorithmics remains in its infancy. It is conditioned by political decisions to include algorithms in the mathematics curriculum. Finding ways to help students access an algorithmic language together with adequate mental models of data representation and processing appears to be a condition in order that they could tackle central questions like complexity or proof of algorithms. This is consistent with Djiskra's (ibid.) epistemological view that a suitable formal notation is needed to study algorithms as mathematical objects. It is also a stimulating challenge that the abovementioned research studies just started to take up.

## Cross-References

▶ Algorithms

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## Algorithms

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### Keywords

Algorithm; Computing; History of mathematics; Instrumental understanding; Representation

### Definition

The word algorithm probably comes from a transliterated version of the name al-Khwarizmi (c. 825 CE), the Arabic mathematician who described how to solve equations in his publication *al-jabr w'al-muqabala*. An algorithm comprises a step-by-step set of instructions in logical order that enable a specific task to be accomplished. Due to its nature it can be programmed into a computer, although some problems may not be computable or solvable by an algorithm. In his famous paper, Turing (1936) showed, among other things, that Hilbert's Entscheidungsproblem can have no solution. He did this by proving "that there can be no general process for determining whether a given formula $U$ of the functional calculus $K$ is provable, i.e., that there can be no machine which, supplied with any one $U$ of these formulae, will eventually say whether $U$ is provable" (1936, p. 259).

An example of a simple well-known algorithm is that for sorting a sequence of real numbers into descending (or with a minor change, ascending) order, sometimes called a bubble sort. In this we perform something similar to the following steps, which describe the algorithm:

1. Set the count to 0.
2. Compare the first two numbers $a_1$ and $a_2$ in the sequence. If $a_1 < a_2$ then swap $a_1$ and $a_2$ and add 1 to the count. If $a_1 > a_2$ then proceed directly to step 3.
3. Compare the numbers $a_2$ and $a_3$ in the sequence and repeat as in step 2.
4. When the last two numbers in the sequence have been compared, consider the count of the number of changes. If the count is zero then the sequence is sorted into order. If the count is greater than zero repeat from step 1.

We note that two algorithms to accomplish the same task may vary or be entirely different. For example, there are a number of different algorithms for sorting numbers into order, often much more efficiently than the bubble sort, such as the quicksort algorithm.

Another common example referred to as the Euclidean algorithm for finding the greatest common divisor (gcd) of two integers $n$ and $m$ may be stated as:

1. If $n = m$ then output $n$ as the gcd $(n, m)$ and end.
2. If $n > m$ the initialize $a = n$ and $b = m$. Otherwise, initialize $a = m$ and $b = n$.
3. Apply the division theorem to $a$ and $b$ by finding integers $q$ and $r$ such that $a = q.b + r$, where $0 \leq r \leq b$.
4. If $r = 0$ then output $b$ as the gcd $(n, m)$ and stop. Otherwise set $a = b$ and $b = r$. Go to step 3.

Based on Khoussainov and Khoussainova (2012), p. 29.

In this case we illustrate how algorithms to accomplish the same task may be equivalent but presented differently, and thus not necessarily appear to be the same. Consider, for example, a second version of the Euclidean algorithm (based on the version found at http://www.math.rutgers.edu/~greenfie/gs2004/euclid.html):

1. If $m < n$, exchange $m$ and $n$.
2. Divide $m$ by $n$ and get the remainder, $r$. If $r = 0$, report $n$ as the gcd.

3. Replace $m$ by $n$ and replace $n$ by $r$. Return to the previous step.

What do we notice about these two versions? While they are the same algorithm, that is, they accomplish the same task in the same way, the first one appears more complex. This is because it uses function notation (gcd $(n, m)$); it is not self-contained, but refers to a previous result (the division theorem); and it introduces more variables (an extra $a, b$) than the second. These differences may be the result of attempts to be rigorous or to make the algorithm more amenable to computerization.

It is perfectly possible to be able to carry out an algorithm, such as the quicksort or Euclidean algorithms above, without understanding how it works. In this case an individual would demonstrate what Skemp (1976) called *instrumental understanding*, whereas knowing the reasons why it works would constitute *relational understanding*. It would also be a mistake to think that mathematics may be reduced to a series of algorithms. The idea of an algorithm is closely related to what, in mathematics education terms, are often called procedures, since these may be accomplished using algorithms. They contrast with other crucial elements of mathematics, such as objects, constructs, or concepts. While both procedures and concepts are important in learning mathematics (Hiebert and Lefevre 1986), teaching algorithms is often easier than addressing concepts and so this approach may prevail in school (and sometimes university) teaching. For example, the formula for solving a quadratic equation $ax^2 + bx + c = 0$ with real roots $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ leads to an algorithm for solving these equations. However, it may be the case that students who can successfully find the roots of a quadratic equation $ax^2 + bx + c = 0$ only have instrumental understanding and do not understand well why the formula works, what an equation is (Godfrey and Thomas 2008), or even what a solution of an equation is. They may not appreciate, for example, that the formula arises from completing the square on $ax^2 + bx + c = 0$; that if $p$ and $q$ are real roots of $ax^2 + bx + c = 0$, then $ap^2 + bp + c = 0$ and $aq^2 + bp + c = 0$ by definition; and that a factorization of the form $ax^2 + bx + c = a(x - p)(x - q)[=0]$ is possible.

One drawback of the step-by-step nature of an algorithm is that it leaves no room for deviation from the method. Hence, it cannot encourage or promote the versatile thinking (Thomas 2008; Graham et al. 2009) that is needed in order to understand some mathematical constructs and hence to solve certain mathematical problems. For example, it may be both useful and enlightening to switch representations or registers to comprehend an idea better (Duval 2006) or to view a written symbolism (described as a *procept*) as either a process or an object (Gray and Tall 1994) depending on the context. One example of this is appreciating the relationship between the roots of the quadratic equation above and the graph of the function. Another is the calculation of integrals through the use of limits of Riemann sums. Algorithms can be constructed for processes that allow students to find a Riemann sum or its limit, but there is evidence that far fewer students understand the nature of the limit object itself (Tall 1992; Williams 1991).

## Cross-References

▶ Algorithmics
▶ Mathematical Approaches

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# Anthropological Approaches in Mathematics Education, French Perspectives

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## Keywords

Anthropology; Didactics; Joint action; Mathematics; Praxeology; School epistemology

## Characteristics

This entry encompasses two interrelated though distinct approaches to mathematics education: the anthropological theory of the didactic (ATD for short) and the joint action theory in didactics (JATD). Historically, the germs of ATD are to be found in the theory of didactic transposition (Chevallard 1991), whose scope was at first limited to the genesis and the ensuing peculiarities of the (mathematical) "contents" studied at school; from this perspective, ATD should be regarded as the result of a definite effort to go further by providing a unitary theory of didactic phenomena as defined in what follows. As for JATD, it has emerged from the theory of didactic situations (Brousseau 1997) and the anthropological theory of the didactic by focusing on the very nature of the communicational epistemic process within didactic transactions. ATD and JATD share a common conception of knowledge as a practice and a discourse on practice together – i.e., as a praxeology – along with a pragmatist epistemology which gives a prominent place to praxis. Their well-thought-out anthropological stance leads the researcher to study didactic facts wherever they are located in social practices. Although these theorizations are by necessity expounded tersely, we hope their forthright presentation will allow the reader to catch the gist of them.

### The Anthropological Theory of the Didactic

The (seemingly) heavy theoretical load of the presentation that follows should not be misinterpreted. On the one hand, every and all notions delineated hereinafter do refer to concrete didactic practice (from which they gradually emerged) and have led, in our view, to some major scientific breakthroughs (Bosch et al. 2011; Bronner et al. 2010; Chevallard 1990, 2006, 2007, to appear; Chevallard and Ladage 2008; Ruiz-Higueras et al. 2007). On the other hand, we strongly believe that, according to a well-known remark by Lewin (1952), p. 169, "there is nothing more practical than a good theory," which is exactly what we aim at providing the interested reader with (Chevallard 1980, 1991, 1992).

### Didactic Systems

*Didactics* can be defined as the (historically incipient) science of knowledge diffusion and acquisition in society. The founding problem of didactics has long been reduced to two characters: some object of knowledge $O$ and some human subject $x$ supposed to "study" $O$. This problem lays in what ATD calls the "relation of $x$ to $O$," written in symbol $R(x, O)$. If $x$ knows nothing about $O$, her relation to $O$ is void: $R(x, O) =$ Æ. How can this relation change, grow, and possibly achieve proficiency, so that one can say that $x$ "knows" $O$? Such is the key question in classical didactics. But the traditional two-character play to which it applies has long since been challenged by the theory of didactic transposition, that forerunner of ATD which questions the nature of object $O$, its genesis. and alterations during the process that leads to the face-to-face encounter of $x$ and $O$. The meeting between $x$ and $O$ takes place in some *institution* – another keyword of ATD – which imposes upon $O$ a number of *conditions* that reshape $O$ and "fix" the conditions under which $x$ will study $O$. ATD is critical of the common view of the two-character didactic scene. Its main tenet holds that in order to "explain" $x$, $O$, and $R(x, O)$, one has to take into account a greater number of conditions. First of all, the "binomial" arrangement made up of $x$ and $O$ is generally part of a "trinomial" layout, including a third character, $y$, to constitute a *didactic system* $S(x, y, O)$, where $y$ is a person supposed to help $x$ to study $O$. When $y$ is missing, the system reduces to an *autodidactic* system, $S(x, O)$, the basic "binomial arrangement" we started with. What is the use of such a formal description? Before answering this question, let's generalize a bit our symbolic gobbledygook: instead of a person $x$, let's consider a *group* of persons $X$; instead of $y$, let's introduce a *team* of $y$, $Y$, so that a didactic system is now denoted by $S(X, Y, O)$. Didactic systems $S(x, y, O)$ are particular cases of this general form. When no $y$ helps $X$, we'll denote the corresponding system by $S(X$, Æ, $O)$ or simply $S(X, O)$. Any class of students $X$ studying some object $O$ under the supervision of some "official" teacher $y$ can be written as $S(X, y, O)$. Two students $x_1$ and $x_2$ working together on their homework $O$ are part of the didactic system $S(\{x_1, x_2\}$, Æ, $O)$ or $S(\{x_1, x_2\}, O)$. When a student $x$ receives help from her mother $y$, they form together a system $S(x, y, O)$. The symbolic notation used thus allows us to "see" not only the didactic systems insistently shown to us – within classrooms, basically – but also the more or less informal, but no less crucial, didactic systems that may appear *almost everywhere* in society: at school in and outside classrooms, in the family, on the telephone yesterday, on the Internet today, etc.

## The Didactic

In ATD, the adjective "didactic" applies to any action induced by the intention to help someone study something. In any didactic system $S(x, y, O)$, $x$ and $y$ act to help $x$ study $O$. It is customary to say that $x$ performs didactic *moves* (or didactic gestures) with respect to the *didactic stake* $O$, to help herself study $O$, and that $y$ achieves didactic moves or gestures with respect to $x$ and $O$ with the same intention. Most didactic moves occurring in a system $S(x, y, O)$ involve both $x$ and $y$, who work together to produce some determined didactic effect: *didactic tasks* are generally *cooperative* tasks, jointly performed by $x$ and $y$ (or by $X$ and $y$, etc.). Considered within the larger frame of society, the fuzzy set of didactic moves, which ATD calls *the didactic* (as one can speak of *the religious*, *the economic*, etc.), is thus everywhere around us, and it is the specific object of study of the science we call didactics.

## Praxeological Analysis

By definition, any didactic study refers to some stake $O$ and some category of "students" $x$. Both $O$ and $x$ impose conditions on the moves that can appropriately be performed by $x$ and $y$. Traditionally, $O$ pertains to some *discipline* such as mathematics, physics, or geography. All these entities are human made: they are, up to a point, artifacts, i.e., "works of art," this expression being understood here in its most primitive meaning. We shall say for short that $O$ is "a work." Analyzing any work $O$ amounts to making clear its *structure*, its *functioning*, and its distinct *uses*. It has become common practice in ATD to describe the set of conditions borne by the disciplinary field to which $O$ belongs – as partaking of a four-scale hierarchy of disciplinary subfields. Firstly, $O$ is situated within some *domain* of the discipline – say, algebra – which in turn is dissociated into a number of *sectors*, each of which is made up of *themes* (or *topics*) that finally separate into *subjects*. Any work $O$ that can be offered for study in a system $S(X, y, O)$ may fall under any of these disciplinary levels: *O* may be "algebra" (domain) or "equations" (sector) or "quadratic equations" (theme) or "incomplete quadratic equations with no constant term" (subject). This four-level structure is one part of the story. For ATD purports that the ultimate building block of all works is a four-component structure called *a praxeology*. The first component of a praxeology is a *type of tasks T* (e.g., "to solve a quadratic equation"). The second component is a *technique* t(*tau*), i.e., a way of performing the tasks of type *T* (or at least some of them). The third component is a *technology* q(*theta*), i.e., a way of explaining and justifying or even of "designing" the aforesaid technique t. Last but not least (although often ignored), there is a fourth component, the *theory* Q("big theta"), which should explain, justify, or generate whatever part of technology q may sound unobvious or missing. It is a crucial precept of classical didactics – one that severs didactics from the old "pedagogy" – that the didactic generated in any system *S*(*x*, *y*, *O*) depends *essentially* on the conditions ingrained in the stake *O*, acting in this respect as a quasi-autonomous system. ATD posits that *O* is a combination of a number of praxeologies (*T*, t, q, Q) that usually share parts of their theory Q and of their technology q. Of course, *O* can be also a "mere" detail of a praxeology, e.g., some instrument used to carry out a given technique t. More often than not, praxeologies are identified by some emblematic "detail": studying "Pythagoras' theorem," for instance, usually *does not* boil down to learning a bare statement ("In any right-angled triangle, the area...") but amounts to studying at least a whole praxeology whose technological component q crucially features Pythagoras' theorem.

## Didactic Analysis

The study of work *O* consists in providing some praxeological analysis of *O*. To do so, *x* (helped by *y*) engages in hard didactic work to analyze the praxeological structure of *O* as well as the *raisons d'être* of *O*, i.e., the role *O* plays in the functioning of the praxeologies of which it is an ingredient and, by the same token, the raisons d'être or ultimate purpose of these praxeologies. Analyzing what *x* and *y* can do in this respect – their possible moves – is tantamount to producing a *didactic analysis*, i.e., an analysis of the *didactic situations* that the system *S*(*x*, *y*, *O*) goes through. Any didactic analysis implies some degree of praxeological analysis of *O* (e.g., even if *O* is praxeologically far from complete: the raisons d'être of *O* are almost always lacking in today's mathematics education). Provided this is done appropriately, ATD offers a model to guide didactic analysis. In the case of a praxeology *O* = (*T*, t, q, Q), to which much can be reduced, there comes a *moment* when, in some didactic situation, *x* meets the type of tasks *T for the first time* – also, there will come a moment when *x* tries to design and then master a technique t relating to *T*; this *model of didactic moments* is essentially dictated by the aforementioned praxeological model. To this model, ATD adds another decisive multilevel structure without which this theory would not fully deserve to be called "anthropological": the *scale of levels of didactic codetermination* that can be sketched as follows: humankind, civilizations, societies, schools, pedagogies, disciplines ... *O*. The conditions to be taken into account in any didactic analysis should not be limited to conditions carried by the stake *O* (or by the "discipline" to which *O* "belongs"). Contrary to other approaches, which regard higher-level conditions as "neutralized variables," ATD allows for conditions originating at the levels of pedagogy, school, society, civilization, and even humankind, *in so far as* they determine (think, e.g., of class and gender) the didactic opportunities open to *x* and *y*. At the same time, ATD classically holds that the manipulation of conditions of higher levels (starting from the pedagogic level) is of little avail if we ignore the conditions properly pertaining to *O*. This anthropological turn results in a deep change in didactics' theory and practice.

## Toward an Anthropology of Didactic Inquiry

In ATD, a school is any institutional arrangement devoted to study, i.e., in which it is legitimate to study some works *O* – a family is thus usually a school for some of its members. The contract linking a society and a school can be of one of two kinds. The traditional contract decides in advance which works $O$ will be studied, a work $O$ being in this case some praxeological entity supposed to allow one to answer questions of a given type – e.g., "What are the roots of this quadratic equation?" Such works are classified beforehand as belonging to mathematics, physics, biology, music, etc. But a different kind of contract can be considered, more akin to the mores of scientific research, in which the stake $O$ is no longer a tool for answering questions but is itself a *question* $Q$, which is a (human made) work as well. In this case, the study of $Q$ ceases to be a priori governed by a discipline determined in advance: these primary questions $Q$ are not clearly introduced as questions of mathematics, of biology, etc. It is the role of $x$ to find out the secondary questions $Q'$, the tertiary questions $Q''$, etc., and the other works $O$ that will prove useful to answer $Q$. The study of $Q$, i.e., the *inquiry* into question $Q$ led by $x$ (with some help from $y$), is now "co-disciplinary" in that it generally requires combined contributions from several, known as well as unknown, disciplinary fields. As of today, ATD is increasingly concerned with the analysis of conditions of every level that may hinder or facilitate the advent of such an anthropological pedagogy of inquiry.

### The Joint Action Theory in Didactics

One cannot understand the didactic system $S(X, Y, O)$ without taking into account the relationships between the three subsystems (teacher, student, the piece of knowledge at stake) as a whole. With this respect, the JATD (Sensevy 2012) puts the emphasis on the "actional turn" in didactics. Emerging from a comparative approach in didactics (Ligozat and Schubauer-Leoni 2009), the JATD institutes a specific unit of analysis that we call an *epistemic joint act*. The linguistic criterion of the description of such an act is that it is impossible to describe it without describing at the same time the teacher's action, the student's action, and the way the knowledge at stake shapes these actions. This assertion is a very general and anthropological one. For example, if a parent holds her hands out to a young child, who is learning to walk, as an incentive to make her walk towards these hands, while the young child tries to take some steps to reach these hands, this is an epistemic joint act. One cannot understand each behavior (parent/teacher or child/student) without taking into account the joint process and the knowledge (walking) that gives its form to the enacted gestures. In this perspective, in the JATD, *knowledge* is always seen as a *power of acting*, in a specific situation, within a given institution. When a person knows something, she becomes able to do something that she was previously unable to do.

### The Didactic Game

We aim to describe the didactic interactions between the teacher and the students as a game of a particular kind, a *didactic* game (Sensevy 2011a). What are the prominent features of this game?

It involves two players, $X$ and $Y$.

$Y$ wins if and only if $X$ wins, but $Y$ cannot give the winning strategy to $X$ directly.

$Y$ is the teacher (the teaching pole). $X$ is the student (the studying pole). Under this description, the didactic game is a collaborative game, a *joint* game, within a joint action. To identify the very nature of the didactic game, we have to consider it as a *conditional game*, in which the teacher's success is conditioned by the student's success. This structure logically entails a fundamental characteristic of the didactic game. In order to win the game, the teacher cannot act directly. For example, in general, she cannot ask a question to the student and immediately answer this question. She needs a certain kind of "autonomy" from the student. In order to win, $Y$ (the teaching pole) has to lead $X$ (the studying pole) to a certain point, a particular "state of knowledge" which allows the student to play the "right moves" in the game, which can ensure the teacher that the student has built the right knowledge. At the core of this process, there is a fundamental condition: in order to be sure that $X$ has really won, $Y$ must remain tacit on the main knowledge at stake. She has to be *reticent*. On her side, the student must act proprio motu; the teacher's help must not allow the student to produce a "good" behavior without calling on the adequate knowledge. This proprio motu clause is necessarily related to the reticence of the teacher (Sensevy 2011b). Indeed the proprio motu clause and the teacher's reticence compose the general pattern of didactic transactions and give them their strongly asymmetrical nature.

### Learning Games

We call *learning game* the didactic game we modelize by using the concepts of didactic contract and didactic milieu (Sensevy et al. 2005). Consider this example: at primary school, students have to reproduce a puzzle by enlarging it, in such a way that a segment which measures 4 cm on the model will measure 7 cm on the reproduction. The pieces of this puzzle constitute the *milieu* that the students face for this "enlargement problem." The didactic contract (Brousseau 1997; Sensevy 2012) refers to the strategic system the student uses in order to work out the problem at stake. This strategic system has been shaped mainly in the previous joint didactic action. In our example, it is mainly an "additive" contract, in that students try to add 3 to every dimension of the puzzle. The milieu (Brousseau 1997; Sensevy 2012a) refers to the set of symbolic forms that the didactic experience transforms in an epistemic system. In our example, the fact that the puzzle pieces are not compatible has first to be an incentive to refute the additive strategy. Modelizing the teaching process by using the concept of learning game enables the researcher to identify the teacher's game on the student's game. When teaching a piece of knowledge, the teacher may rely on the contract properties (by having the students recognize the previous taught knowledge necessary to deal with the problem at stake) or on the milieu structure (by orienting the students so that they experience some epistemic features of this milieu, in our example, the fact that the puzzle pieces do not fit together). The JATD considers such a joint work as a didactic equilibration process, relying on the research of equilibrium, in the teacher's discursive work, between expression and reticence (Mercier et al. 2000; Sensevy et al. 2012b). Documenting this joint action needs a specific methodological instrumentation process (Sensevy and Forest 2012; Tiberghien and Sensevy 2012).

### Epistemic Games

In a nutshell, the notion of learning game is a way of modelizing what the teacher and the student jointly do in order for the student to learn something. The notion of epistemic game is a way of modelizing this *something*, i.e., what has to be learned.

Speaking of epistemic game rather than of "knowledge" or "subject content" is a way of actualizing the JATD's actional turn. An epistemic game is a modelization of what we can call a knowledge practice (the practice of a mathematician, a fiction writer, an historian, etc.). We argue that these practices have to be carefully scrutinized in a comprehensive way that may express their fundamental principles, rules, and strategies. For example, if one intends to some extent to have students as mathematicians, one has to modelize this practice (that of the mathematician) so that the teachers may monitor students' activity in a relevant way by relying on this model. In this respect, an epistemic game is a model (Sensevy et al. 2008), which attempts to grasp the fundamental dynamic structure of a knowledge practice and which can help the designers of a curriculum in the didactic transposition process.

### Cooperative Engineering

In order to contribute to the elaboration of new forms of schooling, the JATD aims at theorizing a specific process of design-based research, called cooperative engineering (Sensevy 2012) in which teachers and researchers jointly act to build teaching-learning sequences grounded on learning games nurtured by specific epistemic games. This process rests on the dilution of dualisms between theory and practice, ends and means. In this way, teachers and researchers may temporally occupy the same position, that of didactician engineer, by sharing the same educational ends and by working out together the means which will allow to reach these ends and to reconceptualize them. In this respect, the JATD endeavors to overcome the classic distinction between applied and fundamental research by proposing concrete curriculum designs.

### Concluding Remarks

Beyond different results and uses, ATD and JATD suggest a new school epistemology and urge a thorough reconstruction of the form of schooling, more open to the basics of cooperative studying and learning that they jointly advocate.

## Cross-References


▶ Design Research in Mathematics Education
▶ Didactic Contract in Mathematics Education
▶ Didactic Engineering in Mathematics Education
▶ Didactic Situations in Mathematics Education
▶ Didactic Transposition in Mathematics Education
▶ Didactical Phenomenology (Freudenthal)


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## Argumentation in Mathematics

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### Keywords

Argumentation; History of logic; Logic; Modal logic; Proof theory

### Definition

Argumentation refers to the process of making an argument, that is, drawing conclusions based on a chain of reasoning. Götz Krummheuer suggests that argumentation can be thought of as a social process in which the cooperating individuals "adjust their intentions and interpretations by verbally presenting the rationales for their actions" (Cobb and Bauersfeld 1995, p. 13). In mathematics, unlike any empirically based discipline, the validity of an argument in its final form is judged solely on whether it is logically consistent.

### Characteristics of Argumentation

The origins of logic, a key component of mathematical argumentation, can be traced back to Aristotelian logic and his use of syllogisms, with thinkers making improvements to this method over time as they were confronted with paradoxes. Argumentation was primarily the domain of theologians and medieval and postmedieval scholastics for over 1,700 years after Aristotle. Some well-known examples of theological argumentation are the Italian prelate St. Anselm of Canterbury's (1033–1109) "ontological argument" in the *Proslogion*, which was later revised by Leibniz and Gödel. Today, sophisticated versions of the ontological argument are written in terms of *modal logic*, a branch of logic which was familiar to the medieval scholastics. Modal logic today is a useful language for *proof theory*, the study of what can and cannot be proved in mathematical systems of deduction. Issues of completeness of mathematical systems, the independence of axioms from other axioms, and the consistency of formal mathematical systems are all part of proof theory. One also finds the use of logical argumentation to prove the existence of God in the theological works of Descartes, Leibniz, and Pascal.

The importance of the role of formal logic in mathematical argumentation continued to increase and reached its apex with the work of David Hilbert and other formalists in the nineteenth and first half of the twentieth century. The *Principia Mathematica*, by Alfred North Whitehead and Bertrand Russell, was a three-volume work that attempted to put the foundations of mathematics on a solid logical basis (Whitehead and Russell, 1927). However, this program came to a definitive end with the publication of Gödel's incompleteness theorems in 1931, which subsequently opened the door for more complex views of mathematical argumentation to develop.

Given this historical preview of the development of logic and its role in mathematical argumentation, we now turn our attention to contemporary views of mathematical argumentation, and in particular its constituent elements. Efraim Fischbein claimed that intuition is an essential component of all levels of an argument, with qualitative differences in the role of intuition between novices [students] and experts [mathematicians]. For novices, it exists as a primary component of the argument. Fischbein (1980) referred to this use of intuition as anticipatory, i.e., "…while trying to solve a problem one suddenly has the feeling that one has grasped the solution even before one can offer an explicit, complete justification for that solution" (p. 10). For example, in response to why a given solution to a problem is correct, the novice may respond "just because … it has to be." The person using this type of intuition accepts the given solution as the truth and believes nothing more needs to be said. In a more advanced argument, intuition plays the role of an "advanced organizer" and is only the beginning of an individual's argument. In this sense, a personal belief about the truth of an idea is formed and acts as a guide for more formal analytic methods of establishing truth. For example, a student may "see" that the result of a theorem is obvious, but realize that deduction is needed to establish truth publicly. Thus intuition serves to convince oneself about the truth of an idea while serving to organize the direction of more formal methods.

In an attempt to determine how mathematicians establish the truth of a statement in mathematics, Kline (1976) found that a group of mathematicians said they began with an informal trial and error approach guided by intuition. It is this process which helped these mathematicians convince themselves of the truth of a mathematical idea. After the initial conviction, formal methods were pursued. "The logical approach to any branch of mathematics is usually a sophisticated, artificial reconstruction of discoveries that are refashioned many times and then forced into a deductive system." (p. 451). There definitely exists a distinction between how mathematicians convince themselves and how they convince others of the truth of mathematical ideas. Another good exposition of what constitutes argumentation in mathematics is found in Imre Lakatos' (1976) *Proofs and Refutations*, in the form of a thought experiment. The essence of Lakatos' method lies in paying attention to the casting out of mathematical pathologies in the pursuit of truth. Typically one starts with a rule and clearly identifies the hypothesis. This is followed by an exploration of the possibility of its truth or falsity. The process of conjecture-proof-refutation results in the refinement of the hypothesis in the pursuit of truth in addition to the pursuit of all tangential hypotheses that arise during the course of discourse. The Lakatosian exposition of mathematical argumentation brings into focus the issue of fallibility of a proof, either due to human error or inconsistencies in an axiomatic system. However, there are self-correcting mechanisms in mathematics, i.e., proofs get fixed or made more rigorous and axiomatic systems get refined to resolve inconsistencies. For example, non-Euclidean geometries arose through work that resolved the question of whether the parallel postulate is logically independent of the other axioms of Euclidean geometry; category theory is a refinement of set theory that resolves set theoretic paradoxes; and the axioms of nonstandard analysis are a reorganization of analysis that eliminates the use of the law of the excluded middle.

However, the mathematical community has on numerous occasions placed epistemic value on results before they were logically consistent with other related results that lend credence to its logical value. For instance, many of Euler and Ramanujan's results derived through their phenomenal intuition and self-devised methods of argumentation (and proof) were accepted as true in an epistemic sense but only proved much later by mathematicians using a more rigorous form of mathematical argumentation to meet contemporary standards of proof. If one considers Weyl's mathematical formulation of the general theory of relativity by using the parallel displacement of vectors to derive the Riemann tensor, one observes the interplay between the intuitive and the deductive (the constructed object). The continued evolution of the notion of tensors in physics/Riemannian geometry can be viewed as a culmination or a result of the flaws discovered in Euclidean geometry. Although the sheer beauty of the general theory of relativity was tarnished by the numerous refutations that arose when it was proposed, one cannot deny the present day value of the mathematics resulting from the interplay of the intuitive and the logical. Many of Euler's results on infinite series have been proven correct according to modern standards of rigor. Yet, they were already established as valid results in Euler's work. This suggests that mathematical argumentation can be thought of as successive levels of formalizations as embodied in Lakatos' thought experiment. Such a view has been expressed in the writings of prominent mathematicians in Hersh's (2006) *18 Unconventional Essays on Mathematics*.

## Cross-References

▶ Argumentation in Mathematics Education
▶ Deductive Reasoning in Mathematics Education

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# Argumentation in Mathematics Education


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## Keywords

Argumentation; Beliefs; Heuristics; Lakatos; Proof


## Definition

"Argumentation in mathematics education" can mean two things:

1. The mathematical arguments that students and teachers produce in mathematics classrooms
2. The arguments that mathematics education researchers produce regarding the nature of mathematics learning and the efficacy of mathematics teaching in various contexts.

This entry is about the first of these two interpretations.

### Mathematics Classrooms and Argumentation

In the context of a mathematics classroom, we will take a "mathematical argument" to be a line of reasoning that intends to show or explain why a mathematical result is true. The mathematical result might be a general statement about some class of mathematical objects or it might simply be the solution to a mathematical problem that has been posed. Taken in this sense, a mathematical argument might be a formal or informal proof, an explanation of how a student or teacher came to make a particular conjecture, how a student or teacher reasoned through a problem to arrive at a solution, or simply a sequence of computations that led to a numerical result. The quantity and nature of mathematical arguments that students and teachers produce in mathematics classrooms varies widely. Observational studies of mathematics classrooms indicate that in some there is essentially no dialogue between students and students and teacher that would constitute an argument that is more complex than a series of calculations. In some classrooms, the teacher produces the majority of arguments, in others the teachers and students coproduce arguments, while in a very few, students spend time working together to develop arguments which they then present or even defend to the entire class.

The different ways in which mathematical arguments are enacted in classrooms reflect different philosophies about the kinds of mathematical arguments that belong in there and the different belief systems held by teacher related to how students develop the knowledge and skill to produce such arguments. These philosophies and belief systems are largely cultural in that teachers learn them implicitly through their own schooling; such knowledge is often tacitly held. In some cases, however, teachers believe that students should be engaging in more complex argumentation but do not have the practical skills to structure classroom episodes so that students are successful in creating or defending more complex mathematical arguments.

## Approaches to Argumentation in Mathematics Education

An exemplary case study of student's successfully creating and defending mathematical arguments is found in Fawcett's (1938) classic book *The Nature of Proof*, in which students are guided to create their own version of Euclidean geometry. This 2-year teaching experiment with high school students highlighted the role of argumentation in choosing definitions and axioms and illustrated the pedagogical value of working with a "limited tool kit." The students in Fawcett's study created suitable definitions, chose relevant axioms when necessary, and created Euclidean geometry by using the available mathematics of Euclid's time period (Sriraman 2006). The glimpses of the discourse one finds in Fawcett's study also illustrate the Lakatosian elements of the possibilities in an "ideal" classroom for argumentation. In the case of Lakatos, the argumentation (or classroom discourse) occurs in his rich imagination in the context of a teacher classifying regular polyhedra and constructing a proof for the relationship between the vertices, faces, and edges of regular polyhedra given by Leonhard Euler as $V + F - E = 2$. The essence of the "Lakatosian" method lies in paying attention to the casting out of mathematical pathologies in the pursuit of truth. Typically one starts with a rule and clearly identifies the hypothesis. This is followed by an exploration of the possibility of its truth or falsity. The process of conjecture-proof-refutation results in the refinement of the hypothesis in the pursuit of truth in addition to the pursuit of all tangential hypotheses that arise during the course of discourse. Mathematics educators have attempted to implement the technique of conjecture-proof-refutation with varying degrees of success in the context of number theoretic or combinatorial problems (see Sriraman 2003, 2006). An important aspect of argumentation in the context of Fawcett's (1938) study is that while the proofs themselves are student created, the format they take on is largely orchestrated by the teacher. The first objective of the class in Fawcett's study was to emphasize the importance of definitions and accepted rules. The class was also trained in identifying hidden assumptions and terms that need no definition. That is, students were trained to start with agreed upon premises (be they axioms, definitions, or generally accepted criteria outside of mathematics) and produce steps that lead to the sought conclusions. Included in this training is the analysis of other arguments on the basis of how well they do the same. This is a deductivist approach to argumentation (Sriraman et al. 2010) and allows for only a single method of proof. Direct proof is given to students with little regard to the way in which they will internalize the method. In the book *Proofs and Refutations* (1976), Lakatos makes the point that this sort of "Euclidean methodology" is detrimental to the exploratory spirit of mathematics. Not only can an overreliance on deduction dampen the discovery aspect of mathematics; it can also ignore the needs of students as they learn argumentation that constitutes a proof.

In *Patterns of Plausible Inference*, Polya (1954) lays out heuristics via which the plausibility of mathematical statements may be tested for validity. By doing so, he gives a guide for students as they go about exploring the validity of a statement. "I address myself to teachers of mathematics of all grades and say: *Let us teach guessing*" (Polya 1954, p. 158). This is quite different from the deductive view which holds fast to inferences that can be logically concluded, where inconclusive but suggestive evidence has no place. While we do not doubt that the deductivist approach leaves room for guessing, it is not its primary emphasis. This is not to say, either, that the heuristic approach would abandon demonstrative proof. In Polya's (1954) heuristic approach, students are exposed to ways familiar to mathematicians when they are judging the potential validity of a statement and looking for proof. Lakatos (1976) makes a similar case. In his fictional class, the students argue in a manner that mirrors the argument the mathematical community had when considering Euler's formula for polyhedra. He states that an overly deductive approach misrepresents the ways the mathematics community really works. Fawcett shows, however, a way in which a deductivist classroom can model the mathematical community to a certain extent. Like in the mathematics community, disagreements arise and the need for convincing fosters the need for proof (Sriraman et al. 2010).

Over the past several decades, philosophers of mathematics have been attempting to describe the nature of mathematical argumentation (e.g., Lakatos, Hersh, others; see ► Argumentation in Mathematics). Many mathematics education researchers have called for teachers to engage students in the practice of doing mathematics as mathematicians, which has mathematical argumentation at its core, for example, Deborah Ball http://ncrtl.msu.edu/http/craftp/html/pdf/cp903.pdf, Schoenfeld (1985).

In the USA, this call was brought to the national conversation through the inclusion of the process standards in the NCTM Principles and Standards for School Mathematics and has evolved to become more specific and concrete in the recent Standards for Mathematical Practice in the Common Core State Standards for Mathematics. Such standards, when coupled with the picture painted in US classrooms, show a wide gulf between the vision the mathematics education community has for how mathematical argumentation might look and what actually transpires in classrooms, at least in the USA. However, the US mathematics education researcher community is not alone; other countries' educational systems also grapple with similar issues, although the framing and details vary as they reflect cultural attitudes about the appropriate nature of mathematical argumentation in mathematics classrooms.

Caveat emptor: Neither Lakatos nor Polya were mathematics educators in the contemporary sense of the word. The former was a philosopher of science who was trying to address his community to pay attention to the history of mathematics, whereas the latter an exemplary mathematician that became interested in pedagogy. Both Lakatos and Polya's work has found an important place in the canon of literature in mathematics education that addresses discourse, argumentation, and proof and hence made central in this encyclopedia entry.

## Cross-References

► Argumentation in Mathematics
► Deductive Reasoning in Mathematics Education
► Quasi-empirical Reasoning (Lakatos)

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# Assessment of Mathematics Teacher Knowledge

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## Keywords

Teacher assessment; Teacher knowledge; Mathematical knowledge for teaching

## Definition

Practices and processes used to assess the mathematical knowledge of teachers. This information is frequently used to establish certification of new teachers, to give promotion and recognition to current teachers, to determine the need and content of professional development for current teachers, and to provide information to researchers about teacher knowledge. Methods to assess the mathematical knowledge of teachers include paper and pencil or oral examinations with multiple choice, short answer, or open-ended questions; portfolios; interviews; and demonstrations of teaching.

The need to systematically assess mathematics teacher knowledge was initiated by the publication in the early 1980s of reports about the low quality of mathematics in schools (e.g., the Cockcroft report in the UK, *Nation at Risk* in the US; see Howson et al. 1981). These reports stirred the need for reform in mathematics classrooms, and in particular to attend to "salary, promotion, tenure, and retention decisions [of teachers, which] should be tied to an effective evaluation system that includes peer review" (National Commission on Excellence in Education 1983, Recommendation D.2 Teaching). Reports of the low attainment of students in international comparisons of mathematics achievement in the studies conducted by the IEA, the OECD, and the UNESCO also have heightened awareness of the need to assess teachers' knowledge and to find its connections to student performance. Several decades later, many educational systems have passed resolutions that impose stringent requirement to certify teachers, to maintain them in the profession, and that have led researchers to investigate methods to measure this knowledge with a goal of producing valid results that are useful in policymaking.

## Certification of New Teachers

Assessment of mathematics teacher knowledge can be associated with processes of certification or licensing of teachers. Certification ensures that people who wish to work as mathematics teachers have sufficient knowledge and competence to practice the profession. Certification processes have changed over time (see Ravitch, n.d., http://www2.ed.gov/admins/tchrqual/learn/preparingteachersconference/ravitch.html for a brief history in the US), from requiring a demonstration of moral character, to demonstration of competency in elementary subjects (e.g., arithmetic, reading, history, and geography), to more specialized processes that may include demonstrations of teaching specific mathematics topics. In countries without a centralized system for regulating certification, more than one process can exist. The processes of certification vary across educational systems, with some requiring various examinations in several selection stages (e.g., written, oral, microteaching in Korea, http://www.MEST.go.kr) and some requiring a written test only (e.g., PRAXIS, in the USA, http://www.ets.org/praxis/about/praxisii).

## Promotion and Recognition of Practicing Teachers

Assessment of mathematics teacher knowledge is relevant to the employment status of current teachers. An educational system may use evidence that teachers hold or have gained sufficient knowledge and competence to retain the teachers in their current jobs, to recognize them, and to promote them. Current teachers may also use the processes to guide their professional development. These processes include peer reviews or observation of instruction by administrators, or in more formal cases, teachers may document their knowledge and create a portfolio that is evaluated by a national board (e.g., National Board for Professional Teaching Standards, http://www.nbpts.org/for_candidates/certificate_areas1?ID=3&x=57&y=8).

## Research on Teacher Knowledge

Assessment of mathematics teacher knowledge has lately been associated with measures of *mathematical knowledge for teaching* (MKT). The impetus for this work can be traced to Shulman's (1986) categorization of teachers' knowledge into content, pedagogical, and curricular. Prior to Shulman's publication, a standard way to measure teachers' knowledge was by the number of subject-matter courses teachers had been exposed to during training or the number of hours of professional development in which they have engaged as practicing teachers.

**Assessment of Mathematics Teacher Knowledge, Table 1** Examples of assessments of teacher knowledge

| Assessment, location | Level, purpose | What is assessed | Format of assessment |
|---|---|---|---|
| Teacher education test, South Korea | Preservice teachers, certification | Mathematics knowledge<br>General pedagogical knowledge<br>Specific knowledge for teaching mathematics | Three tests: multiple choice (for all areas), open ended (mathematics and general), oral and microteaching (general and specific) |
| PRAXIS, USA | Preservice teachers, certification | Varies from state to state, primarily content, although pedagogy is offered | Multiple choice, usually 2-h long. Requirements vary by state |
| NBPTS, USA | Practicing teachers, National Board Certification | Knowledge of mathematics, students, and teaching | Four portfolio entries, two of which are video, followed by six short answer assessment exercises, which are 30 min each |
| MKT, USA, other countries | Elementary teachers, research, and professional improvement | Mathematical knowledge for teaching, with six subcategories | Primarily multiple choice, occasional short answer with optional interviews depending on purpose of test (validation, research, and/or professional improvement) |
| COACTIV, Germany | Secondary teachers whose students participated in PISA, research | Content knowledge and three areas of pedagogical content knowledge | Two short answer paper and pencil tests (70 min for pedagogical content knowledge and 50 min for content). There 2 h more available for follow-up questions |
| TEDS-M, international | Preservice teacher education programs, research, comparative studies | Content knowledge, pedagogical content knowledge, and beliefs | 60-min paper and pencil, some open-ended questions |

The need for measuring teacher knowledge has become more prominent as demands for establishing links between teacher behaviors and student achievement have increased. Initial attempts to establish connections using characteristics such as the number of mathematics courses taken or the number of hours of professional development as proxies for teacher knowledge led to inconclusive results (Blömeke and Delaney 2012).

Research in this area has proposed that mathematics teacher knowledge includes six areas, three related to subject-matter knowledge (common content knowledge, knowledge at the mathematical horizon, and specialized content knowledge) and three related to pedagogical content knowledge (knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum, Ball et al. 2008). Research since the late 1990s has focused on the construction of an instrument that can measure specialized content knowledge. This instrument has been successfully validated with US-practicing elementary teachers (Hill et al. 2008). The instrument is not meant to be used for certification or promotion, rather for establishing a connection between teacher knowledge, student achievement, and quality of instruction (Hill et al. 2005). Because teaching is a highly contextualized practice, current research on the instrument focuses on validity of the instrument in other countries (see the 44th issue of *ZDM Mathematics Education*, 2012 on assessment of teacher knowledge).

Similar efforts to measure teacher knowledge with the purpose of connecting it to student achievement have been pursued in other countries. In Germany the impetus for the **Co**gnitive **Activ**ation in the Classroom (COACTIVE) project (Krauss et al. 2008) was German students' lower than expected performance in the Program of International Student Assessment (PISA) compared to other European countries. Other recent efforts to assess mathematics teacher knowledge in other countries come from the international Teacher Education and Development Study (TEDS), which is designed to describe the quality of teacher education programs in the 16 participating countries (http://www.iea.nl/teds-m.html). As part of the data collected, an instrument to assess mathematics teachers' knowledge and beliefs was used.

In Table 1 we present an overview of different types of processes to assess mathematics teacher knowledge.

Future research on this area of assessment of teacher knowledge will be in three fronts: calibration of the instrument for different contexts, validation of the construct with local definitions of instructional quality, and connections between the measures of teacher knowledge obtained and student performance within educational systems and as part of the international studies of student achievement.

## Cross-References


- ▶ Competency Frameworks in Mathematics Education
- ▶ Frameworks for Conceptualizing Mathematics Teacher Knowledge
- ▶ Mathematics Teacher as Learner
- ▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes
- ▶ Teacher Education Development Study-Mathematics (TEDS-M)
- ▶ Teacher Supply and Retention in Mathematics Education


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# Authority and Mathematics Education


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## Keywords

Authority of mathematics; Democratic values; Expert authority; Sociological perspectives


## Definition

The role of authority relations in mathematics education.

## Characteristics

As a topic for inquiry, authority enters into mathematics education by way of two main arguments. The first is that because sociology, anthropology, and politics are relevant to understanding mathematics education, as is discussed elsewhere in this encyclopedia, authority must be as well, being a central construct in all these attendant fields; indeed, any treatment of power, hierarchy, and social regulation and relations must refer to the notion of authority in some way. The second argument, more specific to mathematics education per se, is that, owing to the perception of mathematics as certain and final, the discipline itself is authoritarian, at least in a manner of speaking. Whether or not authority can be attributed to mathematics *strictly speaking* is moot of course; however, because of this perception of mathematics, authority, often in matters having little to do with mathematics, tends to be transferred to those who are considered mathematical experts. The latter links the second argument with the first, but it also shows how difficulties with authority can arise in classroom situations, for the authority of the mathematics teacher may trump the authority of the discipline, however that is understood.

## The Social Science Context

In the social sciences generally, the locus classicus for the treatment of authority is surely Max Weber's *The Theory of Social and Economic Organization* (Weber 1947). There, Weber describes "authority" (*Herrschaft*) as "...the probability that a command with a given specific content will be obeyed by a given group of persons" (p. 139). Weber's definition stresses that true authority involves more than power of one person or body over another, more than mere coercion: it involves "...a certain minimum of voluntary submission" on the part of the controlled and an interest in obedience on the part of the authority (p. 247). The crucial point is that for authority to be authority, it must be recognized as *legitimate* by those who submit to it; it is this that distinguishes it from mere power (*Macht*) (p. 139).

Weber identifies three grounds of legitimacy and three concomitant "ideal types" of authority: traditional, charismatic, and legal authority. *Traditional authority* is the authority of parents or of village elders. *Charismatic authority* is the authority of one endowed with superhuman powers, a shaman for example. *Legal authority* is authority within an "established impersonal order," a legal or bureaucratic system; the system within the legal authority acts is considered rational, and, accordingly, so too are the grounds of authority and the obedience it commands. These "ideal types" are not necessarily descriptions of given individual authority figures. Weber's claim is that authority can be analyzed *into* these types: the authority of any given individual is almost always an amalgam of various types.

*Expert authority*, which is an essential aspect of teachers' authority, does not appear in Weber's writings, but it is clear that because the grounds of such authority are rational and sanctioned by official actions, for example, the bestowing of an academic degree or a license, Weber could reasonably categorize it as a form of "legal authority." Still, it is different enough and important enough for educational purposes to distinguish expert authority as a distinct type with its legitimacy founded on the possession of knowledge by the authority figure (regardless of whether the knowledge is true or truly possessed).

Students' lives are influenced by a broad web of authorities, but the teachers' authority is the most immediate of these and arguably the most important. It has been suggested too that teachers' authority manifests elements not only of expert authority, but also traditional, legal, and even charismatic authority (Amit and Fried 2005). It is not by accident, then, that early sociological studies of education, such as Willard Waller's classic 1932 sociological study of education (see Amit and Fried 2005) and Durkheim's works on education (Durkheim 1961), underlined the authority of teachers, nor is it surprising that these sociological studies particularly emphasized the function of authority as a socializing force and its connection, accordingly, with moral instruction and discipline.

Because of the strength of teachers' authority, it can conflict with modes of teaching and learning which mathematics education has come to value. Such a conflict arises naturally between teachers' authority and democratic values. This was studied by Renuka Vithal (1999), who concluded that the teachers' authority, although opposed to democracy, could actually live with democracy in a relationship of complementarity. She suggests that the very fact of the teacher's authority, if treated appropriately, could provide an opportunity for students to develop a critical attitude toward authority (see also Skovsmose 1994).

To take full advantage of authority as Vithal suggests, or in any other way, it is essential to understand the mechanisms by which relations of authority are established and reproduced. Indeed, these may be embedded not only in social structures already in place when students enter a classroom, but in subtle aspects of classroom discourse. Herbel-Eisenmann and Wagner (2010), for example, have looked at lexical-bundles, small segments of spoken text, reflecting one's position in an authority relationship. These lexical-bundles are as much a part of the students' discourse as the teachers', recalling how authority relations are always a two-way street, as Weber was at pains to stress.

Paul Ernest's study of social semiotics (Ernest 2008) gives much support to Herbel-Eisenmann and Wagner's approach. Ernest shows how the analysis of classroom-spoken texts brings out the existence of overlapping forms of teachers' authority, different roles in which teachers' authority is manifest. In particular, he says, the teacher is both one *in* authority, a "social regulator" determining how a class is run, and also *an* authority, a "knowledge expert" (p. 42) determining, for example, what tasks are set to the students.

## The Authoritarian Nature of Mathematics

The role of teachers as expert authorities, as task controllers, to use Ernest's term, has very much to do with mathematical content and how it is passed on to students. We are brought, thus, to the second argument concerning authority and mathematics education, for the degree of the overlap Ernest refers to is very much related to the authoritarian nature of mathematics itself. This is not a new phenomenon. Judith Grabiner (2004), writing about Colin Maclaurin (1698–1746), has argued that mathematics in the eighteenth century attained an authority greater even than that of religion, since mathematics was perceived as having the power to achieve agreement with a universality and finality unavailable to religion. How far that authority was transferred to a mathematician like Maclaurin can be judged by the remark of a contemporary referring to actuarial work carried out by Maclaurin, not strictly mathematical work, that "The authority of [Maclaurin's] name was of great use. . .removing any doubt" (quoted in Grabiner 2004, p. 847).

The tension between teachers' authority and democratic modes of teaching has already been noted. But that had little to do with mathematics as such. The authority of mathematics combined with the authority subsequently transferred to practitioners and teachers of mathematics, however, creates a tension arising directly from the nature of mathematical authority. This is because what is essential about mathematical authority is precisely its independence from any human authority: a great mathematician must yield even to a child who has discovered a flaw in the mathematician's work. But Keith Weber and Juan Mejia-Ramos (in press) have shown that mathematicians themselves are influenced by human authorities or by authoritarian institutions – all the more so with students.

How this plays out in a specific mathematical context can be seen in Harel and Sowder's (1998) category of proof schemes based on external conviction, which includes a subcategory called "authoritarian proofs." Typical behavior associated with this proof scheme is that students ". . .expect to be told the proof rather than take part in its construction" (Harel and Sowder 1998, p. 247). The authority of a teacher presenting a proof can thus take precedence over the internal logic behind the authority of the discipline: the whole notion of "proof" is vitiated when this happens, since the truth of a claim becomes established not because of argument but because of a teacher's authoritative voice.

Reminiscent of Vithal's (1999) argument above, the challenge of mathematics teachers must then be, paradoxically, to use their authority to release students from teachers' authority. Jo Boaler (2003) suggests as much when she remarks favorably about a teacher in her study that she "employed an important teaching practice—that of deflecting her authority to the discipline [of mathematics]" (p. 8).

Since the authority of mathematics as a discipline becomes ultimately the possession of the student as the teacher deflects her own authority, we see that the problem of authority in mathematics education is how to devolve authority. The problem of authority, in this way, becomes the mirror problem of agency.

## Future Avenues of Research

One of the important conclusions from Vithal's (1999) work as well as Amit and Fried's (2005) work is that authority may be more than a necessary evil in mathematics education. But exactly how authority can be used to create a more democratic classroom and a more autonomous student needs to be investigated: what the actual mechanisms are through which this is achieved. This will be particularly important for teacher education, since it is teachers who command authority in the most explicit way. This presupposes that researchers have ways of tracing authority relations in the classroom. In this regard, a second necessary avenue of research is the identification of how authority relations are reproduced, research of the sort represented here by Herbel-Eisenmann and Wagner (2010).

## Cross-References

▶ Sociological Approaches in Mathematics Education

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# Autism, Special Needs, and Mathematics Learning

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## Keywords

Autism; Autistic spectrum condition; Autistic spectrum disorders (ASD); Asperger syndrome; Individual differences; Special needs; Inclusion

## Definition

Autism spectrum conditions are lifelong neurodevelopmental conditions that are characterized by often striking difficulties in social communication and repetitive and rigid patterns of behavior (American Psychiatric Association APA 2000). Current estimates indicate that 1 in every 100 children is on the autism spectrum, meaning that *all* schools and colleges are likely to include pupils who lie somewhere on the autism spectrum.

## Characteristics

Although autism is now considered a highly heritable disorder of neural development (Levy et al. 2009), specific genes, and the ways that these genes interact with the environment, are not yet fully understood (Frith 2003). The diagnosis of autism therefore relies on a constellation of behavioral symptoms, which can vary substantially from individual to individual. This variability includes marked differences in the degree of language skills: some individuals do not use oral language to communicate, while others use grammatically correct speech, but the way that they use language within social contexts can be odd and often one sided. Also, a substantial minority, roughly a third, meet the criterion for intellectual disability (Levy et al. 2009). Furthermore, there is wide variation in developmental outcomes: while some individuals with autism will go on to live independently and gain qualifications, many individuals are unable to live on their own or enjoy friendships and social contacts (Howlin et al. 2004).

The unusual abilities of some people with ASD show, such as Dustin Hoffman portrayed in the film Rain Man, have captured public attention. The most common ASD ability is calendar calculation, the ability to name weekdays corresponding to dates in the past or present. Some mathematicians have delighted in calendar calculation (e.g., Berlekamp et al. 1982), but autistic calendar calculation does not reflect any substantial mathematical abilities. Instead, autistic calendar calculators seem to solve date calculation problems by using a combination of memory for day-date combinations, addition and subtraction, and knowledge of calendrical patterns, such as the 28-year rule, i.e., 2 years 28 years apart are the same unless the interval contains a non-leap century year such as 2100 (Cowan and Frith 2009). The degree of skill they exhibit may result from practice. Several autistic calendar calculators do not appear to know how to multiply or divide. Most children with autism do not show any exceptional numerical ability.

There are remarkably few studies of the mathematical progress of children with ASD and most have relied on standardized tests that use arithmetic word problems to assess mathematical skill. The results should be interpreted cautiously as standardized tests can be extremely limited in the skills they assess (Ridgway 1987) and difficulties with arithmetic word problems may reflect autistic children's problems in verbal comprehension rather than difficulties in their computational skill. Nevertheless a recent review concludes that most children with autism show arithmetical skills slightly below those expected from their general ability with some doing markedly worse in arithmetic and others doing markedly better (Chiang and Lin 2007). The reasons for this variation have not been examined and individual differences in mathematical learning by children with ASD are as little understood as the reasons for individual differences in typically developing children.

Some of the core features of autism – including rigid and repetitive ways of thinking and behaving and heightened responses to environmental features (such as the sound of the school bell) – can make learning difficult for many children. Guidance on teaching children with autism therefore emphasizes the need for educators both to help the individual child/young person to develop skills and strategies to understand situations and communicate needs *and* to adapt the environment to enable the child to function and learn within it (Jordan and Powell 1995; Jones 2006; Freedman 2010; Charman et al. 2011; see also websites run by the National Autistic Society and the Autism Society of America). As we have stressed, children with autism differ enormously. For this reason, mathematical educators must be adept at understanding each student's individual needs and use innovative methods of modifying the curriculum, exploiting autistic students' strengths and interests, to make mathematics accessible and rewarding for such students.

## Cross-References


- ▶ 22q11.2 Deletion Syndrome, Special Needs, and Mathematics Learning
- ▶ Blind Students, Special Needs, and Mathematics Learning
- ▶ Deaf Children, Special Needs, and Mathematics Learning
- ▶ Down Syndrome, Special Needs, and Mathematics Learning
- ▶ Giftedness and high ability in mathematics
- ▶ Inclusive Mathematics Classrooms
- ▶ Language Disorders, Special Needs and Mathematics Learning
- ▶ Learner-Centered Teaching in Mathematics Education
- ▶ Learning Difficulties, Special Needs, and Mathematics Learning
- ▶ Mathematical Ability
- ▶ Word Problems in Mathematics Education


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# B

## Bilingual/Multilingual Issues in Learning Mathematics

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### Keywords

Bilingual; Bilingualism; Code-switching; Communication; Language; Learners; Linguistic; Linguistics; Monolingual; Multilingual; Multilingualism; Sociolinguistics; Students; Switching Languages

### Definition

Bilingual and multilingual issues in learning mathematics refer to questions regarding bilingual and multilingual learners as they learn mathematics. Research in mathematics education focusing on bilingual and multilingual issues in learning mathematics is primarily concerned with the study of bilingual and multilingual mathematics learners. Below is an overview of some key issues, ideas, and findings that focus on research on learners rather than on teaching practices, although the two are clearly connected.

### Characteristics

#### Theoretical Perspectives

The study of bilingual and multilingual mathematics learners requires theoretical notions that simultaneously address not only the cognitive and domain-specific aspects of learning mathematics but also the linguistic and cross-cultural nature of this work. Therefore, research addressing these issues draws on work from outside mathematics education. For example, educational anthropology and cultural psychology have been used to ground cross-cultural aspects of this work. Similarly, linguistics, especially approaches to bilingualism and multilingualism, has been used to ground linguistic aspects of this work. In particular, psycholinguistics and sociolinguistics are two theoretical perspectives frequently used in the study of bilingual and multilingual issues in learning mathematics.

### Bilingualism

"Bilingualism" (Peña and Bedore 2010) is an example of a concept that has different meanings depending on the theoretical perspective used. Definitions of bilingualism range from native-like fluency in two languages to alternating use of two languages, to participation in a bilingual community. A researcher working

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from a psycholinguistic perspective would define a bilingual person as an individual who is in some way proficient in more than one language. This definition would include someone who has learned a second language in school with some level of proficiency but does not participate in a bilingual community. In contrast, a researcher working from a sociolinguistic perspective would define a bilingual person as someone who participates in multiple language communities and is "the product of a specific linguistic community that uses one of its languages for certain functions and the other for other functions or situations" (Valdés-Fallis 1978, p. 4). The second definition frames bilingualism not as an individual but as a social and cultural phenomenon that involves participation in the language practices of one or more communities. Some researchers propose using "monolingual" and "bilingual" not as labels for individuals but as labels for modes of communicating (Grosjean 1999).

A common misunderstanding of bilingualism is the assumption that bilinguals are equally fluent in their two languages. If they are not, then they have been described as not truly bilingual or labeled as "semilingual" or "limited bilingual." In contrast, current scholars of bilingualism see "native-like control of two or more languages" as an unrealistic definition. Researchers have recently strongly criticized the concept of semilingualism (Cummins 2000) and propose we leave that notion behind.

## Research Findings

There are several research findings relevant to bilingual and multilingual issues in learning mathematics. Overall, there is strong evidence suggesting that bilingualism does not impact mathematical reasoning or problem solving. There are also relevant findings regarding two common practices among bilingual and multilingual mathematics learners, switching languages during arithmetic computation and code-switching.

Older bilingual students may carry out arithmetic computations in a preferred language, usually the language in which they learned arithmetic. There is evidence that adult bilinguals sometimes switch languages when carrying out arithmetic computations and that adult bilinguals may have a preferred language for carrying out arithmetic computation, usually the language of arithmetic instruction. Language switching can be swift, highly automatic, and facilitate rather than inhibit solving word problems in the language of instruction, provided the student's proficiency in the language of instruction is sufficient for understanding the text of a word problem. These findings suggest that classroom instruction should allow bilingual and multilingual students to choose the language they prefer for arithmetic computation and support all students in learning to read and understand the text of word problems in the language of instruction (Moschkovich 2007).

Another common practice among bilinguals is switching languages during a sentence or conversation, a phenomenon linguists call "code-switching" (Mercado 2010). Bilingual and multilingual mathematics students may use two languages during classroom conversations. In mathematics classrooms, children will use one or another language. Which language children use principally depends on the language ability and choice of the person addressing them. After the age of five, young bilinguals (beyond age 5) tend to "speak as they are spoken to". If Spanish–English bilinguals are addressed in English, they reply in English; if they are addressed in Spanish, they reply in Spanish; and if they are addressing a bilingual speaker, they may code-switch.

Another common misunderstanding is that code-switching is somehow a sign of deficiency. However, empirical research in sociolinguistics has shown that code-switching is a complex language practice and not evidence of deficiencies. In general, code-switching is not primarily a reflection of language proficiency, discourse proficiency, or the ability to recall (Valdés-Fallis 1978). Bilinguals use the two codes differently depending on the interlocutor, domain, topic, role, and function. Choosing and mixing two codes also involves a speaker's cultural identities.

Research does not support a view of code-switching as a deficit itself or as a sign of any deficiency in mathematical reasoning. Researchers in linguistics agree that code-switching is not random or a reflection of language deficiency – forgetting a word or not knowing a concept. Therefore, we cannot use someone's code-switching to reach conclusions about their language proficiency, ability to recall a word, knowledge of a particular mathematics word or concept, mathematical reasoning, or mathematical proficiency. It is crucial to avoid superficial conclusions regarding code-switching and mathematical cognition. For example, we should not conclude that bilingual and multilingual students switch into their first language because they do not remember a word, are missing vocabulary, or do not understand a mathematical concept. Rather than viewing code-switching as a deficiency, instruction for bilingual mathematics learners should consider how this practice serves as a resource for communicating mathematically. Bilingual speakers have been documented using their two languages and code-switching as a resource for mathematical discussions, for example, first giving an explanation in one language and then switching to the second language to repeat the explanation (Moschkovich 2002).

## History

Research on bilingual mathematics learners dates back to the 1970s. Early research focused on the disadvantages that bilinguals face, focusing, for example, on comparing response times between monolinguals and bilinguals (for examples and a review see Moschkovich 2007) or the obstacles the mathematics register in English presents for English learners (for some examples see Cocking and Mestre 1988). Studies focused on the disadvantages bilingual learners faced did not consider any possible advantages of bilingualism, for example the documented "enhanced ability to selectively attend to information and inhibit misleading cues" (Bialystok 2001, p. 245). Studies that focused on the differences between bilinguals and monolinguals may also have missed or de-emphasized any similarities, for example, that both groups may have similar responses to syntactic aspects of algebra word problems.

Some early research used vague notions of language and narrow conceptions of mathematics as arithmetic or word problems and focused on two scenarios, carrying out arithmetic computation and solving word problems (Moschkovich 2002, 2010). Later studies developed a broader view of mathematical activity, examining not only responses to arithmetic computation but also reasoning and problem solving, detailed protocols of students solving word problems, the strategies children used to solve arithmetic word problems, and student conceptions of two digit quantities. (The volume "*Linguistic and cultural influences on learning mathematics*" edited by Cocking and Mestre includes both types of research studies.)

More recent research uses broader notions of mathematics and language, in particular by using sociocultural, sociolinguistic, and ethnomathematical perspectives. A central concern has been to shift away from deficit models of bilingual and multilingual students to theoretical frameworks and practices that value the resources these students bring to the mathematics classroom from their previous experiences and their homes. More recently, researchers have studied language, bilingualism, and mathematics learning in many different settings (for examples see Adler 1998; Barton et al. 1998; Barwell et al. 2007, 2009; Barwell 2003b and 2009; Clarkson and Galbraith 1992; Dawe 1983; Kazima 2007; Roberts 1998; Setati 1998).

This work can provide important resources for addressing issues for bilingual and multilingual students in other settings, as long as differences among settings are considered. One difference is how languages are used in the classroom. Barwell (2003a) provides some useful distinctions among different language settings, using the terms *monopolist*, *pluralist*, and *globalist*. In monopolist classrooms, all teaching and learning take place in one dominant language; in pluralist classrooms, several languages used in the local community are also used for teaching and learning; in globalist classrooms, teaching and learning are conducted in an internationally used language that is not used in the surrounding community.

Another difference to consider across settings is the nature of the mathematics register in students' first language. For example, the mathematics register in Spanish is used to express many types of mathematical ideas from everyday to advanced academic mathematics. This may not be the case for the home languages of students in other settings. Barwell (2008) makes two crucial observations: (1) "all languages are equally capable of developing mathematics registers, although there is variation in the extent to which this has happened" and (2) "the mathematics registers of different languages... stress different mathematical meanings." These differences in mathematics registers, however, should not be construed as a reflection of differences in learner's abilities to reason mathematically or to express mathematical ideas. Furthermore, we should not assume that there is a hierarchical relationship among languages with different ways to express academic mathematical ideas, for example using one word versus using (or inventing) multiple word phrases.

## Issues in Designing Research

One challenge researchers face when designing research with bilingual and multilingual learners is that these labels are used in ambiguous ways and with multiple meanings. Research studies need to specify how the labels bilingual or multilingual are used, when applied to learners or classrooms. These labels do not describe exactly what happens in the classroom in terms of how teachers and students use languages. Studies should document students' language proficiencies in both oral and written modes and also describe students' histories, practices, and experiences with each language across a range of settings and mathematical tasks.

"Language proficiency" is a complex construct that can reflect proficiency in multiple contexts, modes, and academic disciplines. Current measures of language proficiency may not give an accurate picture of an individual's language competence. We do not have measures or assessments for language proficiency related to competence in mathematics for different ages or mathematical topics. There are serious challenges that research still needs to address, given the complexity of defining a construct such as language "proficiency": (a) the lack of instruments sensitive to both oral and written modes for mathematical communication and (b) the scarcity of instruments that address features of the mathematics register for specific mathematical topics. Studies should not assess language proficiency in general but rather specifically for communicating in writing and orally about a particular mathematical topic. Students have different opportunities to talk and write about mathematics in each language, in informal or instructional settings, and about different mathematical topics. Assessments of language proficiency, then, should consider not only proficiency in each language but also proficiency for using each language to talk or write about a particular mathematical topic.

## Future Issues and Questions

Research on bilingual and multilingual issues in mathematics learning is still in a developing stage. A central issue is grounding research in mathematics education on theoretical perspectives and findings from relevant fields such as linguistics and anthropology. Future work should avoid reinventing wheels or, worse, reifying myths or misunderstandings about bilingualism/multilingualism. This is best accomplished through repeated and extended interactions between scholars who study mathematics learning and scholars who study bilingual and multilingual learners. Future studies should avoid deficit-oriented models of bilingual and multilingual learners and consider any advantages that bilingualism might provide for learning mathematics.

## Cross-References

- ▶ Cultural Diversity in Mathematics Education
- ▶ Discourse Analytic Approaches in Mathematics Education

- ▶ Immigrant Students in Mathematics Education
- ▶ Language Background in Mathematics Education
- ▶ Language Disorders, Special Needs and Mathematics Learning
- ▶ Mathematical Language
- ▶ Mathematical Representations
- ▶ Semiotics in Mathematics Education

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# Blind Students, Special Needs, and Mathematics Learning

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## Keywords

Blind mathematics learners; Perception and cognition; Visualization; Auditory representations; Tactile representations

## Characteristics

Blindness, in itself, does not seem to be an impediment to learning mathematics. Indeed, history shows that there have been a number of very successful blind mathematicians, perhaps the most well known being Euler (1707–1783), who became blind in the latter part of his life, and Saunderson (1682–1739) who lost his sight during his first year. Jackson (2002), in his consideration of the work of these and more contemporary blind mathematicians, suggests that the lack of access to the visual field does not diminish a person's ability to visualize – but modifies it, since spatial imagination amongst those who do not see with their eyes relies on tactile and auditory activity. This would suggest that to understand the learning processes of blind mathematics learners, it is important to investigate how the particular ways in which they access and process information shapes their mathematical knowledge and the learning trajectories through which it is attained.

Vygotsky's work with disabled learners, in general, and those with visual impairments, in particular, during the 1920s and 1930s represented an early attempt to do just this. Rather than associating disability with deficit and focusing on quantitative differences in achievements between those with and without certain abilities, he proposed that a qualitative perspective should be adopted to research how access to different mediating resources impacts upon development (1997). The key to understanding and supporting the practices of blind learners, he argued, lies in investigating how the substitution of the eyes by other tools both permits and shapes their participation in social and cultural activities, such as mathematics learning.

For the study of mathematical topics that involve working with spatial representations and information, the hands represent the most obvious substitute for the eyes, and hence it is not surprising that research involving blind geometry learners has focused on how explorations of tactile representation of geometrical objects contribute to the particular conceptions that emerge. While vision is synthetic and global, with touch the whole emerges from relationships between its parts, a difference which Healy and Fernandes (2011) suggest might explain the tendency amongst blind learners to describe geometrical properties and relations using dynamic rather than static means, which simultaneously correspond to and generalize their physical actions upon the objects in question.

Hands also play an important role in blind students' access to written materials, with Braille codes substituting text in documents for blind readers. There are, however, a number of particular challenges associated with learning and doing mathematics using Braille. First, there is no one universally accepted Braille code for mathematics, with different notations used in different countries. The coding systems are complex and can take considerable time to master (Marcone and Penteado 2013). An additional complication is that Braille is a strictly linear notation, whereas conventional mathematical notations make use of visual features – fractions provide a case in point. The linear versions of conventional notations require additional symbols, making expressions in Braille lengthy; compounded by the fact that Braille readers can only perceive what is under their fingers at a particular moment in time, it can be very difficult for them to obtain a general view of algebraic expressions. Digital technologies are facilitating conversions between Braille and text and offering the blind learner spoken versions of written mathematics, but research is needed to investigate how such alternative notation forms might impact differently on mathematical understandings and practices.

Use of spoken rather than written materials suggests that the ears can also be used as substitutes for the eyes. But auditory learning materials need not be limited to speech. Leuders (2012) argues that auditory perception represents an important modality for processing mathematical structures that has been under-explored. Here, too, digital technologies are bringing new forms of representing and exploring mathematical objects; one example is a musical calculator which enables students to hear as well as see structures of rational and irrational numbers (Fernandes et al. 2011).

In short, although the practice of blind mathematics learners is a topic that has been relatively under-researched in the field of mathematics education, the evidence that does exist suggests that in the absence of the visual field, information received through other sensory and perceptual apparatuses provides alternative forms of experiencing mathematics. Deepening our understandings of how those who do not see with their eyes learn and do mathematics may hence contribute to furthering our understanding of the relationships between perception and mathematical cognition more generally.

## Cross-References

- ▶ Deaf Children, Special Needs, and Mathematics Learning
- ▶ Equity and Access in Mathematics Education
- ▶ Inclusive Mathematics Classrooms
- ▶ Mathematical Representations
- ▶ Political Perspectives in Mathematics Education
- ▶ Psychological Approaches in Mathematics Education
- ▶ Visualization and Learning in Mathematics Education

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# Bloom's Taxonomy in Mathematics Education


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## Keywords

Cognition; Evaluation; Educational objectives; Student achievement; Assessment


## Definition

An approach to classifying reasoning goals with respect to mathematics education.

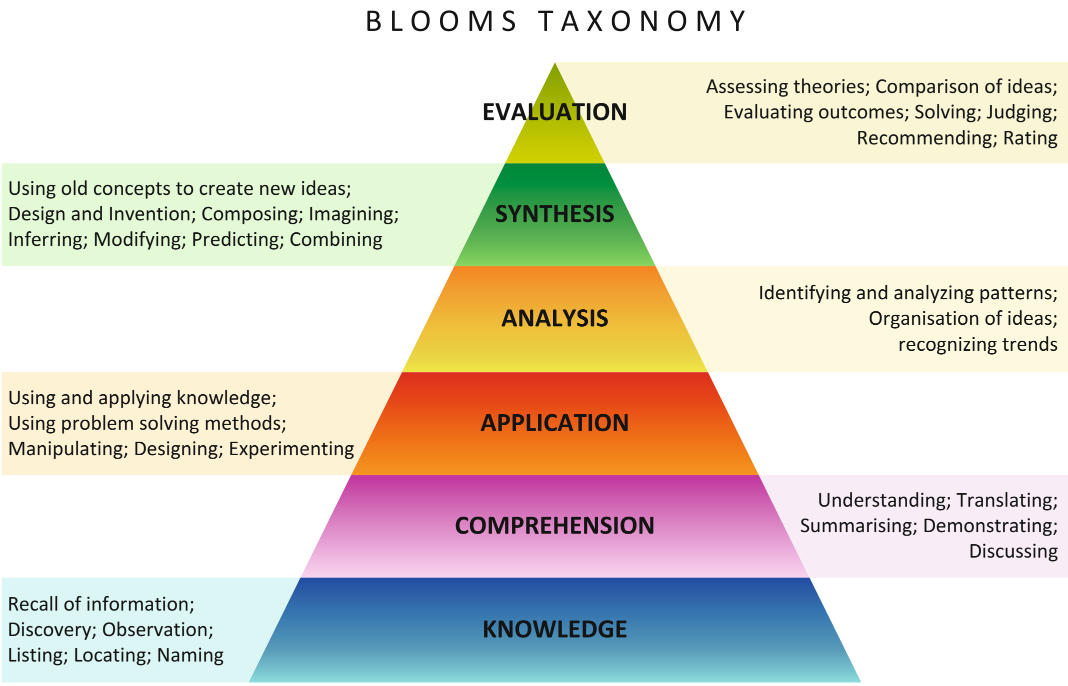
## Overview

Bloom's Taxonomy is arguably one of the most recognized educational references published in the twentieth century. As noted in a 40-year retrospective by Benjamin Bloom (1994), "it has been used by curriculum planners, administrators, researchers, and classroom teachers at all levels of education" (p. 1), and it has been referenced in academic publications representing virtually every academic discipline. Given the prevalence of testing in mathematics and the regular use of mathematics as a context for studying student reasoning and problem solving, Bloom's Taxonomy has been applied and adapted by mathematics educators since its publication.

### Historical Development

Originally designed as a resource to support the development of examinations, Bloom et al. (1956) wrote their taxonomy to insure greater accuracy of communication among educators in a manner similar to the taxonomies used in biology to organize species of flora and fauna. The ubiquitous reference to Bloom's Taxonomy is a triangle with six levels of named educational objectives for the cognitive domain: knowledge, comprehension, application, analysis, synthesis, and evaluation (Fig. 1; Office of Community Engagement and Service 2012).

Because of this reductivist use of *Handbook 1: Cognitive Domain* in which the taxonomy appeared (Bloom et al. 1956), few will recall that the knowledge category included multiple "knowledge of" subcategories such as knowledge of conventions, knowledge of trends and sequences, and knowledge of methodology. The writing team recognized that even knowledge ranges in complexity and is quite nuanced and detailed in ways that belie its perfunctory contemporary placement on the base of the




**Bloom's Taxonomy in Mathematics Education, Fig. 1** Bloom's Taxonomy

triangle. It is also worth noting that the *Handbook* includes many examples of "illustrative test items," suggesting both its intended use as a resource for evaluation and the importance of using content-specific examples to communicate objectives for student learning. Few of these illustrative test items, however, were in the domain of mathematics.

The authors of Bloom's Taxonomy acknowledged that it was imperfect and subject to adaptation and critique. Since these criticisms are relevant to the use and misuse of the Taxonomy in mathematics education, they are presented here to frame the section that follows.

Postlethwaite (1994) summarized the major criticisms as:

1. The distinctions between any two levels of the Taxonomy may be blurred.
2. The Taxonomy is not hierarchical; rather it is just a set of categories.
3. The lockstep sequence underlying the Taxonomy based on one dimension (e.g., complexity or difficulty) is naïve (p. 175).

A revision of the Taxonomy, which took into account recent advances in educational psychology and potential applications in curriculum and instruction, was published by Anderson et al. (2001); however, since the influence of the revised Taxonomy is difficult to determine, it is not discussed here in reference to mathematics education.

## Influence on Mathematics Education

Much of the influence of the Taxonomy on mathematics education has been on evaluation and more specifically in the design and interpretation achievement tests (e.g., Webb 1996). Since these aspects of school mathematics often influence the curricular goals, there has also been some indirect influence on curriculum development and classroom assessment in mathematics (Sosniak 1994).

Many assessment frameworks for mathematics have utilized the Taxonomy for guidance regarding the distribution of items on achievement tests. In Korea in the late 1950s, "teacher-made achievement tests and . . . entrance examinations," including those in mathematics, were analyzed for the distribution of test items across the six categories of Taxonomy (Chung 1994, p. 165). Since its inception in 1958, the International Association for the Evaluation of Educational Achievement has used the Taxonomy to support curriculum analysis, test construction, and data analysis, which precipitated its widespread use internationally (Lewy and Báthory 1994, p. 147). A familiar international achievement test to most mathematics educators is the Trends in Mathematics and Science Study (TIMSS). The TIMSS framework for mathematics (Mullis et al. 2003) includes four cognitive domains along with several subcategories (Table 1):

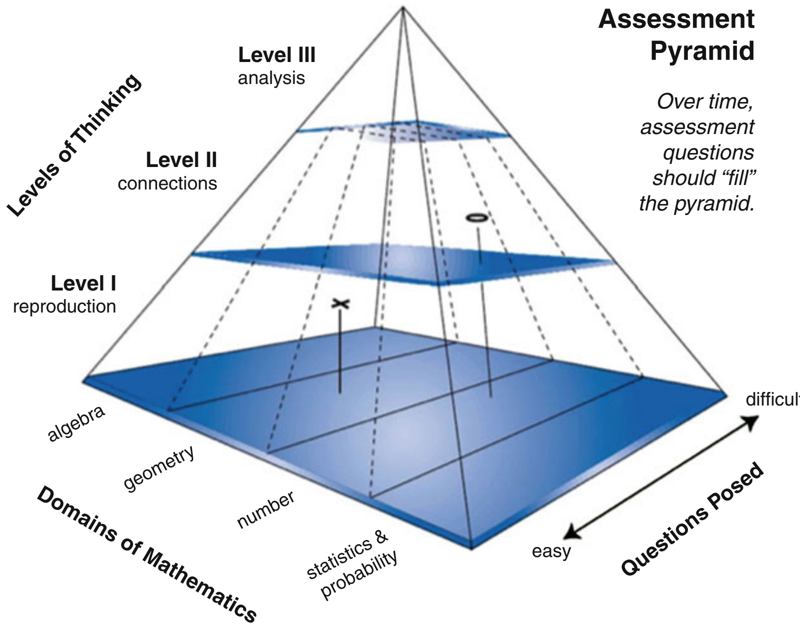
**Bloom's Taxonomy in Mathematics Education, Table 1** TIMSS 2003 mathematics framework (cognitive domains)

| TIMSS math cognitive domains | Subcategories |
|---|---|
| Knowing facts and procedures | Recall |
| | Recognize/identify |
| | Compute |
| | Use tools |
| Using concepts | Know |
| | Classify |
| | Represent |
| | Formulate |
| | Distinguish |
| Solving routine problems | Select |
| | Model |
| | Interpret |
| | Apply |
| | Verify/check |
| Reasoning | Hypothesize/conjecture/predict |
| | Analyze |
| | Evaluate |
| | Generalize |
| | Connect |
| | Synthesize/integrate |
| | Solve nonroutine problems |
| | Justify/prove |

When taking into account both the TIMSS domains and subcategories, several similarities are found with respect to Bloom's Taxonomy: (a) the hierarchical representation of knowledge to more complex forms of mathematical reasoning, (b) a large base of knowledge-related subcategories in the first two TIMSS domains, (c) application in the Taxonomy is synonymous with the TIMSS domain solving routine problems, and (d) the Taxonomy domains of analysis, synthesis, and evaluation are all named in the reasoning domain. Even though Bloom's Taxonomy is not explicitly named in the narrative for the TIMSS framework, it is evident that the Taxonomy influenced the organization and subcategories of the TIMSS framework. This serves as one example, although there are many, of the ways in which the Taxonomy has permeated the way evaluation in mathematics education is conceived and communicated.

With respect to Postlethwaite's summary of major criticisms, the TIMSS framework does caution the reader in incorrectly perceiving these four domains as hierarchical or organized as a lockstep sequence. Mullis et al. (2003) state, "cognitive complexity should not be confused with item difficulty. For nearly all of the cognitive skills listed, it is possible to create relatively easy items as well as very challenging items" (p. 25). Likewise, to counter the perception that reasoning goals are hierarchical, the Mathematics Framework for the *Program for International Student Assessment* (OECD 2003) organized the reasoning goals of reproduction, connections, and analysis as a horizontal set of mathematical competencies. Yet, in spite of the various ways in which cognitive domains or competencies are represented, results from studies of teachers' classroom assessment practices suggest that the general perception of mathematics teachers is that knowledge of skills and procedures is a prerequisite for student engagement in any of the other cognitive domains (Dekker and Feijs 2005; Webb 2012).

One of the more outspoken critics of Bloom's Taxonomy was the Dutch mathematician Hans Freudenthal, who was noteworthy for his contributions to both mathematics and mathematics education. By the mid-1970s, Freudenthal had argued that the simplification of reasoning into the taxonomic categories had a detrimental effect on test development. As summarized by Marja




**Bloom's Taxonomy in Mathematics Education, Fig. 2** Dutch assessment pyramid

van den Heuvel-Panhuizen (1996), "In a nutshell, Bloom sees the capacity to solve a given problem as being indicative of a certain level, while, in Freudenthal's eyes, it is the way in which the student works on a problem that determines the level. The latter illustrates this viewpoint using the following example:

> A child that figures out 8 + 7 by counting 7 further from 8 on the abacus, acts as it were on a senso-motoric level. The discovery that 8 + 7 is simplified by 8 + (2 + 5) = (8 + 2) + 5 witnesses a high comprehension level. Once this is grasped, it becomes mere knowledge of the method; as soon as the child has memorized 8 + 7 = 15, it is knowledge of facts. At the same moment figuring out 38 + 47 may still require comprehension; later on, knowledge of method can suffice; for the skilled calculator it is mere knowledge of facts" (Freudenthal 1978, p. 91, as cited in van den Heuvel-Panhuizen 1996, p. 21).

At issue in Freudenthal's remarks are not the categories themselves, but the way in which a taxonomy implies levels, orders of sophistication, and artificially imposed limits on educators' perceptions of children's mathematical reasoning (also see Kreitzer and Madaus 1994).

## Contemporary Classroom Applications

A primary motivation in publishing and disseminating Bloom's Taxonomy was the need to advance the design of achievement measures to assess more than recall of skills, facts, and procedures. A similar argument could be made for investigating different examples of mathematical reasoning with teachers. In one 3-year study conducted by Webb (2012) with middle-school mathematics teachers, analyses of over 10,000 assessment tasks used by 19 teachers revealed that greater than 85 % of the tasks assessed knowledge of skills and procedures.

To motivate teachers to use tasks assessing a broader range of mathematical reasoning goals, teachers categorized the assessment tasks they used using the Dutch assessment pyramid (Fig. 2; Shafer and Foster 1997; adapted from Verhage and de Lange 1997).

Even though the pyramid hints at the triangular representation of Bloom's Taxonomy in Fig. 1, the additional dimension of Questions Posed (i.e., from easy to difficult) illustrates that questions that elicit student reasoning at different levels are not necessarily more difficult. This was the identical argument made in the TIMSS framework. This work has since been extended into the design of professional development activities that support teacher change in classroom assessment (Her and Webb 2005; Webb 2009). As Black and Wiliam (1998) and Hattie and Timperley (2007) meta-analyses have given greater attention, respectively, to formative assessment and instructional feedback, there will be a continued need among mathematics educators to communicate goals for student learning. Over 50 years ago, Bloom's Taxonomy offered a compelling and influential example to address this need.

## Cross-References

- ▶ Abstraction in Mathematics Education
- ▶ Critical Thinking in Mathematics Education
- ▶ Deductive Reasoning in Mathematics Education
- ▶ Mathematical Knowledge for Teaching
- ▶ Mathematical Modelling and Applications in Education
- ▶ Mathematical Proof, Argumentation, and Reasoning
- ▶ Mathematics Classroom Assessment
- ▶ Problem Solving in Mathematics Education
- ▶ Questioning in Mathematics Education

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## Calculus Teaching and Learning

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### Keywords

Calculus key concepts; Intuitive representations; Formal definitions; Intuition of infinity; Notion of limit; Cognitive difficulties; Theoretical dimensions; Epistemological dimension; Research in teaching and learning calculus; Role of technology; Visualization; Coordination between semiotic registers; Role of historical perspective; Sociocultural approach; Institutional approach; Teaching practices; Role of the teacher; Transition between secondary school and university

## Definition: What Teaching and Learning Calculus Is About?

The differential and integral calculus is considered as one of the greatest inventions in mathematics. Calculus is taught at secondary school and at university. Learning calculus includes the analysis of problems of changes and motion. Previous related concepts like the concept of a variable and the concept of function are necessary for the understanding of calculus concepts. However, the learning of calculus includes new notions like the notion of limit and limiting processes, which intrinsically contain changing quantities. The differential and integral calculus is based upon the fundamental concept of limit. The mathematical concept of limit is a particularly difficult notion, typical of the kind of thought required in advanced mathematics.

## Characteristics

### Calculus Curriculum

There have been efforts in many parts of the world to reform the teaching of calculus. In France, the syllabus changed in the 1960s and 1970s due to the influence of the Bourbaki group. The limit concept with its rigorous basis has penetrated even into the school curriculum: in 1972, the classical definition of the derivative in terms of the limit of a quotient of differences was introduced. Another change occurred in the French calculus curriculum in 1982, this time influenced by the findings of mathematics education research, and the curriculum focused on more intuitive approaches. As a result, the formalization of the limit has been omitted at secondary school. This is the situation in most countries today: at high school level, there is an effort to develop a first approach to calculus concepts without relying on formal definitions and proofs. An intuitive and pragmatic approach to calculus at the high senior level at school (age 16–18) precedes the formal approach introduced at university.

At university level, calculus is among the more challenging topics faced by new undergraduates.

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In the United States, the calculus reform movement took place during the late 1980s. The recommendation was that calculus courses should address fewer topics in more depth, and students should learn through active engagement with the material. The standard course syllabus was revised, and new projects arose which incorporated technology into instruction.

In most countries, the transition towards more formal approach that takes place at university is accompanied with conceptual difficulties.

### Early Research in Learning Calculus: The Cognitive Difficulties

The cognitive difficulties that accompany the learning of central notions like functions, limit, tangent, derivative, and integral at the different stages of mathematics education are well reported in the research literature on calculus learning. These concepts are key concepts that appear and reappear in different contexts in calculus. The students meet some of these central topics at school, then the same topics appear again, with a different degree of depth at university. We might attribute the high school students' cognitive difficulties to the fact that the notions were presented to them in an informal way. In other words, we might expect that the difficulties will disappear when the students will learn the formal definition of the concepts. Undergraduate mathematics education research suggests otherwise. The cognitive difficulties that accompany the key concepts in calculus are well described in Sierpinska (1985); Davis and Vinner (1986); Cornu (1991); Williams (1991); Tall (1992), as well as in the book *Advanced mathematical thinking* edited by Tall (1991). The main source of difficulty resides in the fact that many students' intuitive ideas are in conflict with the formal definition of the calculus concepts like the notion of limit.

In these early researches on learning calculus, the theoretical dimensions are essentially cognitive and epistemological. The cognitive difficulties that accompany the learning of the key concepts in calculus like the limit concept are inherent to the epistemological nature of the mathematics domain. In the following we consider some facets of the dynamic interaction between formal and intuitive representations as they were discussed in these early studies. We encounter the first expression of the dynamic interaction between intuition and formal reasoning in the terms *concept definition and concept image*. For example, the intuitive thinking, the visual intuitions, and verbal descriptions of the limit concept that precede its definition are necessary for understanding the concept. However, research on learning calculus demonstrates that there exists a gap between the mathematical definition of the limit concept and the way one perceives it. In this case, we may say that there is a gap between the concept definition and the concept image (Tall and Vinner 1981; Vinner 1983). Vinner also found that students' intuitive ideas of the tangent to a curve are in conflict with the formal definition. This observation might explain students' conceptual difficulties to visualize a tangent as the limiting case of a secant.

Conceptual problems in learning calculus are also related to infinite processes. Research demonstrates that some of the cognitive difficulties that accompany the understanding of the concept of limit might be a consequence of the learners' intuition of infinity. Fischbein et al. (1979) observed that the natural concept of infinity is the concept of *potential infinity*, for example, the non-limited possibility to increase an interval or to divide it. The *actual infinity*, for example, the infinity of the number of points in a segment, the infinity of real numbers as *existing*, as *given* is, according to Fischbein, more difficult to grasp and leads to contradictions. For example, "If one has 1/3 it is easy to accept the equality $1/3 = 0.33\ldots$ The number 0.333…represents a potential (or dynamic) infinity. On the other hand, students questioned whether 0.333… is equal to 1/3 or tends to 1/3 answer usually that 0.333…tends to 1/3."

Among the theoretical constructs that accompany the early strands in research on learning calculus, we mention the *process-object duality*. The lenses offered by this framework highlight students' dynamic process view in relation to concepts such as limit and infinite sums and help researchers to understand the cognitive difficulties that accompany the learning of the limit concept. Gray and Tall (1994) introduced the notion of *procept*, referring to the manner in which learners cope with symbols representing both mathematical processes and mathematical concepts. Function, derivative, integral, and the fundamental limit notion are all examples of procepts. The limit concept is a procept: the same notation represents both the process of tending to the limit and also the value of the limit.

## Research and Alternative Approaches to Teaching and Learning Calculus

Different directions of research were investigated in the last decades. The use of technology offered a new mean in the effort to overcome some of the conceptual difficulties: the power of technology is particularly important to facilitate students' work with epistemological double strands like discrete/continuous and finite/infinite. Visualization and especially dynamic graphics were also used. Some researchers based their research on the historical development of the calculus. Other researchers used additional theoretical lenses that include the sociocultural approach, the institutional approach, or the semiotic approach. In the following we relate to these different directions of research.

## The Role of Technology

A key aspect of nearly all the reform projects has been the use of graphics calculators, or computers with graphical software, to help students develop a better intuitive understanding. Since learning calculus includes the analysis of changing quantities, technology has a crucial role in enabling dynamic graphical representations and animations. Technology was first incorporated as support for visualization and coordination between semiotic registers. The possibility of computer magnification of graphs allows the limiting process to be implicit in the computer magnification, rather than explicit in the limit concept. In his plenary paper, Dreyfus (1991) analyzed the powerful role for visual reasoning in learning several mathematical concepts and processes. With the new technologies there was a rapid succession of new ideas for use in calculus and its teaching. Calculus uses numerical calculations, symbolic manipulations, and graphical representations, and the introduction of technology in calculus allows these different registers. Researches on the role of technology in teaching and learning calculus are described, for example, in Artigue (2006); Robert and Speer (2001), and in Ferrera et al. in the 2006 handbook of research on the psychology of mathematics education (pp. 256–266). In the study by Ferrera et al., some researches that relate to using a CAS towards the conceptualization of ***limit*** are described. For example, Kidron and Zehavi use symbolic computation and dynamic graphics to enhance students' ability for passing from visual interpretation of the limit concept to formal reasoning. In this research a sort of balance between the conception of an infinite sum as a process and as an object was supported by the software. The research by Kidron as reported in the study by Ferrera et al. (2006) describes some situations in which the combination of dynamic graphics, algorithms, and historical perspective enabled students to improve their understanding of concepts such as limit, convergence, and the quality of approximation. Most researches offer an analysis of teaching experiments that promote the conceptual understanding of key notions like limits, derivatives, and integral. For example, in a research project by Artigue (2006), the calculator was used towards conceptualization of the notion of **derivative**. One of the aims of the project was to enable grade 11 students to enter the interplay between local and global points of view on functional objects.

Thompson (1994) investigated the concept of rate of change and infinitesimal change which are central to understanding the fundamental theorem of calculus. Thompson's study suggests that students' difficulties with the theorem stem from impoverished concepts of rate of change. In the last two decades, Thompson published several studies which demonstrate that a reconstruction of the ideas of calculus is made possible by its uses of computing technology. The concept of accumulation is central to the idea of integration and therefore is at the core of understanding many ideas and applications in calculus. Thompson et al. (2013) describe a course that approaches introductory calculus with the aim that students build a reflexive relationship between concepts of accumulation and rate of change, symbolize that relationship, and then extend it. In a first phase, students develop accumulation functions from rate of change functions. In the first phase, students "restore" the integral to the fundamental theorem of calculus. In the second phase, students develop rate of change functions from accumulation functions. The main idea is that accumulation and rate of change are never treated separately: the fundamental theorem of calculus is present all the time. Rate is an important, but difficult, mathematical concept. Despite more than 20 years of research, especially with calculus students, difficulties are still reported with this concept.

Tall (2010) reflects on the ongoing development of the teaching and learning calculus since his first thinking about the calculus 35 years ago. During these years, Tall's research described how the computer can be used to show dynamic visual graphics and to offer a remarkable power of numeric and symbolic computation. As a consequence of the cognitive difficulties that accompany the conceptual understanding of the key notions in calculus, Tall's quest is for a "sensible approach" to the calculus which builds on the evidence of our human senses and uses these insights as a meaningful basis for later development from calculus to analysis and even to a logical approach in using infinitesimals. Reflecting on the many years in which reform of calculus teaching has been considered around the world and the different approaches and reform projects using technology, Tall points out that *what has occurred is largely a retention of traditional calculus ideas now supported by dynamic graphics for illustration and symbolic manipulation for computation.*

## The Role of Historical Perspective and Other Approaches

The idea to use a historical perspective in approaching calculus was also demonstrated in other studies not necessarily in a technological environment. Taking into account the long way in which the calculus concepts were developed and then defined, appropriate historically inspired teaching sequences were elaborated.

Recent approaches in learning and teaching calculus refer to the social dimension like the approach to teach calculus called "scientific debate" which is based on a specific form of discussion among students regarding the validity of theorems. The increasing influence taken by sociocultural and anthropological approaches towards learning processes is well expressed in research on learning and teaching calculus. Even the construct concept image and concept definition, which was born in an era where the theories of learning were essentially cognitive theories, was revisited (Bingolbali and Monaghan 2008) and used in interpreting data in a sociocultural study. This was done in a research which investigated students' conceptual development of the derivative with particular reference to rate of change and tangent aspects.

In more recent studies, the role of different theoretical approaches in research on learning calculus is analyzed. Kidron (2008) describes a research process on the conceptualization of the notion of limit by means of the discrete continuous interplay. The paper reflects many years of research on the conceptualization of the notion of limit, and the focus on the complementary role of different theories reflects the evolution of this research.

## The Role of the Teacher

In the previous section, different educational environments were described. Educational environments depend on several factors, including teaching practices. As mentioned by Artigue (2001), reconstructions have been proved to play a crucial role in calculus especially at the secondary/tertiary transition. Some of these reconstructions deal with mathematical objects already familiar to students before the teaching of calculus at university. In some cases, reconstructions result from the fact that only some facets of a mathematical concept can be introduced at the first contact with it. The reconstruction cannot result from a mere presentation of the theory and formal definitions. Research shows that teaching practices underestimate the conceptual difficulties associated with this reconstruction and that teaching cannot leave the responsibility for most of the corresponding reorganization to students.

Research shows that alternative strategies can be developed fruitfully especially with the help of the technology but successful integration of technology at a large-scale level is still a major problem (Artigue 2010). Technology cannot be considered only as a kind of educational assistant. It was demonstrated how it deeply shapes what we learn and the way we learn it.

Artigue points out the importance of the teacher's dimension. Kendal and Stacey (2001) describe teachers' practices in technology-based mathematics lessons. The integration of technology into mathematics teachers' classroom practices is a complex undertaking (Monaghan 2004; Lagrange 2013). Monaghan wrote and cowrote a number of papers in which teachers' activities in using technology in their calculus classrooms were analyzed but there were still difficulties that the teachers had experienced in their practices that were difficult to explain in a satisfactory manner. Investigating the reasons for the discrepancy between the potentialities of technology in learning calculus and the actual uses in the classroom, Lagrange (2013) searches for theoretical frameworks that could help to focus on the teacher using technology; the research on the role of the teacher strengthened the idea of a difficult integration in contrast with research centered on epistemological and cognitive aspects. An activity theory framework seems helpful to give insight on how teachers' activity and professional knowledge evolve during the use of technology in teaching calculus.

## The Transition Between Secondary and Tertiary Education

A detailed analysis of the transition from secondary calculus to university analysis is offered by Thomas et al. (2012). A number of researchers have studied the problems of the learning of calculus in the transition between secondary school and university. Some of these studies focus on the specific topics of real numbers, functions, limits, continuity, and sequences and series. They were located in several countries (Brazil, Canada, Denmark, France, Israel, Tunisia) and use different frameworks. Some have shown that calculus conflicts that emerged from experiments with first-year students could have their roots in a limited understanding of the concept of function, as well as suggesting the need for a more intensive exploration of the dynamical nature of the differential calculus. Results of the survey suggest that there is some room for improvement in school preparation for university study of calculus.

The transition to advanced calculus as taught at the university level has been extensively investigated within the Francophone community, with the research developed displaying a diversity of approaches and themes but a shared vision of the importance to be attached to epistemological and mathematical analyses.

Analyzing the transition between the secondary school and the university, French researchers reflect on approaches to teaching and learning calculus in which the consideration of sociocultural and institutional practices plays an essential role. These approaches offer complementary insights to the understanding of teaching and learning calculus. The theoretical influence of the theory of didactic situations which led to a long-term Francophone tradition of didactical engineering research has been designed in the last decade to support this transition from secondary calculus to university analysis.

## New Directions of Research

New directions of research in teaching and learning calculus were investigated in the last decades. We observe the need for additional theoretical lenses as well as a need to link different theoretical frameworks in the research on learning and teaching calculus. In particular, we observe the need to add additional theoretical dimensions, like the social and cultural dimensions, to the epistemological analyses that were done in the early research. In some cases, we notice the evolution of research during many years with the same researchers facing the challenging questions concerning the cognitive difficulties in learning calculus. The questions are still challenging.

The theoretical dimension is essential for research on calculus teaching and learning, but we should not neglect the practice. As pointed out by Robert and Speer (2001), there are some efforts towards a convergence of theory-driven and practice-driven researches. Further research on how to consider meaningfully theoretical and pragmatic issues is indicated.

As mentioned earlier, reconstructions have been proved to play a crucial role in calculus, essentially these reconstructions that deal with mathematical objects already familiar to students before the teaching of calculus. Further research should underline the important role of teaching practices for such successful reorganization of previous related concepts towards the learning of calculus.

## Cross-References


- ▶ Actions, Processes, Objects, Schemas (APOS) in Mathematics Education
- ▶ Algebra Teaching and Learning
- ▶ Anthropological Approaches in Mathematics Education, French Perspectives
- ▶ Concept Development in Mathematics Education
- ▶ Epistemological Obstacles in Mathematics Education
- ▶ Intuition in Mathematics Education
- ▶ Technology and Curricula in Mathematics Education
- ▶ Technology Design in Mathematics Education
- ▶ Theories of Learning Mathematics
- ▶ Visualization and Learning in Mathematics Education


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## Collaborative Learning in Mathematics Education

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### Keywords

Collaborative learning; Cooperative learning; Project-based learning

Collaborative learning (CL) involves a team of students who learn through working together to share ideas, solve a problem, or accomplish a common goal. In mathematics education, CL's popularity surged in the 1980s, but it has since continued to evolve (Artzt and Newman 1997; Davidson 1990). The terms collaborative/cooperative learning are often used interchangeably, although some claim the former requires giving students considerable autonomy (more appropriate for older students), while the latter is more clearly orchestrated by the teacher (appropriate for all ages) (Panitz 1999).

Three dimensions seem to define collaborative learning (CL) and help distinguish among its many different models: the *structure* of the CL environment (including assessments and rewards), the *teacher and student roles*, and the types of *tasks*.

The CL structure defines how student groups are formed (usually by teacher assignment) and how group members are expected to interact. Research generally recommends mixed ability grouping. Carefully designed assessment and reward structures document student learning and provide incentives for students to work productively together. All models of CL involve group accountability, but some models also include some individual rewards, while others may pit groups against each other in a competitive reward structure.

The teacher's role is to determine the CL structure and task, then serve as facilitator. In some CL models, students are assigned specific group roles (e.g., recorder, calculator); other models require students to tackle portions of the task independently, then pool their efforts toward a common solution. Individual accountability requires that each student be responsible not only for his/her own learning but also for sharing the burden for all group members' learning.

CL tasks must be carefully chosen: amenable to group work and designed so that success depends on contributions from all group members. Particular attention to task difficulty ensures all students can engage at an appropriate level.

CL is grounded in a social constructivist model of learning (Yackel et al. 2011). Some CL models involve peer tutoring (e.g., *Student Team Learning*: Slavin 1994). In the more common investigative CL models (e.g., *Learning Together*: Johnson and Johnson 1998), the emphasis is on learning through problem solving, but higher-order skills such as interpretation, synthesis, or investigation are also required.

Project-based learning (PBL) – a twenty-first-century group-investigation CL model – involves cross-disciplinary, multifaceted, open-ended tasks, usually set in a real-world context, with results presented via oral or written presentation.

PBL tasks often take several weeks because students must grapple with defining, delimiting, and planning the project; conducting research; and determining both the solution and how best to present it (Buck Institute 2012). A stated PBL goal is to help students develop "twenty-first-century skills" relating to collaboration, time management, self-assessment, leadership, and presentation concurrently with engaging in critical thinking and mastering traditional academic concepts and skills (e.g., mathematics).

Research has found student learning is accelerated when students work collaboratively on tasks that are well structured, carefully implemented, and have individual accountability. There is also evidence that affective outcomes, such as interest in school, respect for others, and self-esteem, are also positively impacted (Slavin 1992).

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# Communities of Inquiry in Mathematics Teacher Education

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## Keywords

Mathematics teacher education; Community; Inquiry

## Definition

Mathematics teacher education (MTE) consists of processes and practices through which teachers or student teachers learn to teach mathematics. It involves as participants, primarily, student teachers, teachers, and teacher educators; other stakeholders such as school principals or policy officials with regulatory responsibilities can be involved to differing degrees. Thus a *community* in MTE consists of people who engage in these processes and practices and who have perspectives and knowledge in what it means to learn and to educate in mathematics and an interest in the outcomes of engagement. An *inquiry community*, or *community of inquiry*, in MTE is a community which brings *inquiry* into *practices* of teacher education in mathematics – where inquiry implies questioning and seeking answers to questions, problem solving, exploring, and investigating – and in which inquiry is the basis of an epistemological stance on practice, leading to "metaknowing" (Wells 1999; Jaworski 2006). The very nature of a "community" of inquiry rooted in communities of practice (Wenger 1998) implies a sociohistorical frame in which knowledge grows and learning takes place through participation and dialogue in social settings (Wells 1999).

## Characteristics

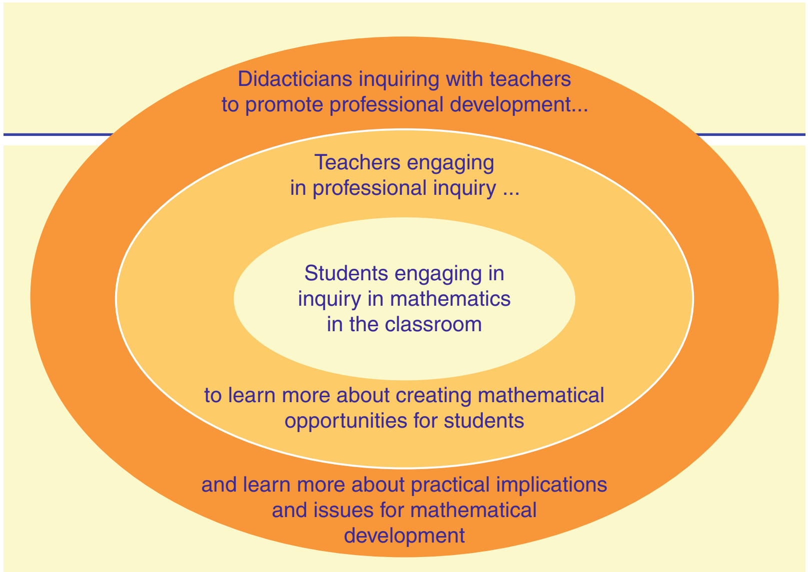
Rather than seeing knowledge as objective, pre-given and immutable (an *absolutist* stance:

Ernest 1991) with learning as a gaining of such knowledge and teaching as a conveyance of knowledge from one who knows to one who learns, an *inquiry stance* sees knowledge as fluid, flexible and *fallible* (Ernest 1991). These positions apply to mathematical knowledge and to knowledge in teaching: teachers of mathematics need both kinds of knowledge. Knowledge is seen variously as formal and external, consisting of general theories and research-based findings to be gained and put into practice; or as craft knowledge, intrinsic to the knower, often tacit, and growing through action, engagement, and experience in practice; or yet again as growing through inquiry in practice so that the knower and the knowledge are inseparable. Cochran-Smith and Lytle (1999) call these three ways of conceptualizing knowledge as knowledge *for* teaching, *in* teaching, and *of* teaching. With regard to knowledge-*of*-teaching, they use the term "inquiry as stance" to describe the positions teachers take towards knowledge and its relationships towards practice. This parallels the notion of "inquiry as a way of being" in which teachers take on the mantle of inquiry as central to how they think, act, and develop in practice and encourage their students to do so as well (Jaworski 2006).

An inquiry community in mathematics teacher education therefore involves teachers (including student teachers who are considered as less experienced teachers) engaging together in inquiry into teaching processes to promote students' learning of mathematics and, moreover, involving students in inquiry in mathematics. The main purpose of inquiry is to call into question aspects of a source (such as mathematics) which encourages a deeper engagement as critical questioning takes place and knowledge grows within the community. When the source is mathematics, inquiry in mathematics allows students to address mathematical questions in ways that seek out answers and lead to new knowledge. Thus mathematics itself becomes accessible, no longer perceived as only right or wrong, and its revealed fallibility is an encouragement to the learner to explore further and understand more deeply. Similarly as teachers explore into aspects of mathematics teaching – for example, the design of inquiry-based mathematical tasks for students – their critical attitude to their practice generates new knowledge in practice and new practice-based understandings (Jaworski 2006).

In a *community of practice*, Wenger (1998) suggests that "belonging" to the community involves "engagement," "imagination," and "alignment." Participants *engage* with the practice, use imagination in weaving a personal trajectory in the practice and *align* with norms and expectations within the practice. The transformation of a community of practice to a community of inquiry requires participant to *look critically* at their practices as they engage with them, to question what they do as they do it, and to explore new elements of practice. Such inquiry-based forms of engagement have been called "critical alignment" (Jaworski 2006). Critical alignment is a necessity for developing an inquiry way of being within a community of inquiry.

Like teachers, teacher educators in mathematics (sometimes called didacticians, due to their practices in relation to the didactics of mathematics) are participants in communities of inquiry in which they too need to develop knowledge in practice through inquiry. Their practices are different from those of teachers, but there are common layers of engagement in which teachers and teacher educators side by side explore practices in learning and teaching of mathematics in order to develop practice and generate new knowledge. Teacher educators also have responsibilities in linking theoretical perspectives to development of practice and to engaging in research formally for generation of academic knowledge. Thus it is possible to see three (nested) layers of inquiry community in generating new understandings of teaching to develop the learning of mathematics: inquiry by students into mathematics in the classroom, inquiry by teachers into the processes and practices of creating mathematical learning in classrooms, and inquiry by teacher educators into the processes by which teachers learn through inquiry and promote the mathematical learning of their students (Jaworski and Wood 2008) (Fig. 1).




**Communities of Inquiry in Mathematics Teacher Education, Fig. 1** Three layers of inquiry in mathematics teaching development

## Cross-References


- ▶ Communities of Practice in Mathematics Education
- ▶ Inquiry-Based Mathematics Education
- ▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes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# Communities of Practice in Mathematics Education


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## Keywords

Ethnomathematics; Informal learning; Situated cognition


## Definition and Originators

Communities of practice (CoP) are an important component of an emerging social theory of learning. Lave and Wenger (1991) originally envisioned this social learning theory as a way to deepen and extend the notion of situated learning that occurs in traditional craft apprenticeships, contexts in which education occurs outside of formal schools. Drawing upon evidence from ethnographic investigations of apprenticeships in a range of settings (e.g., tailoring), they have frequently argued that it is important to separate learning from formal school contexts to understand that most human activities involve some form of teaching and learning. Wenger (1998) argued that CoP's two components (community and practice) are inherently connected by three dimensions: "(1) mutual engagement; (2) a joint enterprise; (3) a shared repertoire" (p. 73). One important aim of a CoP is the negotiation of meaning among participants. This is one way to differentiate groups of people who live or work in the same location from other groups who are actively involved in communicating with each other about important issues and working together towards common goals. Another important aspect of CoP is that learning may be demonstrated by changes in the personal identities of the community members. Changes in identity are accompanied by increasing participation in the valued practices of this particular CoP as newcomers become old-timers in the community.

## Characteristics

### How the Problem Was Identified and Why

Social theories of learning have a long history in psychology (Cole 1996). Nevertheless, more experimental and reductionist theories were the predominant form of psychology until the late twentieth century. The reemergence of social theories of learning has occurred in numerous places, such as discursive psychology (Harré and Gillett 1994), as well as in mathematics education (Lerman 2001; van Oers 2001). The reasons why we need a social learning theory in mathematics education have been outlined by Sfard (1998). She contrasted two key metaphors: learning as acquisition versus learning as participation. Most research conducted during the last century in mathematics education used the acquisition metaphor. In contrast, the participation metaphor shifts the focus from individual ownership of skills or ideas to the notion that learners are fundamentally social beings who live and work as members of communities. Teaching and learning within CoP depend upon social processes (collaboration or expert guidance) as well as social products (e.g., tools, language, laws) in order to help newcomers master the important practices of their community. In addition, social theories of learning are needed to address some of the fundamental quandaries of educational research and practice (Sfard 2008). These enduring dilemmas include the unwillingness of some students to expend enough energy to master difficult mathematical concepts and the puzzling discrepancy in performance on in-school and out-of-school mathematical problems.

### History of Use

Lave's (1988, 2011) own empirical research began with a focus on mathematical proficiency in out-of-school settings (tailoring garments). She initially chose situated cognition tasks that required mathematical computations so that she could more easily compare them with school-like tasks. Similar work in ethnomathematics was conducted by other colleagues for a range of cultural activities (e.g., selling candy on the street) (Nunes et al. 1993). One recurrent finding of this research has been that children, adolescents, and adults can demonstrate higher levels of mathematical proficiency in their out-of-school activities than in school, even when the actual mathematical computations are the same (Forman 2003). Another finding was that social processes (e.g., guided participation) and cultural tools (e.g., currency) were important resources for people as they solved mathematical problems outside of school (Saxe 1991). This research forces one to question the validity of formal assessments of mathematical proficiency and to wonder how mathematical concepts and procedures are developed in everyday contexts of work and play. Many of these investigators began to question the basic assumptions of our individual learning theories and turn their attention to developing new social theories of learning, like those proposed by Lave, Wenger, Saxe, and Sfard. It also led them to rely, to an increasing degree on ethnographic studies of everyday life (Lave 2011).

When Lave and others began their research in the early 1970s, it had limited impact on research in schools. This has changed as mathematics educators have begun to use this research to improve school instruction, curriculum development, and teacher education (e.g., Nasir and Cobb 2007). For example, Cobb and Hodge (2002) use the notion of CoP to propose that we investigate at least two types of mathematical communities in our work: local (home, school, neighborhood) and broader (district, state, national, international). In the classroom, all of these types of communities affect students' access to high levels of mathematical reasoning and problem solving as well as their own sense of identity as capable mathematical learners. Cobb and Hodge suggest that we try to understand and acknowledge these discontinuities in students' communities that can impact their motivation, self-image, and school achievement. Building, in part, on the work of Luis Moll and his colleagues (Gonzalez et al. 2004), Cobb and Hodge recommend that teachers view their students' families and neighborhood communities as sources of important funds of knowledge that can be accessed in school. By viewing home proficiencies in a positive light (as funds of knowledge) instead of deficits (limited formal education), these educators propose that the discontinuities between classroom and home CoP be minimized and destigmatized, making it easier for students to engage more fully in learning about mathematics and other subjects.

Unfortunately, many students from impoverished neighborhoods experience discontinuities between their home and peer communities and those of schools. As a result, researchers such as Nasir (2007) have tried to forge new connections between students' out-of-school lives (e.g., when they use mathematics to win at basketball) and the mathematics they are taught in school. One key component of these connections is the need to engage students in the process of viewing themselves as capable learners of mathematics and establishing a new identity as successful mathematics problem solvers (Martin 2007). Finally, Cobb and Hodge (2002) remind us that resolving the tensions between local communities of practice for students may also involve acts of imagination when young people work towards a revised and expanded identity for themselves in the future.

## Perspectives on Issue in Different Cultures/Places

The origin of CoP in ethnomathematics means that the earliest research was conducted in cultural settings very different from those of European or North American classrooms such as Brazil, Liberia, Mexico, and Papua New Guinea. In addition to a broad range of national settings, this ethnographic work focused on the mathematical reasoning that occurred in the daily lives of poor and middle-class people who may or may not be enrolled in formal educational institutions. After it began to be applied to school classrooms, many of its research sites were located in Europe or North America (e.g., Seeger et al. 1998). Thus, unlike many educational innovations, the study of CoP began in impoverished locations and later spread to wealthy settings. In addition, the methods of ethnography previously used to study the work and play lives of people in impoverished communities were then applied to classrooms where experimental methods are more often used (Lave 2011).

## Gaps that Need to Be Filled

In Wenger's (1998) expanded formulation of CoP, he clarified its dynamic properties. Forms of mutual engagement change over time within any community. Collective goals evolve as different interpretations clash and new understandings are negotiated. As this occurs, new tools are created and modified, new vocabulary developed, and new routines and narratives invented. Only recently have ethnographers like Lave and Saxe been able to take a long view of the CoP they originally studied in the 1970s and 1980s. As a result they have been able to deepen their theoretical positions and reconceptualize their methods. Clearly this is one area where the theoretical formulations of CoP are ahead of the empirical results in mathematics classrooms.

Another area that has been growing in mathematics education is the study of identity in mathematics classrooms (e.g., Martin 2007). Although this work is also hampered by the limited time span studied, early work suggests that repeated experiences with different classroom CoP may result in negative or positive reactions to their instructional experiences in mathematics by students (Boaler and Greeno 2000). The development of a person's mathematical identity may build slowly over time in homes, communities, and schools through recurrent processes such as social positioning by parents, teachers, and peers (Yamakawa et al. 2009).

Finally, an area of rich growth in mathematics education is the attempt to align classroom communities with those of professional mathematicians (Lampert 1990). For example, studies of argumentation and proof in K-12 classrooms indicate that even young children are capable of articulating their reasons for their mathematical decisions and defending those positions when carefully guided by an experienced teacher (O'Connor 2001). In addition, current research in other content areas such as history or science shows that argumentation can be productively fostered across the curriculum as well as in different grades.

## Cross-References

► Ethnomathematics
► Situated Cognition in Mathematics Education

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# Communities of Practice in Mathematics Teacher Education

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## Keywords

Mathematics teacher education; Communities of practice; Identity

## Definition

**Communities of practice in mathematics teacher education** are informed by a theory of learning as social participation, in which teacher learning and development are conceptualized as increasing participation in social practices that develop an identity as a teacher.

## Background

The idea of learning in a community of practice grew from Jean Lave's and Etienne Wenger's research on learning in apprenticeship contexts (Lave 1988; Lave and Wenger 1991). Drawing on their ethnographic observations of apprentices learning different trades, Lave and Wenger developed a theory of learning as social practice to describe how novices come to participate in the practices of a community. These researchers introduced the term "legitimate peripheral participation" to explain how apprentices, as newcomers, are gradually included in the community through modified forms of participation that are accessible to potential members working alongside master practitioners. Although social practice theory aimed to offer a perspective on learning in out-of-school settings, Lave (1996) afterwards argued that apprenticeship research also has implications for both learning and teaching in schools and for students and teachers as participants in social practices that shape identities.

To further analyze the concepts of identity and community of practice, Wenger (1998) proposed a more elaborated social theory of learning that integrates four components – meaning (learning as experience), practice (learning as doing), community (learning as belonging), and identity (learning as becoming). Wenger explained that communities of practice are everywhere – in people's workplaces, families, and leisure pursuits, as well as in educational institutions. Most people belong to multiple communities of practice at any one time and will be members of different communities throughout their lives. His theory has been applied to organizational learning as well as learning in schools and other formal educational settings.

## Communities of Practice as a Framework for Understanding Mathematics Teacher Learning and Development

Social theories of learning are now well established in research on mathematics education. Lerman (2000) discussed the development of "the social turn" in mathematics education research and proposed that social theories drawing on community of practice models provide insights into the complexities of teacher learning and development. From this perspective, learning to teach involves developing an identity as a teacher through increasing participation in the practices of a professional community (Lerman 2001). At the time of publication of Lerman's (2001) review chapter on research perspectives on mathematics teacher education, there were few studies drawing on Lave's and Wenger's ideas. Reviewing the same field 5 years later, Llinares and Krainer (2006) noted increasing interest in using the idea of a community of practice to conceptualize learning to teach mathematics. Such studies can be classified along several dimensions, according to their focus on:

1. Preservice teacher education or the professional learning and development of practicing teachers
2. Face-to-face or online interaction (or a combination of both)
3. Questions about how a community of practice is formed and sustained compared with questions about the effectiveness of communities of practice in promoting teacher learning

Research has been informed by the two key conceptual strands of Wenger's (1998) social practice theory. One of these strands is related to the idea of learning as increasing *participation in socially situated practices* and the other to learning as developing an *identity* in the context of a community of practice.

### Learning as Participation in Practices

With regard to *participation in practices*, Wenger describes three dimensions that give coherence to communities of practice: mutual engagement of participants, negotiation of a joint enterprise, and development of a shared repertoire of resources for creating meaning. Mutuality of engagement need not require homogeneity, since productive relationships arise from diversity and these may involve tensions, disagreements, and conflicts. Participants negotiate a joint enterprise, finding ways to do things together that coordinate their complementary expertise. This negotiation gives rise to regimes of mutual accountability that regulate participation, whereby members work out who is responsible for what and to whom, what is important and what can safely be ignored, and how to act and speak appropriately. The joint enterprise is linked to the larger social system in which the community is nested. Such communities have a common cultural and historical heritage, and it is through the sharing and reconstruction of this repertoire of resources that individuals come to define their relationships with each other in the context of the community.

This aspect of Wenger's theory has been used to investigate discontinuities that may be experienced in learning to teach mathematics in the different contexts in which prospective and beginning teachers' learning occurs – the university teacher education program, the practicum, and the early years of professional experience (Llinares and Krainer 2006). One of the more common discontinuities is evident in the difficulty many beginning teachers experience in sustaining the innovative practices they learn about in their university courses. This observation can be explained by acknowledging that prospective and beginning teachers participate in separate communities – one based in the university and the other in school – which often have different regimes of accountability that regulate what counts as "good teaching."

Researchers have also investigated how participation in online communities of practice supports the learning of prospective and practicing teachers of mathematics, and insights into principles informing the design of such communities are beginning to emerge (Goos and Geiger 2012). Some caution is needed in interpreting the findings of these studies, since few present evidence that a community of practice has actually been formed: for example, by analyzing the extent of mutual engagement, how a joint enterprise is negotiated, and whether a shared repertoire of meaning-making resources is developed by participants (Goos and Bennison 2008). Nevertheless, studies of online communities of practice demonstrate that technology-mediated collaboration does more than simply increase the amount of knowledge produced by teachers; it also leads to qualitatively different forms of knowledge and different relationships between participants.

### Learning as Developing an Identity

With regard to *identity development*, Wenger wrote of different modes of belonging to a community of practice through engagement, imagination, and alignment. Beyond actually engaging in practice, people can extrapolate from their experience to imagine new possibilities for the self and the social world. Alignment, the third mode of belonging, refers to coordinating one's practices to contribute to the larger enterprise or social system. Alignment can amplify the effects of a practice and increase the scale of belonging experienced by community members, but it can also reinforce normative expectations of practice that leave people powerless to negotiate identities.

Research into teacher *identity* development in communities of practice is perhaps less advanced than studies that analyze evidence of changing participation in the *practices* of a community. This may be due to a lack of well-developed theories of identity that can inform research designs and provide convincing evidence that identities have changed. Jaworski's (2006) work on identity formation in mathematics teacher education proposes a conceptual shift from learning within a community of practice to forming a community of inquiry. The distinguishing characteristic of a community of inquiry is reflexivity, in that participants critically reflect on the activities of the community in developing and reconstructing their practice. This requires a mode of belonging that Jaworski calls "critical alignment" – adopting a critically questioning stance in order to avoid perpetuating undesirable normative states of activity.

## Issues for Future Research

Elements of Wenger's social practice theory resonate with current ways of understanding teachers' learning, and this may explain why his ideas have been taken up so readily by researchers in mathematics teacher education. Nevertheless, the notion of situated learning in a community of practice composed of experts and novices was not originally focused on school classrooms, nor on pedagogy, and so caution is needed in applying this perspective on learning as an informal and tacit process to learning in formal education settings, including preservice and in-service teacher education (Graven and Lerman 2003). Wenger's model was developed from studying learning in apprenticeship contexts, where teaching is incidental rather than deliberate and planned, as in university-based teacher education. It remains to be seen whether community of practice approaches can be applied to understand the role of teacher educators in shaping teachers' learning.

## Cross-References

▶ Communities of Inquiry in Mathematics Teacher Education
▶ Communities of Practice in Mathematics Education
▶ Mathematics Teacher as Learner
▶ Mathematics Teacher Identity
▶ Professional Learning Communities in Mathematics Education

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## Competency Frameworks in Mathematics Education

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### Keywords

Competence; Conceptual framework; Taxonomy; Subject matter; Mental process

### Definition

A structural plan for organizing the cognitive skills and abilities used in learning and doing mathematics.

### Characteristics

The concept of competence is one of the most elusive in the educational literature. Writers often use the term *competence* or *competency* and assume they and their readers know what it means. But arriving at a simple definition is a challenging matter. Dictionaries give such definitions as "the state or quality of being adequately or well qualified"; "the ability to do something successfully or efficiently"; "possession of required skill, knowledge, qualification, or capacity"; "a specific range of skill, knowledge, or ability"; and "the scope of a person's or group's knowledge or ability." *Competence* seems to possess a host of near synonyms: *ability, capability, cognizance, effectuality, efficacy, efficiency, knowledge, mastery, proficiency, skill, and talent* – the list goes on.

Arriving at a common denotation across different usages in social science is even more difficult. "There are many different theoretical approaches, but no single conceptual framework" (Weinert 2001, p. 46). Weinert identifies seven different ways that "competence has been defined, described, or interpreted theoretically" (p. 46). They are as follows: general cognitive competencies, specialized cognitive competencies, the competence-performance model, modifications of the competence-performance model, cognitive competencies and motivational action tendencies, objective and subjective competence concepts, and action competence. Competency frameworks in mathematics education fall primarily into Weinert's specialized-cognitive-competencies category, but they also overlap some of the other categories.

The progenitor of competency frameworks in mathematics education is Bloom's (1956) *Taxonomy of Educational Objectives*, which attempted to lay out, in a neutral way, the cognitive goals of any school subject. The main categories were *knowledge, comprehension, application, analysis, synthesis*, and *evaluation*. These categories were criticized by mathematics educators such as Hans Freudenthal and Chris Ormell as being especially ill suited to the subject of mathematics (see Kilpatrick 1993 on the critiques as well as some antecedents of Bloom's work). Various alternative taxonomies have subsequently been proposed for school mathematics (see Tristán and Molgado 2006, pp. 163–169, for examples). Further, Bloom's taxonomy has been revised (Anderson and Krathwohl 2001) to separate the knowledge dimension (*factual, conceptual, procedural*, and *metacognitive*) from the cognitive process dimension (*remember, understand, apply, analyze, evaluate*, and *create*), which does address one of the complaints of mathematics educators that the original taxonomy neglected content in favor of process. But the revision nonetheless fails to address such criticisms as the isolation of objectives from any context, the low placement of understanding in the hierarchy of processes, and the failure to address important mathematical processes such as representing, conjecturing, and proving.

Whether organized as a taxonomy, with an explicit ordering of categories, or simply as an arbitrary listing of topics, a competency framework for mathematics may include a breakdown of the subject along with the mental processes used to address the subject, or it may simply treat those processes alone, leaving the mathematical content unanalyzed. An example of the former is the model of outcomes for secondary school mathematics proposed by James Wilson (cited by Tristán and Molgado 2006, p. 165). In that model, mathematical content is divided into *number systems, algebra*, and *geometry*; cognitive behaviors are divided into *computation, comprehension, application, and analysis*; and affective behaviors are either *interests and attitudes* or appreciation. Another example is provided by the framework proposed for the Third International Mathematics and Science Study (TIMSS; Robitaille et al., 1933, Appendix A). The main content categories are *numbers; measurement; geometry* (position, visualization, and shape; symmetry, congruence, and similarity); *proportionality; functions, relations, and equations; data representation, probability, and statistics; elementary analysis; validation and structure;* and *other content* (informatics). The performance expectations are *knowing, using routine procedures, investigating and problem solving, mathematical reasoning*, and *communicating*.

Other competency frameworks, like that of Bloom's (1956) taxonomy, do not treat different aspects of mathematical content separately but instead attend primarily to the mental processes used to do mathematics, whether the results of those processes are termed *abilities, achievements, activities, behaviors, performances, practices, proficiencies*, or *skills*. Examples include the five strands of mathematical proficiency identified by the Mathematics Learning Study of the US National Research Council – *conceptual understanding, procedural fluency, strategic competence, adaptive reasoning*, and *productive disposition* – and the five components of mathematical problem-solving ability identified in the Singapore mathematics framework: *concepts, skills, processes, attitudes*, and metacognition (see Kilpatrick 2009, for details of these frameworks).

A final example of a competency framework in mathematics is provided by the KOM project (Niss 2003), which was charged with spearheading the reform of mathematics in the Danish education system. The KOM project committee addressed the following question: What does it mean to master mathematics? They identified eight competencies, which fell into two groups. The first four address the ability to ask and answer questions in and with mathematics:

1. Thinking mathematically
2. Posing and solving mathematical problems
3. Modeling mathematically
4. Reasoning mathematically

The second four address the ability to deal with and manage mathematical language and tools:

5. Representing mathematical entities
6. Handling mathematical symbols and formalisms
7. Communicating in, with, and about mathematics
8. Making use of aids and tools

Niss (2003) observes that each of these competencies has both an analytic and a productive side. The analytic side involves understanding and examining the mathematics, whereas the productive side involves carrying it out. Each competency can be developed and used only by dealing with specific subject matter, but the choice of curriculum topics is not thereby determined. The competencies, though specific to mathematics, cut across the subject and can be addressed in multiple ways.

The KOM project also found it necessary to focus on mathematics as a discipline. The project committee identified three kinds of "overview and judgment" that students should develop through their study of mathematics: its actual application, its historical development, and its special nature. Like the competencies, these qualities are both specific to mathematics and general in scope.

Niss (2003) observes that the competencies and the three kinds of overview and judgment can be used: (a) normatively, to set outcomes for school mathematics; (b) descriptively, to characterize mathematics teaching and learning; and (c) metacognitively, to help teachers and students monitor and control what they are teaching or learning. These three usages apply as well to the other competency frameworks developed for mathematics.

Regardless of whether a competency framework is hierarchical and regardless of whether it addresses topic areas in mathematics, its primary use will be normative. Competency frameworks are designed to demonstrate to the user that learning mathematics is more than acquiring an array of facts and that doing mathematics is more than carrying out well-rehearsed procedures. School mathematics is sometimes portrayed as a simple contest between knowledge and skill. Competency frameworks attempt to shift that portrayal to a more nuanced portrait of a field in which a variety of competences need to be developed.

## Cross-References

- ▶ Bloom's Taxonomy in Mathematics Education
- ▶ Competency Frameworks in Mathematics Education
- ▶ Frameworks for Conceptualizing Mathematics Teacher Knowledge
- ▶ International Comparative Studies in Mathematics: An Overview

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# Complexity in Mathematics Education


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## Keywords

Complexity science; Adaptive learning systems; Collective knowledge-producing systems


## Definition

Over the past half century, "complex systems" perspectives have risen to prominence across many academic domains in both the sciences and the humanities. Mathematics was among the originating domains of complexity research. Education has been a relative latecomer, and so perhaps not surprisingly, mathematics education researchers have been leading the way in the field.

There is no unified definition of complexity, principally because formulations emerge from the study of specific phenomena. One thus finds quite focused definitions in such fields as mathematics and software engineering, more indistinct meanings in chemistry and biology, and quite flexible interpretations in the social sciences (cf. Mitchell 2009). Because mathematics education reaches across several domains, conceptions of complexity within the field vary from the precise to the vague, depending on how and where the notion is taken up.

Diverse interpretations do collect around a few key qualities, however. In particular, *complex* systems adapt and are thus distinguishable from *complicated*, mechanical systems. A complicated system is one that comprises many interacting components and whose global character can be adequately described and predicted by applying laws of physics. A complex system comprises many interacting agents – and those agents, in turn, may comprise many interacting subagents – presenting the possibility of global behaviors that are rooted in but that cannot be reduced to the actions or qualities of the constituting agents. In other words, a complex system is better described by using Darwinian principles than Newtonian ones. It is thus that each complex phenomenon must be studied in its own right. For each complex unity, new laws emerge that cannot be anticipated or explained strictly by reference to prior, subsequent, or similar systems. Popularly cited examples include anthills, economies, and brains, which are more than the collected activities of ants, consumers, and neurons. In brief, whereas the opposite of complicated is simple, opposites of complex include reducible and decomposable. Hence, prominent efforts toward a coherent, unified description of complexity revolve around such terms as *emergent*, *noncompressible*, *self-organizing*, *context-sensitive*, and *adaptive*.

The balance of this discussion is organized around four categories of usage within mathematics education – namely, complexity as a theoretical discourse, a historical discourse, a disciplinary discourse, and a pragmatic discourse.

## Characteristics

### Complexity as a Theoretical Discourse

Among educationists interested in complexity, there is frequent resonance with the notions that a complex system is one that *knows* (i.e., perceives, acts, engages, and develops) and/or *learns* (adapts, evolves, maintains self-coherence, etc.). This interpretation reaches across many systems that are of interest among educators, including physiological, personal, social, institutional, epistemological, cultural, and ecological systems. Unfolding from and enfolding in one another, it is impossible to study one of these phenomena without studying all of the others.

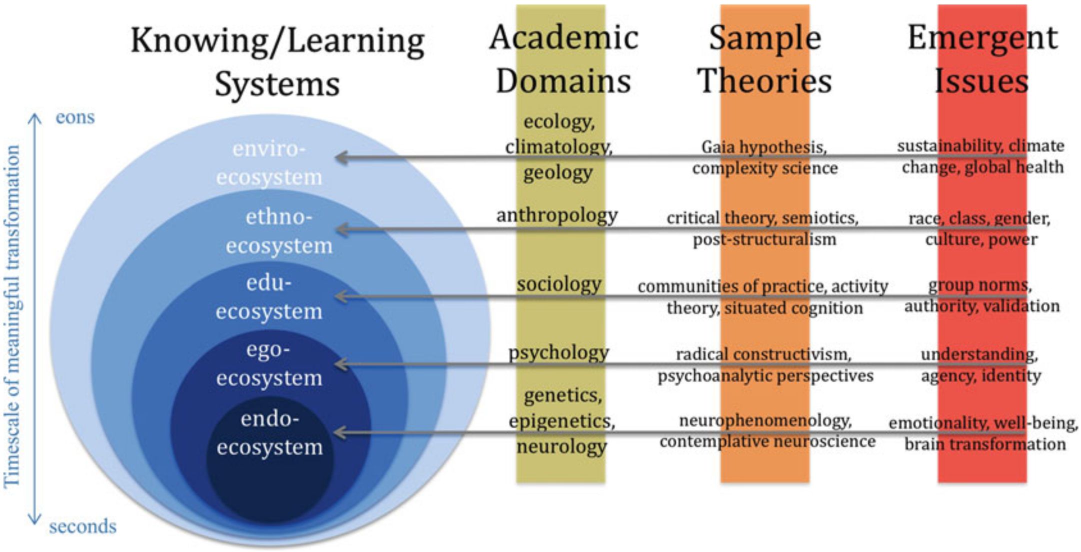
This is a sensibility that has been well represented in the mathematics education research literature for decades in the form of varied theories of learning. Among others, radical constructivism, sociocultural theories of learning, embodied, and critical theories can all be read as instances of complexity theories. That is, they all invoke bodily metaphors, systemic concerns, evolutionary dynamics, emergent possibility, and self-maintaining properties.

This is not to assert some manner of hidden uniformity to the theories just mentioned. On the contrary, much of their value is to be found in their diversity. As illustrated in Fig. 1, when learning phenomena of interest to mathematics educators are understood as nested systems, a range of theories become necessary to grapple with the many issues the field must address. Sophisticated and effective mathematics pedagogies demand similarly sophisticated insights into the complex dynamics of knowing and learning. More significantly, perhaps, by introducing the time frames of meaningful systemic transformation into discussions of individual knowing and collective knowledge, complexity not only enables but compels a consideration of the manners in which knowers and systems of knowledge are co-implicated (Davis and Simmt 2006).

### Complexity as a Historical Discourse

School mathematics curricula are commonly presented as a-historical and a-cultural. Contra this perception, complexity research offers an instance of emergent mathematics that has arisen and that is evolving in a readily perceptible time frame. As an example of what it describes – a self-organizing, emergent coherence – complexity offers a site to study and interrogate the nature of mathematics, interrupting assumptions of fixed and received knowledge.

To elaborate, the study of complexity in mathematics reaches back the late nineteenth century when Poincaré conjectured about the three-body problem in mechanics. Working qualitatively, from intuition Poincaré recognized the problem of thinking about complex systems with the




**Complexity in Mathematics Education, Fig. 1** Some of the nested complex systems of interest to mathematics educators

assumptions and mathematics of linearity (Bell 1937). The computational power of mathematics was limited to the calculus of the time; however, enabled by digital technologies of the second half of the twentieth century, such problems became tractable and the investigation of dynamical systems began to flourish.

With computers, experimental mathematics was born, and the study of dynamical systems led to new areas in mathematics. For the first time, it was easy and quick to consider the behavior of a function over time by computing thousands and hundreds of thousands of iterations of the function. Numerical results were readily converted into graphical representations (the Lorenz attractor, Julia sets, bifurcation diagrams) which in turn inspired a new generation of mathematicians, scientists, and human scientists to think differently about complex dynamical systems.

The mid-twentieth century brought about great insights into features of dynamical systems that had been overlooked. As mathematicians and physical and computer scientists were exploring dynamical systems (e.g., Smale, Prigogine, Lorenz, Holland), their work and the work of biologists began to intersect. Emerging out of that activity were interdisciplinary workshops, conferences, and think tanks such as the Santa Fe Institute, a research center dedicated to all matters of complexity science.

In brief, the emergence of complexity as a field of study foregrounds that mathematics might be productively viewed as a humanity. More provocatively, the emergence of a mathematics of implicatedness and entanglement alongside the rise of a more sophisticated understanding of humanity's relationship to the more-than-human world might be taken as an indication of the ecological character of mathematics knowledge.

## Complexity as a Disciplinary Discourse

A common criticism of contemporary grade school mathematics curriculum is that little of its content is reflective of mathematics developed after the sixteenth or seventeenth centuries, when publicly funded and mandatory education spread across Europe. A deeper criticism is that the mathematics included in most preuniversity curricula is fitted to a particular worldview of cause-effect and linear relationships. Both these concerns might be addressed by incorporating complexity-based content into programs of study.

Linear mathematics held sway at the time of the emergence of the modern school – that is, during the Scientific and Industrial Revolutions – because it lent itself to calculations that could be done by hand. Put differently, linear mathematics was first championed and taught for pragmatic reasons, not because it was seen to offer accurate depictions of reality. Descartes, Newton, and their contemporaries were well aware of nonlinear phenomena. However, because of the intractability of many nonlinear calculations, when they arose they were routinely replaced by linear approximations. As textbooks omitted nonlinear accounts, generations of students were exposed to over-simplified, linearized versions of natural phenomena. Ultimately that exposure contributed to a resilient worldview of a clockwork reality. However, the advent of powerful computing technologies over the past half century has helped to restore an appreciation of the relentless nonlinearity of the universe. That is, the power of digital technologies have not just opened up new vistas of calculation, they have triggered epistemic shifts as they contribute to redefinitions of what counts as possible and what is expressible (Hoyles and Ness 2008, p. 89).

With the ready access to similar technologies in most school classrooms within a culture of ubiquitous computation, some (e.g., Jacobson and Wilensky 2006) have advocated for the inclusion of such topics as computer-based modeling and simulation languages, including networked collaborative simulations (see Kaput Center for Research and Innovation in STEM Education). In this vein, complexity is understood as a digitally enabled, modeling-based branch of mathematics, creating space in secondary and tertiary education for new themes such as recursive functions, fractal geometry, and modeling of complex phenomena with mathematical tools such as iteration, cobwebbing, and phase diagrams. Others (e.g., English 2006; Lesh and Doerr 2003) have advocated for similarly themed content, but in a less calculation-dependent format, arguing that the shift in sensibility from linearity to complexity is more important than the development of the computational competencies necessary for sophisticated modeling. In either case, the imperative is to provide learners with access to the tools of complexity, along with its affiliated domains of fractal geometry, chaos theory, and dynamic modeling.

New curriculum in mathematics is emerging. More profoundly, when, how, who, and where we teach are also being impacted by the presence of complexity sensibilities in education because they are a means to nurture emergent possibility.

## Complexity as a Pragmatic Discourse

To recap, complexity has emerged in education as a set of mathematical tools for analyzing phenomena, as a theoretical frame for interpreting activity of adaptive and emergent systems, as a new sensibility for orienting oneself to the world, and for considering the conditions for emergent possibilities leading to more productive, "intelligent" classrooms.

In the last of these roles, complexity might be regarded as the pragmatic discourse – and of the applications of complexity discussed here, this one may have the most potential for affecting school mathematics by offering guidance for structuring learning contexts. In particular, complexity offers direct advice for organizing classrooms to support the individual-and-collective generation of insight – by, for example, nurturing the common experiences and other redundancies of learners while making space for specialist roles, varied interpretations, and other diversities. Another strategy is to utilize digital technologies that offer environments for (and systemic memories of) assembling interpretations, strategies, solutions, evaluations, and judgments.

Mobile digital technologies have occasioned other kinds of learning opportunities that lend themselves to both the sorts of analysis and the sorts of advice afforded by complexity research. For example, within platforms such as wikis, Twitter, and Facebook, students can organize along language and interest lines while they connect with and elaborate the contributions of their peers. In a more extreme frame, the emergence of massive online open-learning courses (MOOCs) represents an interesting new example of the impact a complexity sensibility can have in the educational context as they invite large numbers of participants to engage with the thinking of experts. It is not without irony that, within a complexity frame, even the most denigrated of teaching strategies – the large lecture – can serve as a critical part of a vibrant, knowledge-producing system when coupled to connectivity, playback, feedback, and other aspects of a digital environment.

As complexity becomes more prominent in educational discourses and entrenches in the infrastructure of "classrooms," mathematics education can move from a culture of cooperation to one of collaboration, and that has entailments for the outcomes of schooling – articulated in, e.g., movements from generalist preparation to specialist expertise, from independent workers to team-based workplaces, and from individual knowing to social action.

## Cross-References

- ▶ Activity Theory in Mathematics Education
- ▶ Calculus Teaching and Learning
- ▶ Collaborative Learning in Mathematics Education
- ▶ Critical Mathematics Education
- ▶ Critical Thinking in Mathematics Education
- ▶ Enactivist Theories
- ▶ Functions Learning and Teaching
- ▶ Gender in Mathematics Education
- ▶ Information and Communication Technology (ICT) Affordances in Mathematics Education
- ▶ Metaphors in Mathematics Education
- ▶ Mathematical Modelling and Applications in Education
- ▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes
- ▶ Situated Cognition in Mathematics Education
- ▶ Theories of Learning Mathematics
- ▶ Types of Technology in Mathematics Education
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# Concept Development in Mathematics Education

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## Keywords

Notion; Concept; Concept formation in babies; Concrete object; Similarities; Generalization; Ostensive definitions; Mathematical definitions; Intuitive; Concept image; Concept definition; Stereotypical examples; System 1 and system 2; Pseudo-analytical; Pseudo-conceptual; Mathematical objects; Mathematical mind

## Characteristics

Concept formation and development in general is an extremely complicated topic in cognitive psychology. There exists a huge literature about it, classical and current. Among the classical works on it, one can mention for instance, Piaget and Inhelder (1958) and Vygotsky (1986). However, this issue is restricted to concept formation and development in mathematics. Nevertheless, it is suggested not to isolate mathematical concept formation and development from concept formation and development in general.

One terminological clarification should be made before the main discussion. When dealing with concepts, very often, also the term "notion" is involved. A *notion* is a lingual entity – a word, a word combination (written or pronounced); it can also be a symbol. A *concept* is the meaning associated in our mind with a notion. It is an idea in our mind. Thus, *a notion is a concept name.* There might be concepts without names and for sure there are meaningless notions, but discussing them requires subtleties which are absolutely irrelevant to this context. In many discussions people do not bother to distinguish between notions and concepts, and thus the word "notion" becomes ambiguous. The ambiguity is easily resolved by the context.

As recommended above, it will be more useful not to disconnect mathematical concept formation from concept formation in general, and therefore, let us start our discussion with an example of concept formation in babies. How do we teach them, for instance, the concept of chair? The common practice is to point at various chairs in various contexts and to say "chair." Amazingly enough, after some repetitions, the babies understand that the word "chair" is supposed to be related to chairs, which occur to them in their daily experience, and when being asked "what is this?" they understand that they are supposed to say "chair." Later on, they will imitate the entire ritual on their own initiative. They will point at chairs and say "chair." I would like to make a theoretical claim here by saying that, *seemingly*, they have constructed in their mind the class of all possible chairs. Namely, a concept is formed in their mind, and whenever a concrete object is presented to them, they will be able to decide whether it is a chair or not. Of course, some mistakes can occur in that concept formation process. It is because in this process, two *cognitive mechanisms* are involved. The first mechanism is the one that identifies similarities. The mind distinguishes that one particular chair presented to the baby is similar to some particular chairs presented to her or him in the past. The second mechanism is the one which *distinguishes differences*. The mind distinguishes that a certain object is not similar to the chairs which were presented to the baby in the past, and therefore, the baby is not supposed to say "chair" when an object that is not a chair is presented to him or her by the adult. Mistakes about the acquired concept might occur because of two reasons. An object, which is not a chair (say a small table), appears to the baby (or even to an adult) like a chair. In this case, the object will be considered as an element of the class of all chairs while, in fact, it is not an element of this class. The second reason for mistakes is that an object that is really a chair will not be identified as a chair because of its weird shape. Thus, an object which was supposed to be an element of the class is excluded from it. More examples of this type are the following: sometimes, babies consider dogs as cats and vice versa. These are intelligent mistakes because there are some similarities between dogs and cats. They are both animals; sometimes they even have similar size (in the case of small dogs) and so on.

The above process which leads, in our mind, to the construction of the set of all possible objects to which the concept name can be applied is a kind of *generalization*. Thus, generalizations are involved in the formation of any given concept. Therefore, concepts can be considered as generalizations.

The actions by means of which we try to teach children concepts of chair are called *ostensive definitions*. Of course, only narrow class of concepts can be acquired by means of ostensive definitions. Other concepts are acquired by means of *explanations* which can be considered at this stage as definitions. Among these concepts I can point, for instance, at a forest, a school, work, hunger and so on. When I say definitions at this stage, I do not mean definitions which are similar, or even seemingly similar to rigorous mathematical definitions. The only restriction on these definitions is that familiar concepts will be used in order to explain a non-familiar concept. Otherwise, the explanation is useless. (This restriction, by the way, holds also for mathematical definitions, where new concepts are defined by means of previously defined concepts or by primary concepts.) In definitions which we use in non-technical context in order to teach concepts, we can use examples. For instance, in order to define furniture, we can say: A chair is furniture, a bed is furniture, and tables, desks, and couches are furniture.

The description which was just given deals with the primary stage of concept formation. However, concept formation in ordinary language is by far more complicated and very often, contrary to the mathematical language, ends up in a vague notion. Take, for instance, again, the notion of furniture. The child, when facing an object which was not previously introduced to him or to her as furniture, should decide whether this object is furniture or not. He or she may face difficulties doing it. Also adults might have similar difficulties. This is only one example out of many which demonstrates the complexity of concept formation in the child's mind as well as in the adult's mind. There are even greater complexities when concept formation of abstract nouns, adjectives, verbs, and adverbs is involved. Nevertheless, despite that complexity, the majority of children acquire language at an impressive level by the age of six (an elementary level is acquired already at the age of three). The cognitive processes associated with the child's acquisition of language are discussed in details in *cognitive psychology, linguistics, and philosophy of language*. One illuminating source which is relevant to this issue is Quine's (1964) "Word and object." However, a detailed discussion of these processes is not within the scope of this issue.

In addition to the language acquisition, the child acquires also broad knowledge about the world. He or she knows that when it rains, it is cloudy, they know that dogs bark and so on and so forth. In short, they know infinitely many other facts about their environment. And again, it is obtained in a miraculous way, smoothly without any apparent difficulties. Things, however, become awkward when it gets to mathematics. One possible reason for things becoming awkward in mathematics is that, in many cases, *mathematical thinking is essentially different from the natural intuitive mode of thinking* according to which the child's intellectual development takes place. The major problem is that mathematical thinking is shaped by rigorous rules, and in order to think mathematically, children, as well as adults, should be aware of these rules while thinking in mathematical contexts. One crucial difficulty in mathematical thinking is that *mathematical concepts are strictly determined by their definitions*. In the course of their mathematical studies, children, quite often, are presented to mathematical notions with which they were familiar from their past experience. For instance, in Kindergarten they are shown some geometrical figures such as squares and rectangles. The adjacent sides of the rectangle which are shown to the children in Kindergarten have always different length. Thus, the set of all possible rectangles which is constructed in the child's mind includes only rectangles, the adjacent sides of which have different length. In the third grade, in many countries, a definition of a rectangle is presented to the child. It is a quadrangle which has 4 right angles. According to this definition, *a square is also a rectangle*. Thus, a *conflict* may be formed in the child's mind between the suggested definition and the concept he or she already has about rectangles. The concept the child has in mind was formed by the set of examples and the properties of these examples which were presented to the child. It was suggested (Vinner 1983) to call it the *concept image* of that notion. Thus, in the above case of the rectangle, there is a conflict between the *concept image* and the *concept definition*. On the other hand, quite often some concepts are introduced to the learner by means of formal definitions. For instance, an altitude in a triangle. However, a formal definition, generally, remains meaningless unless it is associated with some examples. The examples can be given by a teacher or by a textbook, or they can be formed by the learners themselves. The first examples which are associated with the concept have a crucial impact on the concept image. Unfortunately, quite often, in mathematical thinking, when a task is given to students, in order to carry it out, they consult their concept image and forget to consult the concept definition. It turns out that, in many cases, there are critical examples which shape the concept image. In some cases, these are the first examples which are introduced to the learner. For instance, in the case of the altitude (a segment which is drawn from one vertex or the triangle and it is perpendicular to the opposite side of this vertex *or to its continuation*), it is pedagogically reasonable to give examples of altitudes in acute angle triangles. Later on, in order to form the appropriate concept image of an altitude, the teacher, as well as the textbook, should give examples of altitudes from vertices of acute angles in an obtuse angle triangle. However, before this stage of the teaching takes place, the concept image of the altitude was shaped by the stereotypical examples of altitudes in an acute angle triangle (sometimes, even by the stereotypical examples of altitudes which are perpendicular to a horizontal side of a triangle). Thus, when the learners face a geometrical problem about altitudes which do not meet the stereotypes in their concept image, they are stuck. It does not occur to them to consult the concept definition of the altitude, and if it does occur, they usually recall the first part of the definition ("a segment which is drawn from one vertex or the triangle and it is perpendicular to the opposite side of this vertex") and forget the additional phrase in the definition ("or to its continuation"). Two additional examples of this kind are the following: (1) At the junior high level, in geometry, when a quadrangle is defined as a particular case of a polygon (a quadrangle is a polygon which has 4 sides), the learners have difficulties to accept a concave quadrangle or a quadrangle that intersects itself as quadrangles. (2) At the high school level, when a formal definition of a function is given to the students, eventually, the stereotypical concept image of a function is that of an algebraic formula. A common formal definition of a function can be the following one: a correspondence between two non-empty sets which assigns to every element in the first set (the domain) exactly one element in the second set (the range). Even if some non-mathematical examples are given to the students (for instance, the correspondence which assigns to every living creature its mother), even then, the stereotypical concept image of a function is that of an algebraic formula, as claimed above.

A plausible explanation to these phenomena can be given in terms of the psychological theory about *system 1* and *system 2*. Psychologists, nowadays, speak about two cognitive systems which they call system 1 and system 2. It sounds as if there are different parts in our brain which produce different kinds of thinking. However, this interpretation is wrong. The correct way to look at system 1 and system 2 is to consider them as *thinking modes*. This is summarized very clearly in Stanovich (1999, p. 145). System 1 is characterized there by the following adjectives: *associative, tacit, implicit, inflexible, relatively fast, holistic, and automatic*. System 2 is characterized by: *analytical, explicit, rational, controlled, and relatively slow*. Thus, notions that were used by mathematics educators in the past can be related now to system 1 or system 2, and therefore this terminology is richer than the previously suggested notions. Fischbein (1987) spoke about *intuition* and this can be considered as system 1. Skemp (1979) spoke about two systems which he called *delta one* and *delta two*. They can be considered as *intuitive* and *reflective* or using the new terminology, system 1 and system 2, respectively. Vinner (1997) used the notions *pseudo-analytical* and *pseudo-conceptual* which can be considered as system 1.

*In mathematical contexts the required thinking mode is that of system 2*. This requirement presents some serious difficulties to many people (children and adults) since, most of the time, thought processes are carried out within system 1. Also, in many people, because of various reasons, system 2 has not been developed to the extent which is required for mathematical thinking in particular and for rational thinking in general. Nevertheless, in many contexts, learners succeed in carrying out mathematical tasks which are presented to them by using system 1. This fact does not encourage them to become aware of the need to use system 2 while carrying out mathematical tasks.

When discussing concept development in mathematical thinking, it is worthwhile to mention also some concepts which can be classified as *metacognitive* concepts. Such concepts are *algorithm, heuristics*, and *proof*. While studying mathematics, the learners face many situations in which they or their teachers use algorithms, heuristics, and proof. However, usually, the notions "algorithm" and "heuristics" are not introduced to the learners in their school mathematics. Some of them will be exposed to them in college, in case they choose to take certain advanced mathematics courses. As to the notion of proof, in spite of the fact that this notion is mentioned a lot in school mathematics (especially in geometry), the majority of students do not fully understand it. Many of them try to identify mathematical proof by its *superficial characteristics*. They do it without understanding the logical reasoning associated with these characteristics. A meaningless use of symbols and verbal expressions as "therefore," "it follows," and "if... then" is considered by many students as a mathematical proof (See for instance Healy and Hoyles 1998). It turns out that it takes a lot of mathematical experience until meaningless verbal rituals (as in the case of the baby acquiring the concept of chair) become *meaningful thought processes*. And how do we know that the learners use the above verbal expressions meaningfully? We assume so because their use of these expressions is in absolute agreement with the way we, mathematicians and mathematics educators, use them.

Another important aspect of mathematical concept development is the understanding that certain mathematical concepts are related to each other. Here comes the idea of structure. For instance, from triangles, quadrangles, pentagons, and hexagons, we reach the concept of a polygon. From the general concept of quadrangles, we approach to trapezoids, parallelograms, rhombus, rectangles, and squares, and we realize there all kinds of class inclusions. Thus, we distinguish partial order in the set of mathematical concepts. Finally, and this is perhaps the ultimate stage of mathematical concept development, we conceive mathematics as a collection of various *deductive structures* (Peano's Arithmetic, Euclidean Geometry, Set Theory, Group Theory, etc.). Also, in more advanced mathematical thinking, we conceive mathematical objects (numbers, functions, geometrical figures in Euclidean geometry, etc.) as abstract objects. All these require thought processes within system 2. However, it should be emphasized that all the above concept developments do not occur simultaneously. They also do not occur in all students who study mathematics. One should take many mathematics courses and solve a lot of mathematical problems in order to achieve that level. Those who do it should have special interest in mathematics or what can be called mathematical curiosity. It requires, what some people call, a mathematical mind. Is it genetic (Devlin 2000) or acquired? At this point we have reached a huge domain of psychological research which is far beyond the scope of this particular encyclopedic issue.

## Cross-References


- ▶ Abstraction in Mathematics Education
- ▶ Concept Development in Mathematics Education
- ▶ Critical Thinking in Mathematics Education
- ▶ Intuition in Mathematics Education
- ▶ Mathematical Proof, Argumentation, and Reasoning
- ▶ Metacognition
- ▶ Problem Solving in Mathematics Education
- ▶ Theories of Learning Mathematics
- ▶ Values in Mathematics Education
- ▶ Zone of Proximal Development in Mathematics Education


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## Constructivism in Mathematics Education

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### Keywords

Epistemology; Social constructivism; Radical constructivism; Knowledge; Reality; Truth; Objectivity

### Background

Constructivism is an epistemological stance regarding the nature of human knowledge, having roots in the writings of Epicurus, Lucretius, Vico, Berkeley, Hume, and Kant. Modern constructivism also contains traces of pragmatism (Peirce, Baldwin, and Dewey). In mathematics education the greatest influences are due to Piaget, Vygotsky, and von Glasersfeld. See Confrey and Kazak (2006) and Steffe and Kieren (1994) for related historical accounts of constructivism in mathematics education.

There are two principle schools of thought within constructivism: radical constructivism (some people say individual or psychological) and social constructivism. Within each there is also a range of positions. While radical and social constructivism will be discussed in a later section, it should be noted that both schools are grounded in a strong skeptical stance regarding reality and truth: *Knowledge cannot be thought of as a copy of an external reality, and claims of truth cannot be grounded in claims about reality*.

The justification of this stance toward knowledge, truth, and reality, first voiced by the skeptics of ancient Greece, is that to verify that one's knowledge is correct, or that what one knows is true, one would need access to reality by means other than one's knowledge of it. The importance of this skeptical stance for mathematics educators is to remind them that students have their own mathematical realities that teachers and researchers can understand only via models of them (Steffe et al. 1983, 1988).

Constructivism did not begin within mathematics education. Its allure to mathematics educators is rooted in their long evolving rejection of Thorndike's associationism (Thorndike 1922; Thorndike et al. 1923) and Skinner's behaviorism (Skinner 1972). Thorndike's stance was that learning happens by forming associations between stimuli and appropriate responses. To design instruction from Thorndike's perspective meant to arrange proper stimuli in a proper order and have students respond appropriately to those stimuli repeatedly. The behaviorist stance that mathematics educators found most objectionable evolved from Skinner's claim that all human behavior is due to environmental forces. From a behaviorist perspective, to say that children participate in their own learning, aside from being the recipient of instructional actions, is nonsense. Skinner stated his position clearly:

> Science . . . has simply discovered and used subtle forces which, acting upon a mechanism, give it the direction and apparent spontaneity which make it seem alive. (Skinner 1972, p. 3)

Behaviorism's influence on psychology, and thereby its indirect influence on mathematics education, was also reflected in two stances that were counter to mathematics educators' growing awareness of learning in classrooms. The first stance was that children's learning could be studied in laboratory settings that have no resemblance to environments in which learning actually happens. The second stance was that researchers could adopt the perspective of a universal knower. This second stance was evident in Simon and Newell's highly influential information processing psychology, in which they separated a problem's "task environment" from the problem solver's "problem space."

> We must distinguish, therefore, between the task environment—the omniscient observer's way of describing the actual problem "out there"— and the problem space—the way a particular subject represents the task in order to work on it. (Simon and Newell 1971, p. 151)

Objections to this distinction were twofold: Psychologists considered themselves to be Simon and Newell's omniscient observers (having access to problems "out there"), and students' understandings of the problem were reduced to a subset of an observer's understanding. This stance among psychologists had the effect, in the eyes of mathematics educators, of blinding them to students' ways of thinking that did not conform to psychologists' preconceptions (Thompson 1982; Cobb 1987). Erlwanger (1973) revealed vividly the negative consequences of behaviorist approaches to mathematics education in his case study of a successful student in a behaviorist individualized program who succeeded by inventing mathematically invalid rules to overcome inconsistencies between his answers and an answer key.

The gradual release of mathematics education from the clutches of behaviorism, and infusions of insights from Polya's writings on problem solving (Polya 1945, 1954, 1962), opened mathematics education to new ways of thinking about student learning and the importance of student thinking. Confrey and Kazak (2006) described the influence of research on problem solving, misconceptions, and conceptual development of mathematical ideas as precursors to the emergence of constructivism in mathematics education.

Piaget's writings had a growing influence in mathematics education once English translations became available. In England, Skemp (1961, 1962) championed Piaget's notions of schema, assimilation, accommodation, equilibration, and reflection as ways to conceptualize students' mathematical thinking as having an internal coherence. Piaget's method of clinical interviews also was attractive to researchers of students' learning. However, until 1974 mathematics educators were interested in Piaget's writings largely because they thought of his work as "developmental psychology" or "child psychology," with implications for children's learning. It was in 1974, at a conference at the University of Georgia, that Piaget's work was recognized in mathematics education as a new field, one that leveraged children's cognitive development to study the growth of knowledge. Smock (1974) wrote of *constructivism's* implications for instruction, not *psychology's* implications for instruction. Glasersfeld (1974) wrote of Piaget's genetic epistemology as a theory of knowledge, not as a theory of cognitive development. The 1974 Georgia conference is the first occasion this writer could find where "constructivism" was used to describe the epistemological stance toward mathematical knowing that characterizes constructivism in mathematics education today.

Acceptance of constructivism in mathematics education was not without controversy. Disputes sometimes emerged from competing visions of desired student learning, such as students' performance on accepted measures of competency (Gagné 1977, 1983) versus attendance to the quality of students' mathematics (Steffe and Blake 1983), and others emerged from different conceptions of teaching effectiveness (Brophy 1986; Confrey 1986). Additional objections to constructivism were in reaction to its fundamental aversion to the idea of truth as a correspondence between knowledge and reality (Kilpatrick 1987).

## Radical and Social Constructivism in Mathematics Education

Radical constructivism is based on two tenets: "(1) Knowledge is not passively received but actively built up by the cognizing subject; (2) the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality" (Glasersfeld 1989, p. 114). Glasersfeld's use of "radical" is in the sense of fundamental – that cognition is "a constitutive activity which, alone, is responsible for every type or kind of structure an organism comes to know" (Glasersfeld 1974, p. 10).

Social constructivism is the stance that history and culture precede and preform individual knowledge. As Vygotsky famously stated, "Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first *between* people . . ., then *inside* the child" (Vygotsky 1978, p. 57).

The difference between radical and social constructivism can be seen through contrasting interpretations of the following event. Vygotsky (1978) illustrated his meaning of *internalization* – "the internal reconstruction of an external operation" – by describing the development of pointing:

> The child attempts to grasp an object placed beyond his reach; his hands, stretched toward that object, remain poised in the air. His fingers make grasping movements. At this initial state pointing is represented by the child's movement, which seems to be pointing to an object—that and nothing more. When the mother comes to the child's aid and realizes his movement indicates something, the situation changes fundamentally. Pointing becomes a gesture for others. The child's unsuccessful attempt engenders a reaction not from the object he seeks but *from another person* [sic]. Consequently, *the primary meaning of that unsuccessful grasping movement is established by others* [italics added]. (Vygotsky 1978, p. 56)

Vygotsky clearly meant that meanings originate in society and are transmitted via social interaction to children. Glasersfeld and Piaget would have listened agreeably to Vygotsky's tale – until the last sentence. They instead would have described the child as making a connection between his attempted grasping action and someone fetching what he wanted. Had it been the pet dog bringing the desired item, it would have made little difference to the child in regard to the practical consequences of his action. Rather, the child realized, in a sense, "Look at what I can make others do with this action." This interpretation would fit nicely with the finding that adults mimic infants' speech abundantly (Fernald 1992; Schachner and Hannon 2011). Glasersfeld and Piaget might have thought that adults' imitative speech acts, once children recognize them as imitations, provide occasions for children to have a sense that they can influence actions of others through verbal behavior. This interpretation also would fit well with Bauersfeld's (1980, 1988, 1995) understanding of communication as a reflexive interchange among mutually oriented individuals: "*The [conversation] is constituted at every moment through the interaction of reflective subjects*" (Bauersfeld 1980, p. 30 italics in original).

Paul Ernest (1991, 1994, 1998) introduced the term social constructivism to mathematics education, distinguishing between two forms of it. One form begins with a radical constructivist perspective and then accounts for human interaction in terms of mutual interpretation and adaptation (Bauersfeld 1980, 1988, 1992). Glasersfeld (1995) considered this as just radical constructivism. The other, building from Vygotsky's notion of cultural regeneration, introduced the idea of mathematical objectivity as a social construct.

> Social constructivism links subjective and objective knowledge in a cycle in which each contributes to the renewal of the other. In this cycle, the path followed by new mathematical knowledge is from subjective knowledge (the personal creation of an individual), via publication to objective knowledge (by intersubjective scrutiny, reformulation, and acceptance). Objective knowledge is internalized and reconstructed by individuals, during the learning of mathematics, to become the individuals' subjective knowledge. Using this knowledge, individuals create and publish new mathematical knowledge, thereby completing the cycle. (Ernest 1991, p. 43)

Ernest focused on objectivity of adult mathematics. He did not address the matter of how children's mathematics comes into being or how it might grow into something like an adult's mathematics.

Radical and social constructivists differ somewhat in the theoretical work they ask of constructivism. Radical constructivists concentrate on understanding learners' mathematical realities and the internal mechanisms by which they change.

They conceive, to varying degrees, of learners in social settings, concentrating on the sense that learners make of them. They try to put themselves in the learner's place when analyzing an interaction. Social constructivists focus on social and cultural mathematical and pedagogical practices and attend to individuals' internalization of them. They conceive of learners in social settings, concentrating, to various degrees, on learners' participation in them. They take the stances, however, of an observer of social interactions and that social practices predate individuals' participation.

Conflicts between radical and social constructivism tend to come from two sources: (1) differences in meanings of truth and objectivity and their sources and (2) misunderstandings and miscommunications between people holding contrasting positions. The matter of (1) will be addressed below. Regarding (2), Lerman (1996) claimed that radical constructivism was internally incoherent: How could radical constructivism explain agreement when persons evidently agreeing create their own realities? Steffe and Thompson (2000a) replied that interaction was at the core of Piaget's genetic epistemology and thus the idea of intersubjectivity was entirely coherent with radical constructivism. The core of the misunderstanding was that Lerman on the one hand and Steffe and Thompson on the other had different meanings for "intersubjectivity." Lerman meant "agreement of meanings" – same or similar meanings. Steffe and Thompson meant "nonconflicting mutual interpretations," which might actually entail nonagreement of meanings of which the interacting individuals are unaware. Thus, Lerman's objection was valid relative to the meaning of intersubjectivity he presumed. Lerman on one side and Steffe and Thompson on the other were in a state of intersubjectivity (in the radical constructivist sense) even though they publicly disagreed. They each presumed they understood what the other meant when in fact each understanding of the other's position was faulty.

Other tensions arose because of interlocutors' different objectives. Some mathematics educators focused on understanding individual's mathematical realities. Others focused on the social context of learning. Cobb, Yackel, and Wood (1992) diffused these tensions by refocusing discussions on the work that theories in mathematics education must do – they must contribute to our ability to improve the learning and teaching of mathematics. Cobb et al. first reminded the field that, from any perspective, what happens in mathematics classrooms is important for students' mathematical learning. Thus, a theoretical perspective that can capture more, and more salient, aspects for mathematics learning (including participating in practices) is the more powerful theory. With a focus on the need to understand, explain, and design events within classrooms, they recognized that there are indeed social dimensions to mathematics learning and there are psychological aspects to participating in practices and that researchers must be able to view classrooms from either perspective while holding the other as an active background: "[W]e have proposed the metaphor of mathematics as an evolving social practice that is constituted by, *and does not exist apart from*, the constructive activities of individuals" (Cobb et al. 1992, p. 28, italics added).

Cobb et al.'s perspective is entirely consistent with theories of emergence in complex systems (Schelling 1978; Eppstein and Axtell 1996; Resnick 1997; Davis and Simmt 2003) when taken with Maturana's statement that "anything said is said by an observer" (Maturana 1987). Practices, as stable patterns of social interaction, exist in the eyes of an observer who sees them. The theoretician who understands the behavior of a complex system as entailing simultaneously both microprocesses and macrobehavior is better positioned to affect macrobehavior (by influencing microprocesses) than one who sees just one or the other. It is important to note that this notion of emergence is not the same as Ernest's notion of objectivity as described above.

## Truth and Objectivity

Radical constructivists take the strong position that children have mathematical realities that do not overlap an adult's mathematics (Steffe et al. 1983; Steffe and Thompson 2000).

Social constructivists (of Ernest's second type) take this as pedagogical solipsism.

> The implications of [radical constructivism] are that individual knowers can construct truth that needs no corroboration from outside of the knower, making possible any number of "truths." Consider the pedagogical puzzles this creates. What is the teacher trying to teach students if they are all busy constructing their own private worlds? What are the grounds for getting the world right? Why even care whether these worlds agree? (Howe and Berv 2000, pp. 32–33)

Howe and Berv made explicit the social constructivist stance that there is a "right" world to be got – the world of socially constructed meanings. They also revealed their unawareness that, from its very beginning, radical constructivism addressed what "negotiation" could mean in its framework and how stable patterns of meaning could emerge socially (Glasersfeld 1972, 1975, 1977). Howe and Berv were also unaware of the notion of *epistemic subject* in radical constructivism – the mental construction of a nonspecific person who has particular ways of thinking (Beth and Piaget 1966; Glasersfeld 1995). A teacher need not attend to 30 mathematical realities with regard to teaching the meaning of fractions in a class of 30 children. Rather, she need only attend to perhaps 5 or 6 epistemic children and listen for which fits the ways particular children express themselves (Thompson 2000).

## A Short List: Impact of Constructivism in Mathematics Education

- Mathematics education has a new stance toward learners at all ages. One must attend to learner's mathematical realities, not just their performance.
- Current research on students' and teachers' thinking and learning is largely consistent with constructivism – to the point that articles rarely declare their basis in constructivism. Constructivism is now taken for granted.
- Teaching experiments (Cobb and Steffe 1983; Cobb 2000; Steffe and Thompson 2000) and design experiments (Cobb et al. 2003) are vital and vibrant methodologies in mathematics education theory development.
- Conceptual analysis of mathematical thinking and mathematical ideas is a prominent and widely used analytic tool (Smith et al. 1993; Glasersfeld 1995; Behr et al. 1997; Thompson 2000; Lobato et al. 2012).
- What used to be thought of as *practice* is now conceived as *repeated experience*. Practice focuses on repeated behavior. Repeated experience focuses on repeated reasoning, which can vary in principled ways from setting to setting (Cooper 1991; Harel 2008a, b).
- Constructivism has clear and operationalized implications for the design of instruction (Confrey 1990; Simon 1995; Steffe and D'Ambrosio 1995; Forman 1996; Thompson 2002) and assessment (Carlson et al. 2010; Kersting et al. 2012).

## Cross-References


▶ Constructivist Teaching Experiment
▶ Misconceptions and Alternative Conceptions in Mathematics Education
▶ Sociomathematical Norms in Mathematics Education


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## Constructivist Teaching Experiment

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### Keywords

Constructivism; Methodology; Teaching experiment; Instrumental understanding

### Introduction

> The constructivist is fully aware of the fact that an organism's conceptual constructions are not fancy-free. On the contrary, the process of constructing is constantly curbed and held in check by the constraints it runs into. (Ernst von Glasersfeld 1990, p. 33)

The constructivist teaching experiment emerged in the United States circa 1975 (Steffe et al. 1976) in an attempt to understand children's numerical thinking and how that thinking might change rather than to rely on models that were developed outside of mathematics education for purposes other than educating children (e.g., Piaget and Szeminska 1952; McLellan and Dewey 1895; Brownell 1928). The use of the constructivist teaching experiment in the United State was buttressed by versions of the teaching experiment methodology that were being used already by researchers in the Academy of Pedagogical Sciences in the then Union of Soviet Socialist Republics (Wirszup and Kilpatrick 1975–1978). The work at the Academy of Pedagogical Sciences provided academic respectability for what was then a major departure in the practice of research in mathematics education in the United States, not only in terms of research methods but more crucially in terms of the research orientation of the methodology. In El'konin's (1967) assessment of Vygotsky's (1978) research, the essential function of a teaching experiment is the production of models of student thinking and changes in it:

> Unfortunately, it is still rare to meet with the interpretation of Vygotsky's research as modeling, rather than empirically studying, developmental processes. (El'konin 1967, p. 36)

Similarly, the primary purpose of constructivist teaching experiments is to construct explanations of students' mathematical concepts and operations and changes in them. Without experiences of students' mathematics afforded by teaching, there would be no basis for coming to understand the mathematical concepts and operations students construct or even for suspecting that these concepts and operations may be distinctly different from those of teacher/researchers. The necessity to attribute mathematical concepts and operations to students that are independent of those of teacher/researchers has been captured by Ackermann (1995) in speaking of human relations:

> In human relations, it is vital to attribute autonomy to others and to things—to celebrate their existence independently from our current interaction with them. This is true even if an attribution (of existence) is a mental construct. We can literally rob others of their identity if we deny them an existence beyond our current interests (p. 343).

Students' mathematical concepts and operations constitute first-order models, which are models that students construct to organize, comprehend, and control their own experience (Steffe et al. 1983, p. xvi). Through a process of *conceptual analysis* (Glasersfeld 1995), teacher/researchers construct models of students' mathematical concepts and operations to explain what students say and do. These second-order models (Steffe et al. 1983, p. xvi) are called *mathematics of students* and students' first-order models are called simply *students' mathematics*. While teacher/researchers may write about the schemes and operations that constitute these second-order models as if they are identical to students' mathematics, these constructs, in fact, are a construction of the researcher that only references students' mathematics. Conceptual analysis is based on the belief that mathematics is a product of the functioning of human intelligence (Piaget 1980), so the mathematics of students is a legitimate mathematics to the extent that teacher/researchers can find rational grounds to explain what students say and do.

The overarching goal of the teacher/researchers who use the methodology is to establish the mathematics of students as a conceptual foundation of students' mathematics education (Steffe and Wiegel 1992; Steffe 2012). The mathematics of students opens the way to ground school mathematics in the history of how it is generated by students in the context of teaching. This way of regarding school mathematics casts it as a living subject rather than as a subject of being (Steffe 2007).

## Characteristics: The Elements of Constructivist Teaching Experiments

### Teaching Episodes

A constructivist teaching experiment involves a sequence of teaching episodes (Hunting 1983; Steffe 1983). A teaching episode includes a teacher/researcher, one or more students, a witness of the teaching episodes, and a method of recording what transpires during the episodes. These records can be used in preparing subsequent episodes as well as in conducting conceptual analyses of teaching episodes either during or after the experiment.

### Exploratory Teaching

Any teacher/researcher who hasn't conducted a teaching experiment but who wishes to do so should first engage in exploratory teaching (Steffe and Thompson 2000). It is important that the teacher/researcher becomes acquainted, at an experiential level, with students' ways and means of operating in whatever domain of mathematical concepts and operations are of interest. In exploratory teaching, the teacher/researcher attempts to put aside his or her own concepts and operations and not insist that the students learn what he or she knows (Norton and D'Ambrosio 2008). Otherwise, the teacher/researcher might become caught in what Stolzenberg (1984) called a "trap" – focusing on the mathematics the teacher/researcher takes as given instead of focusing on exploring students' ways and means of operating. The teacher/researcher's mathematical concepts and operations can be orienting, but they should not be regarded, initially at least, as constituting what students should learn until they are modified to include at least aspects of a mathematics of students (Steffe 1991a).

### Meanings of "Experiment"

*Testing Initial Research Hypotheses*. One goal of exploratory teaching is to identify essential differences in students' ways and means of operating within the chosen context in order to establish initial research hypotheses for the teaching experiment (Steffe et al. 1983). These differences are essential in establishing the constructivist teaching experiment as involving an "experiment" in a scientific sense. The established differences can be used to place students in experimental groups and the research hypothesis is that the differences between the students in the different experimental groups would become quite large over the period of time the students participate in the experiment and that the students within the groups would remain essentially alike (Steffe and Cobb 1988). Considerable hypothesis building and testing must happen during the course of a teaching experiment as well. However, one does not embark on the intensive work of a constructivist teaching experiment without having initial research hypotheses to test.

The research hypotheses one formulates prior to a teaching experiment guide the initial selection of the students and the teacher/researcher's overall general intentions. The teacher/researcher does his or her best to set these initial hypotheses aside during the course of the teaching episodes and focus on promoting the greatest progress possible in all participating students. The intention of teacher/researcher is for the students to test the research hypotheses by means of how they differentiate themselves in the trajectory of teaching interactions (Steffe 1992; Steffe and Tzur 1994). A teacher/researcher returns to the initial research hypotheses retrospectively after completing the teaching episodes. This method – setting research hypotheses aside and focusing on what actually happens in teaching episodes – is basic in the ontogenetic justification of school mathematics.

*Generating and Testing Working Hypotheses*. In addition to formulating and testing initial research hypotheses, another modus operandi in a teaching experiment is for a teacher/researcher to generate and test hypotheses during the teaching episodes. Often, these hypotheses are conceived "on the fly," a phrase Ackermann (1995) used to describe how hypotheses are formulated in clinical interviews. Frequently, they are formulated between teaching episodes as well. A teacher/researcher, through reviewing the records of one or more earlier teaching episodes, may formulate hypotheses to be tested in the next episode (Hackenberg 2010). In a teaching episode, the students' language and actions are a source of perturbation for the teacher/researcher. It is the job of the teacher/researcher to continually postulate possible meanings that lie behind students' language and actions. It is in this way that students guide the teacher/researcher. The teacher/researcher may have a set of hypotheses to test before a teaching episode and a sequence of situations planned to test the hypotheses. But because of students' unanticipated ways and means of operating as well as their unexpected mistakes, a teacher/

researcher may be forced to abandon these hypotheses while interacting with the students and to create new hypotheses and situations on the spot (Norton 2008). The teacher/researchers also might interpret the anticipated language and actions of the students in ways that were unexpected prior to teaching. These impromptu interpretations are insights that would be unlikely to happen in the absence of direct, longitudinal interaction with the students in the context of teaching interactions. Here, again, the teacher/researcher is obliged to formulate new hypotheses and to formulate situations of learning to test them (Tzur 1999).

## Living, Experiential Models of Students' Mathematics

Through generating and testing hypotheses, boundaries of the students' ways and means of operating – where the students make what to a teacher/researcher are essential mistakes – can be formulated (Steffe and Thompson 2000). These essential mistakes are of the same nature as those Piaget found in his studies of children, and a teacher/researcher uses them for essentially the same purpose he did. They are observable when students fail to make viable adaptations when interacting in a medium. Operations and meanings a teacher/researcher imputes to students constitute what are called living, experiential models of students' mathematics. Essential mistakes can be thought of as illuminating the boundaries of what kinds of adaptations a living, experiential model can currently make in these operations and meanings. These boundaries are usually fuzzy, and what might be placed just inside or just outside them is always a source of tension and often leads to creative efforts on the part of a teacher/researcher. What students can do is understood better if what they cannot do is also understood. It also helps to understand what a student can do if it is understand what other students, whose knowledge is judged to be at a higher or lower level, can do (Steffe and Olive 2010). In this, we are in accordance with Ackermann (1995) that:

> The focus of the clinician [teacher] is to understand the originality of [the child's] reasoning, to describe its coherence, and to probe its robustness or fragility in a variety of contexts. (p. 346)

## Meanings of Teaching in a Teaching Experiment

Learning how to interact with students through effective teaching actions is a central issue in any teaching experiment (Steffe and Tzur 1994). If teacher/researchers knew ahead of time how to interact with the selected students and what the outcomes of those interactions might be, there would be little reason for conducting a teaching experiment (Steffe and Cobb 1983). There are essentially two types of interaction engaged in by teacher/researchers in a teaching experiment: responsive and intuitive interactions and analytical interactions.

### Responsive and Intuitive Interaction

In responsive and intuitive interactions, teacher/researchers are usually not explicitly aware of how or why they interact as they do. In this role, teacher/researchers are agents of interaction and they strive to harmonize themselves with the students with whom they are working to the extent that they "lose" themselves in their interactions. They make no intentional distinctions between their knowledge and the students' knowledge, and, experientially, everything is the students' knowledge as they strive to feel at one with them. In essence, they become the students and attempt to think as they do (Thompson 1982, 1991; van Manen 1991). Teacher/researchers do not adopt this stance at the beginning of a teaching experiment only. Rather, they maintain it throughout the experiment whenever appropriate. By interacting with students in a responsive and an intuitive way, the goal of teacher/researchers is to engage the students in supportive, nonevaluative mathematical interactivity.

### Analytical Interaction

When teacher/researchers turn to analytical interaction, they "step out" of their role in responsive/intuitive interaction and become observers as well. As first-order observers, teacher/researchers focus on analyzing students' thinking in ongoing interaction (Steffe and Wiegel 1996). All of the teacher/researchers' attention and energy is absorbed in trying to think like the students and produce and then experience mathematical realities that are intersubjective with theirs. The teacher/researchers probes and teaching actions are not to foment adaptation in the students but in themselves. When investigating student learning, teacher/researchers become second-order observers, which Maturana (1978) explained as "the observer's ability . . . to operate as external to the situation in which he or she is, and thus be an observer of his or her circumstance as an observer" (p. 61). As second-order observers, teacher/researchers focus on the accommodations they might engender in the students' ways and means of operating (Steffe 1991b). They become aware of how they interact and of the consequences of interacting in a particular way. Assuming the role of a second-order observer is essential in investigating student learning in a way that explicitly as well as implicitly takes into account the mathematical knowledge of the teacher/researchers as well as the knowledge of the students (Steffe and Wiegel 1996).

### The Role of a Witness of the Teaching Episodes

A teacher/researcher should expect to encounter students operating in unanticipated and apparently novel ways as well as their making unexpected mistakes and becoming unable to operate. In these cases, it is often helpful to be able to appeal to an observer of a teaching episode for an alternative interpretation of events. Being immersed in interaction, a teacher/researcher may not be able to act as a second-order observer and step out of the interaction, reflect on it, and take further action on that basis. In order to do so, a teacher/researcher would have to "be" in the interaction and outside of it, which can be difficult. It is quite impossible to achieve this if there are no conceptual elements available to the teacher/researcher from past teaching experiments that can be used in interpreting the current situation. The result is that teacher/researchers usually react to surprising behavior by switching to a more intuitive mode of interaction. When this happens, the observer may help a teacher/researcher both to understand the student and to posit further interaction. There are also occasions when the observer might make an interpretation of a student's actions that is different from that of a teacher/researcher for any one of several reasons. For example, the observer might catch important elements of a student's actions that apparently are missed by a teacher/researcher. In any case, the witness should suggest but not demand specific teaching interventions.

## Retrospective Conceptual Analysis

Conceptual analysis is intensified during the period of retrospective analysis of the public records of the teaching episodes, which is a critical part of the methodology. Through analyzing the corpus of video records, the teacher/researchers conduct a historical analysis of the living, experiential models of students' mathematics throughout the period of time the teaching episodes were conducted. The activity of model building that was present throughout the teaching episodes is foregrounded, and concepts in the core of a constructivist research program like assimilation, accommodation, scheme (von Glasersfeld 1981), cognitive and mathematical play, communication, spontaneous development (Piaget 1964), interaction (von Foerster 1984), mental operation (von Glasersfeld 1987), and self-regulation emerge in the form of specific and concrete explanations of students' mathematical activity. In this regard, the modeling process in which we engage is compatible with how Maturana (1978) regards scientific explanation:

> As scientists, we want to provide explanations for the phenomena we observe. That is, we want to propose conceptual or concrete systems that can be deemed intentionally isomorphic to the systems that generate the observed phenomena. (p. 29)

However, in the case of a teaching experiment, we seek models that fit within our living, experiential models of students' mathematics without claiming isomorphism because we have no access to students' mathematical realities outside of our own ways and means of operating when bringing the students' mathematics forth. So, we cannot get outside our observations to check if our conceptual constructs are isomorphic to students' mathematics. But we can and do establish viable ways and means of thinking that fit within the experiential constraints that we established when interacting with the students in teaching episodes (Steffe 1988, 1994; Norton and Wilkins 2010).

Since the time of its emergence, the constructivist teaching experiment has been widely used in investigations of students' mathematics as well as in investigations of mathematics teaching (cf. Appendix for sample studies). It has also been adapted to fit within related research programs (e.g., Cobb 2000; Confrey and Lachance 2000; Simon et al. 2010).

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## Cross-References

▶ Constructivism in Mathematics Education
▶ Discourse Analytic Approaches in Mathematics Education
▶ Early Algebra Teaching and Learning
▶ Elkonin and Davydov Curriculum in Mathematics Education
▶ Hypothetical Learning Trajectories in Mathematics Education
▶ Interactionist and Ethnomethodological Approaches in Mathematics Education
▶ Learner-Centered Teaching in Mathematics Education
▶ Mathematics Teacher as Learner
▶ Number Teaching and Learning
▶ Probability Teaching and Learning
▶ Psychological Approaches in Mathematics Education
▶ Realistic Mathematics Education
▶ Teacher as Researcher in Mathematics Education
▶ Theories of Learning Mathematics
▶ Zone of Proximal Development in Mathematics Education

## Appendix: Example Studies Using Teaching Experiment Methodology

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## Creativity in Mathematics Education

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### Keywords

Creativity; Giftedness; Ability

### Definition

In the field of professional mathematics, the creative mathematician is a rarity. At this level mathematical creativity implies mathematical giftedness, but the reverse is not necessarily true (Sriraman 2005). Usiskin's (2000) eight tiered hierarchy of creativity and giftedness in mathematics further shed some light of this view of the relationship between creativity and giftedness in professional mathematics. In his model, mathematically gifted individuals such as professional, working mathematicians are at level five, while creative mathematicians are at level six and seven. However, the relationship between giftedness and creativity has been the subject of much controversy (Leikin 2008; Sternberg and O'Hara 1999) as some see creativity as part of an overall concept of giftedness (Renzulli 1986). In this entry the relationship between mathematical creativity and giftedness and ability will be looked at through a synthesis of some recent articles published in ZDM. First, the concepts of giftedness, ability, and creativity will be discussed. Second, common themes from the three articles will be synthesized that capture the main ideas in the studies. Lastly, the synthesis will be situated into the more generally framed research in psychology.

## Theory

### Creativity

One of the main challenges in investigating mathematical creativity is the lack of a clear and accepted definition of the term mathematical creativity and creativity itself. Previous examinations of the literature have concluded that there is no universally accepted definition of either creativity or mathematical creativity (Sriraman 2005; Mann 2005). Treffinger et al. (2002) write, for instance, that there are more than 100 contemporary definitions of mathematical creativity. Nevertheless, there are certain parameters agreed upon in the literature that helps narrow down the concept of creativity. Most investigations of creativity take one of two directions: extraordinary creativity, known as big C, or everyday creativity, known as little c (Kaufman and Beghetto 2009). Extraordinary creativity refers to exceptional knowledge or products that change our perception of the world. Feldman, Cziksentmihalyi, and Gardner (1994) writes: *"the achievement of something remarkable and new, something which transforms and changes a field of endeavor in a significant way . . . the kinds of things that people do that change the world."* Ordinary, or everyday, creativity is more relevant in a regular school setting. Feldhusen (2006) describes little c as: *"Wherever there is a need to make, create, imagine, produce, or design anew what did not exist before – to innovate – there is adaptive or creative behavior, sometimes called 'small c.*" Investigation into the concept of creativity also distinguishes between creativity as either domain specific or domain general (Kaufman and Beghetto 2009).

Whether or not creativity is domain specific or domain general, or if you look at ordinary or extraordinary creativity, most definitions of creativity include some aspect of usefulness and novelty (Sternberg 1999; Plucker and Beghetto 2004; Mayer 1999). What is useful and novel depends on the context of the creative process of an individual. The criteria for useful and novel in professional arts would differ significantly from what is deemed useful and novel in a mathematics class in lower secondary school. There is therefore a factor of relativeness to creativity. For a professional artist, some new, groundbreaking technique, product, or process that changes his or her field in some significant way would be creative, but for a mathematics student in lower secondary school, an unusual solution to a problem could be creative. Mathematical creativity in a K-12 setting can as such be defined as the process that results in a novel solution or idea to a mathematical problem or the formulation of new questions (Sriraman 2005).

### Giftedness

For decades giftedness was equated with concept of intelligence or IQ (Renzulli 2005; Brown et al. 2005; Coleman and Cross 2005). Terman (1925) claimed that gifted individuals are those who score at the top 1 % of the population on the Stanford-Binet test. This understanding of giftedness has survived to this day in some conceptions. However, most researchers now view giftedness as a more multifaceted concept in which intelligence is but one of several aspects (Renzulli 2005). One example is Renzulli's (1986) three-ring model of giftedness. In an attempt to capture the many facets of giftedness, Renzulli presented giftedness as an interaction between above-average ability, creativity, and task commitment. He went on to separate giftedness into two categories: schoolhouse giftedness and creative productive giftedness. The former refers to the ease of acquiring knowledge and taking standardized tests. The latter involves creating new products and processes, which Renzulli thought was often overlooked in school settings. Many researchers support this notion that creativity should be included in the conception of giftedness in any area (Miller 2012).

For this entry, giftedness will be looked at in the domain of mathematics, as Csikszentmihalyi (2000) pointed out the field-dependent character of the concept of giftedness. Due to the lack of a conceptual clarity regarding giftedness and the heterogeneity of the gifted population, both in general and in mathematics, identification of gifted students has varied (Kontoyianni et al. 2011). Instead, prominent characteristics of giftedness in mathematics are found in the research literature. Krutetskii (1976) noted in his investigation of gifted students in mathematics a number of characteristic features: ability for logical thought with respect to quantitative and spatial relationships, number and letter symbols; the ability for rapid and broad generalization of mathematical relations and operations, flexibility of mental processes and mathematical memory. Similar features of mathematical giftedness have been proposed by other researchers (see for instance Sriraman 2005).

## Ability

Often, mathematical ability has been seen as equivalent to mathematical attainment and to some degree, there is some truth to that notion. There is a statistical relationship between academic attainment in mathematics and high mathematical ability (Benbow and Arjmand 1990). However, Ching (1997) discovered that hidden talent go largely unnoticed in typical classrooms, and Kim et al. (2003) state that traditional tests rarely identify mathematical creativity. Hong and Aqui (2004) compared cognitive and motivational characteristics of high school students who were academically gifted in math, creatively talented in math, and non-gifted. The authors found that the creatively talented students used more cognitive strategies than the academically gifted students. These findings indicate that mathematical ability and mathematical attainment in a K-12 setting are not necessarily synonymous.

In the online thefreedictionary.com, ability is defined as *"the quality of being able to do something, especially the physical, mental, financial, or legal power to accomplish something."* Attainment is defined as "*Something, such as an accomplishment or achievement, that is attained*." The key difference is that ability points to a potential to do something, while attainment refers to something that has been accomplished. In the field of mathematics, mathematical ability then refers to the ability to do mathematics and not the ability to do well on mathematics attainment tests in school. In order to de facto define mathematical ability, mathematics itself has to be defined. It is beyond the scope of this entry to discuss what mathematics itself is (for a K-12 setting, see, for instance, NCTM 2000 or Niss 1999), so mathematical ability will simply be defined as the ability to do mathematics.

## Conceptual Relationships

In some recent ZDM articles the concept of mathematical creativity is linked to other concepts through statistical and qualitative investigation. In Kattou et al. (2013), the relationship between mathematical ability and mathematical creativity was investigated quantitatively with the use of a mathematical ability test and a mathematical creativity test. Data were collected by administering the two tests to 359 elementary school students. The authors concluded, using confirmatory factor analysis, that mathematical creativity is a subcomponent of mathematical ability. Mathematical ability was measured by 29 items in the following categories: quantitative ability, causal ability, spatial ability, qualitative ability, and inductive/deductive ability. The operationalization of mathematical ability was based on the assumption that mathematical ability is a multidimensional construct and Krutetskii's (1976) classification of giftedness in mathematics. Mathematical creativity was measured with five open-ended multiple solution tasks that were assessed on the basis of fluency, flexibility, and originality (Leikin 2007).

Pitta et al. (2013) investigated the relationship between mathematical creativity and cognitive styles. Mathematical creativity was measured similarly to Kattou et al. (2013). A mathematical creativity test consisting of five tasks was given to 96 prospective primary school teachers and was assessed on the basis of fluency, flexibility, and originality. Cognitive style was measured with the *Object-Spatial Imagery and Verbal Questionnaire* (OSIVQ) with respect to three styles: spatial, object, and verbal. Using multiple regression, the authors conclude that spatial and object styles were significant predictors of mathematical creativity, while verbal style was not significant. Spatial cognitive style was positively related to mathematical creativity, while object cognitive style was negatively related to mathematical creativity. Furthermore, spatial cognitive style was positively related to fluency, flexibility, and originality, while object cognitive style was negatively related to originality and verbal cognitive style was negatively related to flexibility.

Using an entirely different methodology, Lev-Zamir and Leikin (2013) analyzed two teachers' declarative conceptions about mathematical creativity in teaching and conceptions-in-action seen in their lessons. The authors write that while declarative statements about mathematical creativity in teaching may seem similar, their conceptions-in-action could differ vastly. In the study, the two teachers used much of the same terminology when relating to originality and flexibility in teaching. However, there was a large gap between the teachers' declarations and actions. One of the teachers displayed a lack of flexibility in the classroom, while the interaction between the other teacher and her students displayed flexibility. The authors point out the distinction between deep beliefs and surface beliefs as a possible explanation for the observed inconsistencies. Teacher-directed conceptions of creativity are associated with surface beliefs and student-directed conceptions of creativity are associated with deep beliefs. The authors go on to state that student-oriented conceptions of creativity are more of a mathematical nature and this attention enables teachers to be more flexible during their lessons.

These three articles all focus on different characteristics of mathematical creativity. They do not explicitly investigate the relationship between giftedness, ability, and creativity. Nevertheless, there are certain similarities that might be inferred on a more structural level, and the studies add to our overall understanding of giftedness, creativity, and ability in mathematics. Certain cognitive styles, mathematical ability, and types of beliefs are all found to predict and have a relationship with mathematical creativity. Student-directed conception of mathematical creativity as a deep belief, spatial cognitive style, and general mathematical ability are all linked to mathematical creativity. As a concept, mathematical creativity does not exist in a vacuum. The literature synthesized in this entry suggests that certain features and factors are required for mathematical creativity to arise.

Although no explicit relationships between mathematical ability, cognitive styles, and beliefs are explored in the three studies, an underlying link may be inferred from them. Pitta et al. (2013) point out that previous research has found that spatial cognitive style can be beneficial for physics, mechanical engineering, and mathematics tasks (see, for instance, Kozhevnikov et al. 2005). In Lev-Zamir and Leikin's study (2013), the teacher with the deep student-directed conceptions of mathematical creativity had a much stronger mathematical background than the other teacher. Both spatial cognitive style and deep student-directed conceptions of mathematical creativity are therefore conceivably connected to mathematical ability and knowledge. Kattou et al. (2013) found a strong correlational relationship between mathematical ability and mathematical creativity. If certain cognitive styles and types of beliefs are connected to mathematical ability, and mathematical ability is linked to mathematical creativity, it stands to reason that mathematical ability, mathematical creativity, certain cognitive styles, and types of beliefs are all linked.

### Who Are Creative?

Closely related to conceptual relationships between mathematical creativity and other concepts is the question of "who are mathematically creative?" Can individuals be distinguished into separate groups according to their mathematical creativity, and what characterizes these groups? Kattou et al. (2013) clustered students into three subgroups: low, average, and high mathematical ability. The high-ability students were also highly creative students, the average-ability students had an average performance on the mathematical creativity test, while low-ability students have a low creative potential in mathematics. Pitta et al. (2013) classified the prospective teachers as spatial visualizers, object visualizers, or verbalizers. The spatial visualizers scored higher on the mathematical creativity test than both other groups. In the third article examined here, the conceptions of creativity of only two teachers were investigated (Lev-Zamir and Leikin 2013). As such, it is difficult to generalize any finding. Nevertheless, the authors point out the different mathematical backgrounds of the two teachers and how the teacher with the stronger mathematical background has deeper beliefs regarding mathematical creativity.

All three studies, through different methodologies, can be said to cluster individuals according to their level of mathematical creativity. As with conceptual relationships, the findings of the three studies synthesized in this entry cannot be unified explicitly. The three studies investigated different aspects of mathematical creativity, using different methodologies. Instead, the findings have to be looked at from a more general and systemic perspective. That means instead of looking at what the specific characteristics of mathematically creative individuals are, the focus is that there are characteristics of mathematically creative individuals. All three studies distinguish individuals into different levels of mathematical creativity according to some other quality or ability.

### Implications for Teaching

Although only one of the articles (Kattou et al. 2013) makes explicit recommendations for mathematics teaching, the implications of the three articles are on some levels related. Kattou et al. conclude that the encouragement of mathematical creativity is important for further development of students' mathematical ability. More importantly, they write, teachers should not limit their teaching to spatial conception, arithmetic, and proper use of methods and operations. Teachers should recognize the importance of creative thinking in the classroom. This is closely related to what Lev-Zamir and Leikin (2013) conclude. Teachers who hold a mathematically student-oriented conception of mathematical creativity were found to be more flexible during lessons and stimulate students' mathematical creativity. In other words, with a student-oriented conception of mathematical creativity, teachers will to a greater degree be able to recognize and encourage creative mathematical thinking during their lessons.

The third article (Pitta et al. 2013) did not make any explicit recommendations or implications for teaching mathematics. However, as they investigated prospective teachers' mathematical creativity, the results and conclusions may be relevant for teaching when seen in a broader perspective. Pitta et al. found that spatial visualizers had a statistical significant higher creative performance than other teachers. The observed differences were related to the different strategies employed by the spatial visualizers, object visualizers, and verbal visualizers. The spatial visualizers employed more flexible and analytic strategies to tasks. This allowed them to be more creative and provide more, different, and unique solutions. In light of Lev-Zamir and Leikin (2013) and Kattou et al. (2013) conclusions, the question becomes whether a flexible and analytic approach to mathematics tasks translates into an analytic and flexible approach to mathematics teaching. If that is the case, then flexible and creative teaching is also related to spatial cognitive style. However, as Pitta et al. (2013) ask, it is unknown whether having a spatial cognitive style is a result of experience or inborn abilities. They go on to recommend further investigation to see if prospective teachers can be trained to use their spatial visualization. It may lead to enhanced spatial imagery and consequently facilitate mathematical creativity, possibly also in their mathematics teaching.

## Giftedness and Creativity in Psychology

The research into the field of general creativity focuses on four different variables: person, process, product, and press. The person category highlights the internal cognitive characteristics of individuals. The process category looks at the internal process that takes place during a creative activity. Product focuses on the characteristics of products thought to be creative. Last, the press category explores the ways environmental factors can influence creativity (Taylor 1988). The articles recently published in ZDM focused their research primarily into the person and press component of mathematical creativity. In other words: what characterizes the mathematically creative individual and how can mathematical creativity be developed in the classroom.

Mathematical creativity is linked to and influenced by ability, beliefs, cognitive style, and the classroom environment (Lev-Zamir and Leikin 2013; Pitta et al. 2013; Kattou et al. 2013). These findings are analogous to much of the research into general creativity and giftedness. The star model of Abraham Tannenbaum (2003) conceptualizes giftedness into five elements, some of which are seen in the studies synthesized in this entry: (a) superior general intellect, (b) distinctive special aptitudes, (c) nonintellective requisites, (d) environmental supports, and (e) chance. Creativity is here included in the nonintellective requisites. Mathematical ability would be placed in the distinctive special aptitudes, as it portrays to domain specific abilities, while both beliefs and cognitive styles would be classified as nonintellective requisites. Flexible teaching that stimulates mathematical creativity falls under the category of environmental support. As such, the observations in the studies synthesized here are in many ways analogous to research into general creativity and giftedness.

Similarly, the dynamic theory of giftedness (Babaeva 1999), which emphasizes the social aspects of the development in giftedness, can also provide a theoretical perspective on the observations synthesized in this entry. This theory consists of three principles that explain the development of giftedness: (a) an obstacle for positive growth is introduced, (b) a process to overcome the obstacle, and (c) alteration and incorporation of the experience (Miller 2012). Kattou et al. (2013) point out how mathematical creativity is essential for the growth of overall mathematical ability (or giftedness), while Lev-Zamir and Leikin (2013) show how challenging mathematical problems and flexible teaching can help the development of mathematical creativity. Both studies show the dynamic aspect of mathematical creativity, in the sense that it evolves and is influenced by other external factors.

## Cross-References

▶ Giftedness and High Ability in Mathematics

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## Critical Mathematics Education

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### Keywords

Mathematics education for social justice; Critical mathematics education; Ethnomathematics; Mathematics in action; Students' foregrounds; Mathemacy; Landscapes of investigation; Mathematization; Dialogic teaching and learning

### Definition

Critical mathematics education can be characterized in terms of concerns: to address social exclusion and suppression, to work for social justice in whatever form possible, to try open new possibilities for students, and to address critically mathematics in all its forms and application.

### Characteristics

#### Critical Education

Inspired by the students' movement, a New Left, peace movements, feminism, antiracism, and critical education proliferated. A huge amount of literature became published, not least in Germany, and certainly the work of Paulo Freire was recognized as crucial for formulating radical educational approaches.

However, critical education was far from expressing any interest in mathematics. In fact, with reference to the Frankfurt School, mathematics was considered almost an obstruction to critical education. Thus, Habermas, Marcuse, and many others associated instrumental reason with, on the one hand, domination, and, on the other hand, the rationality cultivated by natural science and mathematics. Mathematics appeared as the grammar of instrumental reason. How could one imagine any form of emancipatory interests being associated to this subject?

#### Steps into Critical Mathematics Education

Although there were no well-defined theoretical frameworks to draw on, there were from the beginning of the 1970s many attempts in formulating a critical mathematics education. Let me mention some publications.

The book *Elementarmathematik: Lernen für die Praxis* (*Elementary mathematics: Learning for the praxis*) by Peter Damerow, Ulla Elwitz, Christine Keitel, and Jürgen Zimmer from 1974 was crucial for the development of critical mathematics education in a German context. In the article "Plädoyer für einen problemorientierten Mathematikunterrich in emanzipatorisher Absicht" ("Plea for a problem-oriented mathematics education with an emancipatory aim") from 1975, Dieter Volk emphasized that it is possible to establish mathematics education as a critical education. The book *Indlæring som social proces* (*Learning as a social process*) by Stieg Mellin-Olsen was published in 1977. It provided an opening of the political dimension of mathematics education, a dimension that was further explored in Mellin-Olsen (1987). *Indlæring som social proces* was crucial for the development of critical mathematics education in the Scandinavian context. An important overview of Mellin-Olsen's work is found in Kirfel and Lindén (2010). Dieter Volk's *Kritische Stichwörter zum Mathematikunterricht* (*Critical notions for mathematics education*) from 1979 provided a broad overview of what could be called the first wave in critical mathematics education. Soon after followed, in Danish, Skovsmose (1980, 1981a, b).

Marilyn Frankenstein (1983) provided an important connection between critical approaches in mathematics education and the outlook of Freire, and in doing so she was the first in English to formulate a critical mathematics education (see also Frankenstein 1989). Around 1990, together with Arthur Powell and several others, she formed the critical mathematics education group, emphasizing the importance of establishing a united concept of critique and mathematics (see Frankenstein 2012; Powell 2012). Skovsmose (1994) provided an interpretation of critical mathematics education and Skovsmose (2012) a historical perspective.

Critical mathematics education developed rapidly in different directions. As a consequence, the very notion of critical mathematics education came to refer to a broad range of approaches, such as mathematics education for social justice (see, for instance, Sriraman 2008; Penteado and Skovsmose 2009; Gutstein 2012), pedagogy of dialogue and conflict (Vithal 2003), responsive mathematics education (Greer et al. 2009), and, naturally, critical mathematics education. Many ethnomathematical studies also link closely with critical mathematics education (see, for instance, D'Ambrosio 2006; Knijnik 1996; Powell and Frankenstein 1997).

## Some Issues in Critical Mathematics Education

Critical mathematics education can be characterized in terms of concerns, and let me mention some related to mathematics, students, teachers, and society:

- ***Mathematics*** can be brought in action in technology, production, automatization, decision making, management, economic transaction, daily routines, information procession, communication, security procedures, etc. In fact, mathematics in action plays a part in all spheres of life. It is a concern of a critical mathematics education to address mathematics in its very many different forms of applications and practices. There are no qualities, like objectivity and neutrality, that automatically can be associated to mathematics. Mathematics-based actions can have all kind of qualities, being risky, reliable, dangerous, suspicious, misleading, expensive, brutal, profit generating, etc. Mathematics-based action can serve any kind of interest. As with any form of action, so also mathematics in action is in need of being carefully criticized. This applies to any form of mathematics: everyday mathematics, engineering mathematics, academic mathematics, and ethnomathematics.
- ***Students***. To a critical mathematics education, it is important to consider students' interests, expectations, hopes, aspirations, and motives. Thus, Frankenstein (2012) emphasizes the importance of respecting student knowledge. The notion of students' foregrounds has been suggested in order to conceptualize students' perspectives and interests (see, for instance, Skovsmose 2011). A foreground is defined through very many parameters having to do with economic conditions, social-economic processes of inclusion and exclusion, cultural values and traditions, public discourses, and racism. However, a foreground is, as well, defined through the person's experiences of possibilities and obstructions. It is a preoccupation of critical mathematics education to acknowledge the variety of students' foregrounds and to develop a mathematics education that might provide new possibilities for the students. The importance of recognizing students' interest has always been a concern of critical mathematics education.
- ***Teachers***. As it is important to consider the students' interests, it is important to consider the teachers' interests and working conditions as well. Taken more generally, educational systems are structured by the most complex sets of regulations, traditions, and restrictions, which one can refer to as the "logic of schooling." This "logic" reflects (if not represents) the economic order of today, and to a certain degree it determines what can take place in the classroom. It forms the teachers' working conditions. It becomes important to consider the space of possibilities that might be left open by this logic. These considerations have to do with the micro–macro (classroom-society) analyses as in particular addressed by Paola Valero (see, for instance, Valero 2009). Naturally, these comments apply not only to the teachers' working conditions but also to the students' conditions for learning. While the concern about the students' interests has been part of critical

mathematics education right from the beginning, a direct influence from the students' movements, the explicit concern about teaching conditions is a more recent development of critical mathematics education.

- ***Society*** can be changed. This is the most general claim made in politics. It is the explicit claim of any activism. And it is as well a concern of critical mathematics education. Following Freire's formulations, Gutstein (2006) emphasizes that one can develop a mathematics education which makes it possible for students to come to read and write the world: "read it," in the sense that it becomes possible to interpret the world filled with numbers, diagrams, figures, and mathematics, and "write it," in the sense that is becomes possible to make changes. However, a warning has been formulated: one cannot talk about making sociopolitical changes without acknowledge the conditions for making changes (see, for instance, Pais 2012). Thus, the logic of schooling could obstruct many aspirations of critical mathematics education. Anyway, I find that it makes good sense to articulate a mathematics education for social justice, not least in a most unjust society.

## Some Notions in Critical Mathematics Education

Notions such as social justice, mathemacy, dialogue, and uncertainty together with many others are important for formulating concerns of critical mathematics education. In fact we have to concern ourselves with clusters of notions of which I highlight only a few:

- ***Social justice***. Critical mathematics education includes a concern for addressing any form of suppression and exploitation. As already indicated, there is no guarantee that an educational approach might in fact be successful in bringing about any justice. Still, working for social justice is a principal concern of critical mathematics education. Naturally, it needs to be recognized that "social justice" is a most open concept, the meaning of which can be explored in many different directions. Addressing *equity* also represents concerns of critical mathematics education, and the discussion of social justice and equity bring us to address processes of *inclusion* and *exclusion*. Social exclusion can take the most brutal forms being based on violent discourses integrating racism, sexism, and hostility towards "foreigners" or "immigrants." Such discourses might label groups of people as being "disposable," "a burden," or "nonproductive," given the economic order of today. It is a concern of critical mathematics education to address any form of social exclusion. As an example I can refer to Martin (2009). However, social inclusion might also represent a questionable process: it could mean an inclusion into the capitalist mode of production and consumption. So critical mathematics education needs to address inclusion–exclusion as contested processes. However, many forms of inclusion–exclusion have until now not been discussed profoundly in mathematics education: the conditions of blind students, deaf students, and students with different handicaps – in other words, students with particular rights. However, such issues are now being addressed in the research environment created by the Lulu Healy and Miriam Goody Penteado in Brazil. Such initiatives bring new dimensions to critical mathematics education.
- ***Mathemacy*** is closely related to literacy, as formulated by Freire, being a competence in reading and writing the world. Thus, D'Ambrosio (1998) has presented a "New Trivium for the Era of Technology" in terms of literacy, matheracy, and technoracy. Anna Chronaki (2010) provided a multifaceted interpretation of mathemacy, and in this way it is emphasized that this concept needs to be reworked, reinterpreted, and redeveloped in a never ending process. Different other notions have, however, been used as well for these complex competences, including *mathematical literacy* and *mathematical agency*. Eva Jablonka (2003) provides a clarifying presentation of

mathematical literacy, showing how this very notion plays a part in different discourses, including some which hardly represent critical mathematics education. The notion of mathematical agency helps to emphasize the importance of developing a capacity not only with respect to understanding and reflection but also with respect to acting.

- ***Dialogue***. Not least due to the inspiration from Freire, the notion of dialogue has played an important role in the formulation of critical mathematics education. Dialogic teaching and learning has been presented as one way of developing broader critical competences related to mathematics. Dialogic teaching and learning concerns forms of interaction in the classroom. It can be seen as an attempt to break at least some features of the logic of schooling. Dialogic teaching and learning can be seen as a way of establishing conditions for establishing mathemacy (or mathematical literacy, or mathematical agency). *Problem-based learning* and *project work* can also be seen as way of framing a dialogic teaching and learning.
- ***Uncertainty***. Critique cannot be any dogmatic exercise, in the sense that it can be based on any well-defined foundation. One cannot take as given any particular theoretical basis for critical mathematics education; it is always in need of critique (see, for instance, Ernest 2010). In particular one cannot assume any specific interpretation of social justice, mathemacy, inclusion–exclusion, dialogue, critique, etc. They are all contested concepts. There is no particular definition of, say, social justice that one can take as a given. We have to do with concepts under construction.

### Critical Mathematics Education for the Future

The open nature of critical mathematics education is further emphasized by the fact that forms of exploitations, suppressions, environmental problems, and critical situations in general are continuously changing. Critique cannot develop according to any preset program.

For recent developments of critical mathematics education, see, for instance, Alrø, Ravn, and Valero, (Eds.) (2010), Wager, A. A. and Stinson, D. W. (Eds.) (2012); and Skovsmose and Greer (Eds.) (2012). Looking a bit into the future much more is on its way. Let me just refer to some doctoral studies in progress that I am familiar with. Denival Biotto Filho is addressing students in precarious situations and in particular their foregrounds. Raquel Milani explores further the notion of dialogue, while Renato Marcone addresses the notion of inclusion–exclusion, emphasizing that we do not have to do with a straightforward good-bad duality. Inclusion could also mean an inclusion into the most questionable social practices.

Critical mathematics education is an ongoing endeavor. And naturally we have to remember that as well the very notion of critical mathematics education is contested. There are very many different educational endeavors that address critical issues in mathematics education that do not explicitly refer to critical mathematics education. And this is exactly as it should be as the concerns of critical mathematics cannot be limited by choice of terminology.

## Cross-References

▶ Critical Thinking in Mathematics Education
▶ Dialogic Teaching and Learning in Mathematics Education
▶ Mathematical Literacy
▶ Mathematization as Social Process

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# Critical Thinking in Mathematics Education

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## Keywords

Logical thinking; Argumentation; Deductive reasoning; Mathematical problem solving; Critique; Mathematical literacy; Critical judgment; Goals of mathematics education

## Characteristics

Educational psychologists frame critical thinking (CT) as a set of generic thinking and reasoning skills, including a disposition for using them, as well as a commitment to using the outcomes of CT as a basis for decision-making and problem solving. In such descriptions, CT is established as a general standard for making judgments and decisions. Some descriptions of CT activities and skills include a sense for fairness and the assessment of practical consequences of decisions as characteristics of CT (e.g., Paul and Elder 2001). This assumes autonomous subjects who share a common frame of reference for representation of facts and ideas, for their communication, as well as for appropriate (morally "good") action. Important is also the difference as to what extent a critical examination of the criteria for CT is included in the definition: If education for CT is conceptualized as instilling a belief in a more or less fixed and shared system of skills and criteria for judgment, including associated values, then it seems to contradict its very goal. If, on the other hand, education for CT aims at overcoming potentially limiting frames of reference, then it needs to allow for transcending the very criteria assumed for legitimate "critical" judgment. The dimension of not following rules and developing a fantasy for alternatives connects CT with creativity and change. In Asian traditions derived from the Mãdhyamika Buddhist philosophy, critical deconstruction is a method of examining possible alternative standpoints on an issue, which might amount to finding self-contradictions in all of them (Fenner 1994). When combined with meditation, the deconstruction provides for the student a path towards spiritual insight as it amounts to a freeing from any form of dogmatism. The position coincides with some postmodern critiques of purely intellectual perspectives that lack contact with experience and is echoed in some European traditions of skepticism (Garfield 1990). Hence, paradoxical deconstruction appears more radical than CT as it includes overcoming the methods and frames of reference of previous thinking and of purely intellectual plausibility.

The role assigned to CT in mathematics education includes CT as a by-product of mathematics learning, as an explicit goal of mathematics education, as a condition for mathematical problem solving, as well as critical engagement with issues of social, political, and environmental relevance by means of mathematical modeling and statistics. Such engagement can include a critique of the very role mathematics plays in these contexts. In the mathematics education literature, explicit reference to CT as defined in educational psychology is not very widespread, but general mathematical problem-solving skills are commonly associated with critical thinking, even though such association remains under-theorized. On the other hand, the notion of critique, rather than CT, is employed in the mathematics education literature in various programs related to critical mathematics education.

### Critical Thinking and Mathematical Reasoning

Mathematical argumentation features prominently as an example of disciplined reasoning based on clear and concise language, questioning of assumptions, and appreciation of logical inference for deriving conclusions. These features of mathematical reasoning have been contrasted with intuition, associative reasoning, justification by example, or induction from observation. While the latter are also important aspects of mathematical inquiry, a focus on logic is directed towards extinguishing subjective elements from judgments and it is the essence of deductive reasoning. Underpinned by the values of rationalism and objectivity, reasoning with an emphasis on logical inference is opposed to intuition and epiphany as a source of knowledge and viewed as the counterinsurance against dogmatism and opportunism.

The enhancement of students' general reasoning capacity has for quite some time been seen as a by-product of engagement with mathematics. Francis Bacon (1605), for example, wrote that it would "remedy and cure many defects in the wit and faculties intellectual. For if the wit be too dull, they [the mathematics] sharpen it; if too wandering, they fix it; if too inherent in the sense, they abstract it" (VIII (2)). Even though this promotion of mathematics education is based on its alleged value for developing generic thinking or reasoning skills, these skills are not called "critical thinking."

Historically, the notion of critique was tied to the tradition of rhetoric and critical evaluation of texts. Only through the expansion of the function of critique towards general enlightenment, critique became a generic figure of thinking, arguing, and reasoning. This more general notion transcends what is usually associated with accuracy and rigor in mathematical reasoning. Accordingly, CT in mathematics education not only is conceptualized as evaluating rigor in definitions and logical consistency of arguments but also includes attention to informal logic and heuristics, to the point of identifying problem-solving skills with CT (e.g., O'Daffer and Thomquist 1993). Applebaum and Leikin (2007), for example, see the faculty of recognizing contradictory information and inconsistent data in mathematics tasks as a demonstration of CT.

However, as most notions of CT include an awareness of the subject doing it, neither a mere application of logical inference nor successful application of mathematical problem-solving skills would reasonably be labeled as CT. But as a consequence of often identifying CT with general mathematical reasoning processes embedded in mathematical problem solving, there is a large overlap of literature on mathematical reasoning, problem solving, and CT.

There is agreement that CT does not automatically emerge as a by-product of any mathematics curriculum, but only with a pedagogy that draws on students' contributions and affords processes of reasoning and questioning when students collectively engage in intellectually challenging tasks. Fawcett (1938), for example, suggested that teachers (in geometry instruction) should make use of students' disposition for critical thinking and that this capacity can be harnessed and cultivated by an appropriate choice of pedagogy. Reflective thinking practices could be enacted when drawing the students' attention to the need for clear definition of key terms in statements, for examination of alleged evidence, for exposition of assumptions behind their beliefs, and for evaluation of arguments and conclusions. Fawcett's teaching experiments included the critical examination of everyday notions. A more recent example of a pedagogical approach with a focus on argumentation is the organization of a "scientific debate" in the mathematics classroom (Legrand 2001), where students in an open discussion defend their own ideas about a conjecture, which may be prepared by the teacher or emerge spontaneously during class work.

While cultivating some form of discipline-transcending CT has long been promoted by mathematics educators, explicit reference to CT is not very common in official mathematics curriculum documents internationally. For example, "critical thinking" is not mentioned in the US Common Core Standards for Mathematics (Common Core State Standards Initiative 2010). However, in older recommendations from the US National Council of Teachers of Mathematics, mention of "critical thinking" is made in relation to creating a classroom atmosphere that fosters it (NCTM 1989). A comparative analysis of associations made between mathematics education and CT in international curriculum documents remains a research desideratum.

Notions of CT in mathematics education with a focus on argumentation and reasoning skills have in common that the critical competence they promote is directed towards claims, statements, hypotheses, or theories ("texts"), but do include neither a critique of the social realities, in which these texts are produced, nor a critique of the categories, in which these texts describe realities. As it is about learning how to think, but not what to think about, this notion of CT can be taken to implicate a form of thinking without emotional or moral commitment. However, the perspective includes the idea that the same principles that guide critical scientific inquiry could also guide successful problem solving in social and moral matters and this would lead to improvement of society, an idea that was, for example, shared by Dewey (Stallman 2003). Education for CT is then by its nature emancipatory.

## Critical Thinking and Applications of Mathematics

For those who see scientific standards of reasoning as limited, the enculturation of students into a form of CT derived from these standards alone cannot be emancipatory. Such a view is based on a critique of Enlightenment's scientific image of the world. The critique provided by the philosophers of the Frankfurt School is taken up in various projects of critical mathematics education and critical mathematical literacy. This critique is based on the argument that useful things are conflated with calculable things and thus formal reasoning based on quantification, which is made possible through the use of mathematics, is purely instrumental reasoning. Mathematics educators have pointed out that reliance on mathematical models implicates a particular worldview and mathematics education should widen its perspective and take critically into account ethical and social dimensions (e.g., Steiner 1988). In order to cultivate CT in the mathematics classroom, reflection not only of methodological standards of mathematical models but also of the nature of these standards themselves, as well as of the larger social contexts within which mathematical models are used, has been suggested (e.g., Skovsmose 1989; Keitel et al. 1993; Jablonka 1997; Appelbaum and Davila 2009; Fish and Persaud 2012). Such a view is based on acknowledging the interested nature of any application of mathematics. This is not to dismiss rational inquiry, but aims at expanding rationality beyond instrumentality through inclusion of moral and political thought. Such an expansion is seen as necessary by those who see purely formally defined CT as ultimately self-destructive and hence not emancipatory.

## Limitations of Developing CT Through Mathematics Education

The take-up of poststructuralist and psychoanalytic theories by mathematics educators has afforded contributions that hold CT up for scrutiny. Based on the postmodern acknowledgment that all forms of reasoning are only legitimized through the power of some groups in society, and in line with critics who see applied mathematics as the essence of instrumental reason, an enculturation of students into a form of CT embedded in mathematical reasoning must be seen as disempowering. As it excludes imagination, fantasy, emotion, and the particular and metaphoric content of problems, this form of CT is seen as antithetical to political thinking or social commitment (Walkerdine 1988; Pimm 1990; Walshaw 2003; Ernest 2010). Hence, the point has been made that mathematics education, if conceptualized as enculturation into dispassionate reason and analysis, limits critique rather than affording it and it might lead to political apathy.

## Further Unresolved Issues

Engaging students in collaborative CT and reasoning in mathematics classrooms assumes some kind of an ideal democratic classroom environment, in which students are communicating freely. However, classrooms can hardly be seen as ideal speech communities. Depending on their backgrounds and educational biographies, students will not be equally able to express their thoughts and not all will be guaranteed an audience. Further, the teacher usually has the authority to phrase the questions for discussion and, as a representative of the institution, has the obligation to assess students' contributions. Thus, even if a will to cultivate some form of critical reasoning in the mathematics classroom might be shared amongst mathematics educators, more attention to the social, cultural, and institutional conditions under which this is supposed to take place needs to be provided by those who frame CT as an offshoot of mathematical reasoning. Further, taxonomies of CT skills, phrased as metacognitive activities, run the risk of suggesting to treat these explicitly as learning objectives, including the assessment of the extent to which individual students use them. Such a didactical reification of CT into measurable learning outcomes implicates a form of dogmatism and contradicts the very notion of CT.

The antithetical character of the views of what it means to be critical held by those who see CT as a mere habit of thought that can be cultivated through mathematical problem solving, on the one hand, and mathematics educators inspired by critical theory and critical pedagogy, on the other hand, needs further exploration.

Attempts to describe universal elements of critical reasoning, which are neither domain nor context specific, reflect the idea of rationality itself, the standards of which are viewed by many as best modeled by mathematical and scientific inquiry. The extent to which this conception of rationality is culturally biased and implicitly devalues other "rationalities" has been discussed by mathematics educators, but the implications for mathematics education remain under-theorized.

## Cross-References


- ▶ Argumentation in Mathematics Education
- ▶ Authority and Mathematics Education
- ▶ Critical Mathematics Education
- ▶ Dialogic Teaching and Learning in Mathematics Education
- ▶ Didactic Contract in Mathematics Education
- ▶ Logic in Mathematics Education
- ▶ Mathematical Modelling and Applications in Education
- ▶ Mathematical Proof, Argumentation, and Reasoning
- ▶ Mathematization as Social Process
- ▶ Metacognition
- ▶ Problem Solving in Mathematics Education
- ▶ Questioning in Mathematics Education


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## Cultural Diversity in Mathematics Education

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### Keywords

Cultural identity and learning; Vygotsky; Home and school mathematics; Immigrant students; Minority students; Cultural discontinuity; Sociocultural approaches

### Introduction

Cultural diversity in mathematics education is a widely used expression to discuss questions around why students from different cultural, ethnic, social, economic, and linguistic groups perform differently in their school mathematics. These questions are not new in cultural perspectives to mathematics education developed since the late 1980s (Bishop 1988) and in cultural approaches to mathematical cognition (Cole 1996). However, until recently issues of cultural diversity were considered to be out there in other non-Western cultures or to be issues of marginalized and poor groups in society. Globalization changed this perspective. With changes in communication, technologies, and unprecedented levels of migration, cultures have become increasingly complex, connected, and heterogeneous. One of the major impacts on education has been a substantial change in the cultural and ethnic composition of the school population.

Schools and classrooms become places where teachers, students, and parents are exposed to and have to respond to many types of cultural differences. For many these differences are resources enriching the learning opportunities and environments. For many others, diversity is experienced as a problem, which is reflected in school achievement (Secada 1995). The issues cultural diversity poses to education have many facets and have been approached from different perspectives in social sciences (De Haan and Elbers 2008). Conceptions of culture and the role of culture in psychological development inform these perspectives. Examining culture as a way of life of specific cultural groups has contributed to the understanding of cultural discontinuities between schools and the home background of the students. In this perspective, the emphasis has been on the shared cultural practices of the group. A more recent perspective focuses on more dynamic aspects of culture, i.e., on the way a person experiences participation in multiple practices, and the production of new cultural knowledge, meaning, and identities. Mathematics education research draws on these perspectives but also considers issues that are specific to mathematics learning (Cobb and Hodge 2002; Nasir and Cobb 2007; Abreu 2008; Gorgorió and Abreu 2009).

Here the focus is on the development of ideas that examine mathematics as a form of cultural knowledge (Bishop 1988; Asher 2008) and learning as a socioculturally mediated process (Vygotsky 1978). These ideas offer a critique to approaches that locate the sources of diversity in the autonomous individual mind. More importantly, sociocultural approaches have contributed to rethinking cultural diversity as "relational" and "multilayered" phenomena, which can be studied from different angles (Cobb and Hodge 2002; De Haan and Elbers 2008). Empirical research following these approaches has evolved from an examination of diversity between cultural groups, i.e., the nature of mathematical knowledge specific to cultural practices, to an examination of the person as a participant in specific sociocultural practices.

## Diversity and Uses of Cultural Mathematical Tools

A driving force for researching the impact of cultural diversity in mathematics education has been to understand why certain cultural groups experience difficulties in school mathematics. In the culture-free view of mathematics, poor performance in school mathematics was explained in terms of deficits, namely, cognitive deficits that could be the result of cultural deficits. However, since the 1980s, this view has become untenable. Researchers exploring the difficulties non-Western children, such as the Kpelle children in Liberia, experienced with Western-like mathematics introduced with schooling (Cole 1996) realized that their difficulties could not be explained by cognitive deficits or cultural deficits. They discovered that differences in mathematical thinking could be linked to the tools used as mediators. Thus, for instance, the performance in a mathematical task, such as estimating length, was linked to the use of a specific cultural measuring system. With the advance of cultural research and the view of mathematics and cognition as cultural phenomena, alternative explanations of poor performance in school mathematics have been put forward in terms of cultural differences.

Drawing on the insights from examining the mathematics of particular cultural groups research moved to explore cultural differences within societies, which is still the major focus of current research on cultural diversity in mathematics education. A classic example of this research is the "street mathematics" investigations in Brazil by Nunes, Schliemann, and Carraher (1993). In a series of studies that started with street children, Nunes and her colleagues examined differences between school mathematics and out-of-school mathematics. Their findings added support to the notion that mathematical thinking was mediated by cultural tools, such as oral and written arithmetic. The within society studies also highlighted the situated nature of mathematical cognition. Depending on the context of the practice, the same person may draw on different cultural tools; they can call on an oral method to solve a shopping problem and a written method to solve a school problem.

How cultural tools mediate mathematical thinking and learning continues to be a key aspect in investigations in culturally diverse classrooms. Research with minority and immigrant students in different countries shows that the students learned often to use different forms of mathematics at home and at school (Bishop 2002; Gorgorió et al. 2002; Abreu 2008). Similarly, research with parents shows that they refer often to differences in their methods and the ones their children are being taught in school. To sum up, research shows that students from culturally diverse backgrounds are exposed often to different cultural tools in different contexts of mathematical practices. It also suggests that many students experience cultural discontinuities in their transitions between contexts of mathematical practices. A cultural discontinuity perspective offers only a partial account of the impact of diversity, however. The fact that students from similar home cultural groups perform differently at school requires research to consider other aspects of diversity. A fruitful way of continuing to explore the different impacts of diversity in school mathematical learning focuses on how the person as a participant in mathematical practices makes sense of their experiences. The person here can be, for example, an immigrant student in a mathematics classroom, a parent that supports their children with their school homework, and a teacher that is confronted with students from cultural backgrounds they are not familiar with. Here the focus turns to culture as being reconstructed in contexts of practices, and issues of identity and social representations are foregrounded.

## Diversity and Cultural and Mathematical Identities

Many studies with immigrant and minority students have now illustrated that they become aware of the differences between their home culture and their school practices (Bishop 2002; see also ► Immigrant Students in Mathematics Education). Accounts from parents of their experiences of supporting their children's school mathematics at home (e.g., homework) also illustrate the salience of differences between home and school mathematics. These could be experienced in terms of (a) the content of school mathematics and in the strategies used for calculations, (b) the methods of teaching and the tools used in teaching (e.g., methods for learning times tables, use of calculators), (c) the language in which they learned and felt confident doing mathematics, and (d) the parents' and the children's school mathematical identities. Though all the dimensions are important, this research shows that identities take a priority in the way the parents organize their practices to support their children. The societal and institutional valorization of mathematical practices plays a role on this process (Abreu 2008).

Recent studies also show that students talk about differences in relation to how they perceive their home cultural identities as intersecting with their school mathematical learning. Studies with students from minority ethnic backgrounds in England whose parents had been schooled in other countries show that differences between school mathematical practices at home and at school have implications on their mathematical identities. For example, some students report trying to separate home and school, i.e., to use the "home way" at home and the "school way" at school. The reason provided for the separation is that they do not feel that the home ways are valued at school. Other students simply claim that their parents do not know or that their knowledge is old fashioned. In both cases, the construction of a positive school mathematical identity involves suppressing the home mathematical identity (Crafter and Abreu 2010). Identities, as socially constructed, can then be conceptualized as powerful mediators in the way diversities are being constructed in the context of school practices. Indeed, studies examining other types of diversity, such as gender, have also implied similar processes (Boaler 2007).

Studies with immigrant students with a history of success in their school mathematical learning in their home country are also particularly interesting to illustrate the intersection of identities. Firstly, the difficulties of these students cannot be easily attributed to the individual mathematical ability as they have a personal history of being "good mathematics students." Secondly, in this case the cultural diversity is already internalized as part of the student's previous schooling. These students' positive school mathematical identities get disrupted when they receive low grades in the host country school mathematics. Suddenly, the students' common representation that mathematics is just about numbers and formulae and that these are the same everywhere is challenged. It is revealing that young people from different immigrant backgrounds and going to school in different countries report similar experiences (e.g., Portuguese students in England; Ecuadorian students in Catalonia, Spain). This can be interpreted as evidence that when a student joins a mathematical classroom in a new cultural context, their participation is mediated by representations of what counts as mathematical knowledge. These examples illustrate a culture-free view of mathematics that is still predominant in many educational systems but that could be detrimental to immigrant students' academic mathematical careers. Having shown that issues of diversity are very salient in the experiences of students and their parents, the next section briefly examines teachers' representations.

## Diversity and Teachers' Social Representations of Cultural Differences

In many schools, teachers, who have trained to teach monolingual and monocultural students from their own culture, teach students who may speak a different language and come from cultures they are not familiar with. However, in communities with a stronger tradition of receiving immigrants, some teachers themselves have already had to negotiate the practices of the home and school culture. This complex situation may add insight into the ways that cultural differences and identities come to be constructed as significant for the school mathematical learning. An examination of studies carried out in culturally diverse schools in Europe reveals two views in the way teachers make sense of the cultural and ethnic background on their students' mathematical learning (Abreu and Cline 2007; Gorgorió and Abreu 2009). One view stresses "playing down differences" and the other "accepting differences." The view of playing down cultural differences draws upon representations of mathematics as a culture-free subject (that it is the same around the world). This view can also draw on a representation of the child's ability as the key determinant factor in their mathematical learning. The universal construction of children takes priority over their ethnic and cultural backgrounds. Treating everyone as equal based on their merits is also used as a justification for not taking into account cultural differences. The lack of recognition of the cultural nature of mathematical practices may restrict opportunities for students to openly negotiate the differences at school. This way, diversity may become a problem instead of a resource. The alternative positioning of accepting cultural differences represents a minority voice outside the consensus that mathematics is a culture-free subject and that ability is the main factor in the mathematical learning.

## Conclusion

Diversity in mathematics education includes complex and multilayered phenomena that can be explored from different perspectives. Drawing on sociocultural psychology, empirical research on uses and learning of mathematics in different cultural practices offered key insights on understandings of cultural diversity considering (i) mathematical tools (the specific forms of mathematical knowledge associated with cultural groups and sociocultural practices), (ii) identities (the ways differences are experienced by the students and the impact on how they construct themselves as participants in these practices), and (iii) social representations (the images and understandings that enable people to make sense of mathematical practices, such as images of learners and the learning process and views of mathematical knowledge). These understandings emerged from looking at diversity from complementary perspectives. One perspective focuses on the discontinuities between the cultural practices, and the other on how discontinuity is experienced by the person as a participant in school mathematical practices. This second perspective is more recent and is key for the development of approaches where diversity becomes a resource. The extent to which approaches that stress the importance of cultural identities can be used as resources for change from culture-free to culturally sensitive practices in mathematics education is a question for further research. The fact that the views of cultural identities as mediators of school mathematical learning are still marginalized can be seen as a consequence of the dominant cultural practices and representations. For example, this can include practices in teacher training, where little attention is given to preparing teachers to understand the cultural nature of (mathematical) learning and human development (see also, ▶ Immigrant Students in Mathematics Education). Secondly, implicit conceptions of the social and emotional development of the child at school draw on representations of childhood which often do not take into account the cultural diversity of current societies.

## Cross-References

- ▶ Ethnomathematics
- ▶ Immigrant Students in Mathematics Education
- ▶ Situated Cognition in Mathematics Education
- ▶ Theories of Learning Mathematics

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## Cultural Influences in Mathematics Education

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### Keywords

Achievement; Equity; Efficacy; Gender; Gender differences; Social contexts

### Definition

For many students, mathematics is a barrier, often having a profound effect on their further educational and career opportunities. Some authors have looked beyond features of students, curriculum, and pedagogy to argue that the political, economic, and social contexts of schooling also need to be considered in this pedagogical equation (Apple 1992; Tate 1997). Similarly, some scholars have studied cultural issues affecting the study of mathematics, including features of the societal culture in which education is situated and how those cultures affect who succeeds in them (Else-Quest et al. 2010). In this essay, we examine cultural influences, both within and surrounding mathematics, and their effects on who succeeds in mathematics.

### A Review of the Literature

A substantial body of research has been devoted to understanding the relatively small proportion of women and students of color who participate in advanced mathematics and careers. Researchers have explored the reasons for the differences in the achievement, attitudes, learning styles, strategy use, and persistence between girls and boys and among students of different races, ethnicities, social classes, and language proficiencies. Although these differences have generally decreased, differences among groups remain, as do important differences among countries (Else-Quest et al. 2010; Palsdottir and Sriraman 2010). Unfortunately, the work of many researchers has had the paradoxical effect of creating a discourse that females and students of color "can't do math" (Fennema 2000). As a result, the identification of "who" succeeds in mathematics is too often perceived through the lens of a deficit model: When groups of students do not succeed or persist in mathematics, the reason, or so it is sometimes framed in the literature, is a problem with the students in groups themselves, rather than as the result of a broader social or cultural issue.

Building students' sense of belongingness in mathematics has been proposed as a critical feature of an equitable K-12 education (Allexsaht-Snider and Hart 2001; Ladson-Billings 1997). Martha Allexsaht-Snider and Laurie Hart (2001) defined belonging as "the extent to which each student senses that she or he belongs as an important and active participant" in mathematics (p. 97) and have argued that an important purpose of schooling is to facilitate students' sense of belongingness and engagement with mathematics. A similar construct has been proposed at the doctoral level, with several authors arguing belonging to or integrating into the departmental communities is important for student persistence (e.g., Herzig 2002, 2004a, 2010).

There are (at least) two cultural aspects to students' development of a sense that they belong in mathematics: (1) features of mathematics itself, as mathematics is presented in classrooms, and (2) the way the broader society perceives mathematics ability and the students who succeed in math.

First, mathematics is often taught in highly abstracted ways, with little or no explicit connection to other mathematical ideas, ideas outside of mathematics, or the mathematical "big picture" (Herzig 2002; Stage and Maple 1996). Some feminist scholars have challenged the predominance of abstraction in mathematics. Betty Johnston (1995) argued that abstraction in mathematics is a consequence of modern industrial society, which itself is based on the idea of separating things into manageable pieces, distinct from their context. This abstraction of mathematics denies the social nature of mathematics. In an abstract context like the one that is common in Western mathematics, a quest for certain types of understanding can actually interfere with success, as when students look to understand, for example, What does this have to do with the world? With my world? With my life? (Johnston 1995). Mathematics is often taught as a set of manipulations that lead to predetermined results or, at a more advanced level, as sequence of deductive proofs of clearly stated theorems, with little (if any) representation of the roles of intuition, creativity, insight, or trial and error, which give rise to those results and which give them meaning (Herzig 2002, 2010; Sriraman and Steinthorsdottir 2007). Mathematics as it is commonly presented in classrooms in education is isolated from its social and personal contexts and applications, devoid of aesthetic considerations.

Aside from the way mathematics is presented in the classroom, the way that mathematics students are perceived outside the mathematics classroom also affects students' involvement and dedication to mathematics (Campbell 1995; Damarin 2000).

As Noddings (1996) argued, mathematics educators need to find ways to make the social world of mathematics – its culture – more accessible to a broader range of people, and the world outside of mathematics needs to change its perception of those who succeed within it. Only then can more students, including females and people of color, find a way come to feel that they truly belong in some part of the mathematics world.

Suzanne Damarin (2000) compared people with mathematical ability to "marked categories" such as women, people of color, criminals, people of disability, and homosexuals and identified these characteristics:

1. Members of marked categories are ridiculed and maligned, and descriptions of marked categories are used to harass, tease, and discipline members of the larger society.
2. Members of marked categories are portrayed as incompetent in dealing with daily life.
3. In institutions designed to meet the needs of all, the needs of members of marked categories are deferred to the needs of the members of unmarked categories.

4. Members of marked categories are feared as powerful even as they are marked as powerless.
5. Explicit or social marking serves to define communities of the marked.
6. Membership in multiple marked categories places individuals in the margins of each marked community.
7. The study of a marked category leads to the construction and study of the complementary class of people.
8. The unmarked category is generally larger than the marked category; even when this is not the case, the marked category is not recognized as the majority (Damarin 2000, pp. 72–74).

Given the common perceptions of mathematics students as being white, male, childless, and socially inept, having few interests outside of mathematics, students who explicitly do not fit with this described group might conclude that they do not wish to fit in. Thus belonging in mathematics might not be an entirely good thing, as it "marks" a student as deviant and as socially inept. Herzig (2004b) found that some female graduate students described ways that they worked to distance themselves from some of these common constructions of ineptness and social deviance, which, paradoxically, led them to resist belonging in mathematics.

Damarin (2000) argued that membership in the deviant category provides the "deviant" with a community with which to affiliate: Being identified and marked as mathematically able encourages mathematics students to form a community among themselves – if there are enough of them and if they have the social facility needed. Unfortunately, females are members of (at least) two marked categories, and the double marking is not merely additive: That is, females are constructed as deviant as females separately within each marked category in which they are placed. First, they are marked as girls and women, but among girls and women, their mathematical ability defines them as deviant. Second, given common stereotypes of mathematics as a male domain, mathematical women are marked among mathematicians as not actually being mathematicians. For women of color, the marking is threefold and even more complex, making women of color "deviant" within each of the communities to which they belong.

In summary, researchers have made great strides in understanding why mathematics has generally attracted to certain types of students. Rather than studying what is different about women and minorities – groups that have been viewed as unsuccessful in mathematics – studies now strive to ascertain cultural and societal obstacles for these groups. In addition, the literature has shown that students are most engaged when in an educational environment that fosters belonging, which can be difficult in the mathematics field. The stereotypical views of mathematics students can make it particularly challenging for women and minorities to enter the field. In addition, the mathematically capable may not wish to be socially or culturally marked as such due to the preconceived notions many have of mathematics students. However, by understanding the cultural and societal issues in the mathematics field, researchers and educators can begin to implement policies and strategies to create more equitable learning environments and atmospheres.

## Cross-References

▶ Critical Mathematics Education
▶ Cultural Diversity in Mathematics Education

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# Curriculum Resources and Textbooks in Mathematics Education

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## Keywords

Curriculum resources; ICT; Internet; Professional development; Teacher knowledge; Textbooks; Use of resources

## Definition of Curriculum Resources

We define mathematics curriculum resources as all the resources that are developed and used by teachers and pupils in their interaction with mathematics in/for teaching and learning, inside and outside the classroom. Curriculum resources would thus include the following:

- Text resources, such as textbooks, teacher curricular guidelines, websites, student sheets, and syllabi
- Other material resources, such as manipulatives and calculators
- ICT-based resources, such as computer software

Mathematics curriculum resources, and in particular textbooks, are an important part of the environment in which teachers and students work (Haggarty and Pepin 2002). Students spend much of their time in classrooms working with and exposed to prepared resources, such as textbooks, worksheets, and computer software. Teachers often rely on curriculum materials and textbooks in their day-to-day teaching, when they decide what to teach, how to teach it, and when they choose the kinds of tasks, exercises, and activities to assign to their students. In short, curriculum resources considered as educational artifacts are vital tools which are of central importance for both teachers and students.

The concept of curriculum resource can also be viewed in a wider sense, to include "a range of human and material resources, as well as mathematical, cultural and social resources" (Adler 2000, p. 210). This view would include resources such as discussions between teachers (e.g., oral, on a forum), knowledge and qualifications, and contextual/environmental factors (e.g., class size, time, professional leadership, family support). Seen this way, it makes the study of curriculum resources and the interaction with resources a crucial ingredient of teacher education and professional development.

## Curriculum Resources and Teachers' Interaction with Resources

In this text, we particularly focus on curriculum resources and teachers' interactions with such resources. We present a synthesis of the state-of-the-art research, organized under two headings: research about the resources themselves, about their design, and their quality (see section - Conception, Quality, and Design of Resources) and research about the use of resources including the adaptation and transformation by users, in particular teachers (see section The "Use" of Resources). In section Evolutions and Issues for Research, we present current perspectives for research concerning curriculum resources.

## Conception, Quality, and Design of Resources

In terms of analyzing resources (and we include here digital as well as "hard copy" resources), different authors have pursued different lines of inquiry:

1. Analyses of *mathematical intentions* relate to what mathematics is represented, the presentation of mathematical knowledge (such as the content and structure of mathematics curriculum materials, e.g., Valverde et al. 2002, or "complexity," e.g., Schmidt et al. 1997), and also to values and beliefs implicit in curriculum materials (e.g., Haggarty and Pepin 2002).
2. Analyses of *pedagogical intentions* of text materials address the ways in which students are helped (or not) by the text. We can identify at least three themes here: ways in which the learner is helped (or not) within the content of the text to learn the materials (e.g., Van Dormolen 1986), within the methods included in the text, or by the rhetorical voice of the text.
3. *Sociological analyses* of texts investigate mathematics texts, often school texts, with respect to sociocultural factors, such as patterns of social class (e.g., Dowling 1998: differentiation in texts between texts/exercises for "high ability" and "low ability" students).
4. Analyses of curriculum materials with respect to different *mathematical concepts* are numerous (algebra, functions, geometry, etc.). These examine the presentation of the concept itself, for example, the use of different representations in curriculum texts. Equally, there are analyses of curriculum materials with respect to different *mathematical competences*, such as "reasoning" or problem-solving.

All these analyses, more or less explicitly, raise the issue of the quality of curriculum resources and in turn can be reinterpreted as contributions to quality studies. This issue is nevertheless particularly developed in studies concerning digital resources, as the profusion of online resources has created a need for quality criteria. These criteria have to take into account the mathematical content, the didactical aspects, and ergonomic dimension (Trouche et al. 2013).

The quality issue is also relevant in research about resource design (Ruthven et al. 2009), which includes mathematical task design. Research shows that it is crucially important to provide frequent opportunities for students to engage in dynamic mathematical activity that is grounded in rich, worthwhile mathematical tasks. Design-based research (Cobb et al. 2003) is particularly concerned about task design and quality in order to improve educational practices and achievement. This kind of research has clearly identified the involvement of research teams, where researchers and teachers work together, as an essential ingredient for the quality of the tasks designed.

It has become evident that quality and design issues are interrelated. Digital means lead to the development of new design modes and to new possibilities of collaborative work around the design of resources. Research on curriculum resources needs to address questions, such as who are the designers and in which ways does the designer/group of designers impact on the quality of resources? In some countries, national "expert communities/centers" (e.g., NCTEM in the UK, DZLM in Germany, Enciclomedia in Mexico, Enlaces in Chile) "produce" and broadcast resources. In addition, particular communities and associations (e.g., GeoGebra community, Sesamath in France – see Gueudet et al. 2012) make resources available.

In the next section, we address issues involved with the "use" of resources.

## The "Use" of Resources

In this section, we address issues related to the "use" of resources which include the interactions between teachers and students with resources.

In terms of textbooks, large-scale studies, such as TIMSS, recognize the importance of textbooks in teaching and learning and assert that textbooks reflect, to a large extent, official curricular intentions and they are said to play an essential role in the didactical transposition of mathematical knowledge. In many countries, school textbooks need approval from the country's ministry; in other countries, there is a free market for textbooks – textbooks are generally seen as the "translation of policy into practice" (Valverde et al. 2002). In some countries (e.g., USA), textbooks have been published with an explicit intention of influencing teacher practices, and the same holds for digital resources. Nevertheless, the research has also proven that the impact of such attempts, in terms of change of practice, remains limited.

We consider here the interactions between students or teachers and resources from the perspective of mediated activity. This leads to consider a twofold process: on the one hand (1), the resource's features influence the subject's activity and learning (for teachers, this can lead to policy choices, drawing on resources as a means for teacher education); on the other hand (2), the subject shapes his/her resources, according to his/her knowledge and beliefs.

The features of the resources influence students' learning, as well as teachers' practices and professional learning. This has been evidenced by many studies investigating the use of curriculum materials (e.g., Remillard et al. 2009) and of ICT resources (Hoyles and Lagrange 2010) in teachers' and students' work.

Considering the shaping of resources by teachers or pupils, the ways teachers, or students, use, adapt, or transform the resources depend to a large extent on their knowledge and beliefs. The ways students "use," for example, a calculator is said to depend on their knowledge about the calculator and its affordances but also on their knowledge of the mathematics (Hoyles and Lagrange 2010). The same holds for textbooks (Gueudet et al. 2012): in order to find support for solving an exercise, some students will read the course materials, whereas others will search for worked examples. Similarly, two teachers will use the same textbook differently. A teacher can focus on the worksheets, or the provision of exercises, while another will consider the same book as curriculum guide (Remillard et al. 2009). Thus, it can be said that this kind of resources offers personal possibilities for adaptations, and teachers have always adapted and transformed resources: selecting, changing, cutting, and rephrasing. However, the main difference with digital resources, such as digital textbooks, is that these adaptations are technically anticipated and supported with specific technical means (Gueudet et al. 2012).

The two-way process, i.e., the influence of the resources on the teacher and the transformation of the resources by the teacher, can be described as a genesis. Gueudet, Pepin, and Trouche (2012) distinguish between resources, given to the teacher, and documents, developed alongside such a genesis. These geneses are central in teacher professional development. They can be individual but can also involve groups of teachers working collaboratively with resources. Research (e.g., Krainer and Woods 2008) suggests that these evolutions can be supported by teacher development programs which propose the design and testing of their own resources to groups of teachers.

## Evolutions and Issues for Research

Viewing curriculum resources as essential tools for teachers to accomplish their goals has been accepted for a long time. However, the vision of the teacher-tool relationship (Remillard et al. 2009) has changed and needs to be explored in more depth. Moreover, considering the evolution of resources available for teachers and students (e.g., their number, nature, design mode/s), this opens up new directions for research. It leads in particular to view the teacher as a designer of his/her resources. Based on the interpretation of teaching as design, and teachers as designers, existing research emphasizes the vital interaction between the individuals/teachers and the tools/resources to accomplish their goals, an accomplishment inextricably linked to the use of cultural, social, and physical tools. This opens the door for many new avenues of researching mathematics curriculum resources and their interaction with the "learner," may it be the teacher or the student.

Studying resources for the teaching of mathematics requires such a stance, particularly as there have been various recent evolutions linked to the use of the Internet. Teachers increasingly become the designers of their own resources, collecting various materials on the web, transforming them, and discussing them with colleagues around the world. National policies for the design and use of curriculum resources are starting to take these evolutions into account, in particular by collecting users' comments on websites (e.g., dedicated websites for particular textbooks).

Analyzing the quality of available resources, contributing to the design of resources (to be used by students and teachers), and proposing teacher development programs drawing on collaborative resource design are important issues, which need to be addressed by research in mathematics education.

## Cross-References


- ▶ Communities of Inquiry in Mathematics Teacher Education
- ▶ Communities of Practice in Mathematics Education
- ▶ Cultural Diversity in Mathematics Education
- ▶ Design Research in Mathematics Education
- ▶ Didactic Transposition in Mathematics Education
- ▶ Information and Communication Technology (ICT) Affordances in Mathematics Education
- ▶ Instrumentation in Mathematics Education
- ▶ Learning Environments in Mathematics Education
- ▶ Manipulatives in Mathematics Education
- ▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes
- ▶ Mathematics Teachers and Curricula
- ▶ Professional Learning Communities in Mathematics Education
- ▶ Teaching Practices in Digital Environments
- ▶ Technology and Curricula in Mathematics Education
- ▶ Technology Design in Mathematics Education


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# D

## Data Handling and Statistics Teaching and Learning

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### Keywords

Statistics; Data handling; Exploratory data analysis; Teaching and learning statistics; Research on teaching and learning statistics; Statistical reasoning; Statistical literacy; Technological tools in statistics learning

### Definition

Over the past several decades, changes in perspective as to what constitute statistics and how statistics should be taught have occurred, which resulted in new content, pedagogy and technology, and extension of teaching to school level. At the same time, statistics education has emerged as a distinct discipline with its own research base, professional publications, and conferences. There seems to be a large measure of agreement on what content to emphasize in statistics education: exploring data (describing patterns and departures from patterns), sampling and experimentation (planning and conducting a study), anticipating patterns (exploring random phenomena using probability and simulation), and statistical inference (estimating population parameters and testing hypotheses) (Scheaffer 2001). Teaching and learning statistics can differ widely across countries due to cultural, pedagogical, and curricular differences and the availability of skilled teachers, resources, and technology.

### Changing Views on Teaching Statistics Over the Years

By the 1960s statistics began to make its way from being a subject taught for a narrow group of future scientists into the broader tertiary and school curriculum but still with a heavy reliance on probability. In the 1970s, the reinterpretation of statistics into separate practices comprising exploratory data analysis (EDA) and confirmatory data analysis (CDA, inferential statistics) (Tukey 1977) freed certain kinds of data analysis from ties to probability-based models, so that the analysis of data could begin to acquire status as an independent intellectual activity. The introduction of simple data tools, such as stem and leaf plots and boxplots, paved the way for students at all levels to analyze real data interactively without having to spend hours on the underlying theory, calculations, and complicated procedures. Computers and new pedagogies would later complete the "data revolution" in statistics education.

In the 1990s, there was an increasingly strong call for statistics education to focus more on statistical literacy, reasoning, and thinking. Wild and Pfannkuch (1999) provided an

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empirically based comprehensive description of the processes involved in the statisticians' practice of data-based inquiry from problem formulation to conclusions. One of the main arguments presented is that traditional approaches to teaching statistics focus on skills, procedures, and computations, which do not lead students to reason or think statistically.

These changes are implicated in a process of democratization that has broadened and diversified the backgrounds and motivations of those who learn statistics at many levels with very diverse interests and goals. There is a growing recognition that the teaching of statistics is an essential part of sound education since the use of data is increasingly common in science, society, media, everyday life, and almost any profession.

## A Focus on Statistical Literacy and Reasoning

The goal of teaching statistics is to produce statistically educated students who develop statistical literacy and the ability to reason statistically. Statistical literacy is the ability to interpret, critically evaluate, and communicate about statistical information and messages. Statistically literate behavior is predicated on the joint activation of five interrelated knowledge bases – literacy, statistical, mathematical, context, and critical – together with a cluster of supporting dispositions and enabling beliefs (Gal 2002). Statistical reasoning is the way people reason with the "big statistical ideas" and make sense of statistical information during a data-based activity. Statistical reasoning may involve connecting one concept to another (e.g., center and spread) or may combine ideas about data and chance. Statistical reasoning also means understanding and being able to explain statistical processes and being able to interpret statistical results.

The "big ideas" of statistics that are most important for students to understand and use are data, statistical models, distribution, center, variability, comparing groups, sampling and sampling distributions, statistical inference, and covariation. Additional important underlying concepts are uncertainty, randomness, evidence strength, significance, and data production (e.g., experiment design). In the past few years, researchers have been developing ideas of *informal* statistical reasoning in students as a way to build their conceptual understanding of the foundations of more formal ideas of statistics (Garfield and Ben-Zvi 2008).

## What Does Research Tell Us About Teaching and Learning Statistics?

Research on teaching and learning statistics has been conducted by researchers from different disciplines and focused on students at all levels. Common faulty heuristics, biases, and misconceptions were found in adults when they make judgments and decisions under uncertainty, e.g., the representativeness heuristic, law of small numbers, and gambler's fallacy (Kahneman et al. 1982). Recognizing these persistent errors, researchers have explored ways to help people correctly use statistical reasoning, sometimes using specific methods to overcome or correct these types of problems.

Another line of inquiry has focused on how to develop good statistical reasoning and understanding, as part of instruction in elementary and secondary mathematics classes. These studies revealed many difficulties students have with concepts that were believed to be fairly elementary such as data, distribution, center, and variability. The focus of these studies was to investigate how students begin to understand these ideas and how their reasoning develops when using carefully designed activities assisted by technological tools (Shaughnessy 2007).

A newer line of research is the study of preservice or practicing teachers' knowledge of statistics and probability and how that understanding develops in different contexts. The research related to teachers' statistical pedagogical content knowledge suggests that this knowledge is in many cases weak. Many teachers do not consider themselves well prepared to teach statistics nor face their students' difficulties (Batanero et al. 2011).

The studies that focus on teaching and learning statistics at the college level continue to point out the many difficulties tertiary students have in learning, remembering, and using statistics and point to some modest successes. These studies also serve to illustrate the many practical problems faced by college statistics instructors such as how to incorporate active or collaborative learning in a large class, whether or not to use an online or "hybrid" course, or how to select one type of software tool as more effective than another. While teachers would like research studies to convince them that a particular teaching method or instructional tool leads to significantly improved student outcomes, that kind of evidence is not actually available in the research literature. However, recent classroom research studies suggest some practical implications for teachers. For example, developing a deep understanding of statistics concepts is quite challenging and should not be underestimated; it takes time, a well thought-out learning trajectory, and appropriate technological tools, activities, and discussion questions.

## Teaching and Learning

As more and more students study statistics, teachers are faced with many challenges in helping these students succeed in learning and appreciating statistics. The main sources of students' difficulties were identified as: facing statistical ideas and rules that are complex, difficult, and/or counterintuitive, difficulty with the underlying mathematics, the context in many statistical problems may mislead the students, and being uncomfortable with the messiness of data, the different possible interpretations based on different assumptions, and the extensive use of writing and communication skills (Ben-Zvi and Garfield 2004).

The study of statistics should provide students with tools and ideas to use in order to react intelligently to quantitative information in the world around them. Reflecting this need to improve students' ability to reason statistically, teachers of statistics are urged to emphasize statistical reasoning by providing explicit attention to the basic ideas of statistics (such as the need for data, the importance of data production, the omnipresence of variability); focus more on data and concepts, less on theory, and fewer recipes; and foster active learning (Cobb 1992). These recommendations require changes of teaching statistics in *content* (more data analysis, less probability), *pedagogy* (fewer lectures, more active learning), and *technology* (for data analysis and simulations) (Moore 1997).

Statistics at school is usually part of the mathematics curriculum. New K–12 curricular programs set ambitious goals for statistics education, including promoting students' statistical literacy, reasoning, and understanding (e.g., NCTM 2000). These reform curricula weave a strand of *data handling* into the traditional school mathematical strands (number and operations, geometry, algebra). Detailed guidelines for teaching and assessing statistics at different age levels complement these standards. However, school mathematics teachers, which are often not versed in statistics, find it challenging to teach data handling in accordance with these recommendations.

In order to face this challenge and promote statistical reasoning, good instructional practice consists of implementing inquiry or project-based learning environments that stimulate students to construct meaningful knowledge. Garfield and Ben-Zvi (2009) suggest several design principles to develop students' statistical reasoning: focus on developing central statistical ideas rather than on presenting set of tools and procedures; use real and motivating data sets to engage students in making and testing conjectures; use classroom activities to support the development of students' reasoning; integrate the use of appropriate technological tools that allow students to test their conjectures, explore and analyze data, and develop their statistical reasoning; promote classroom discourse that includes statistical arguments and sustained exchanges that focus on significant statistical ideas; and use assessment to learn what students know and to monitor the development of their statistical learning, as well as to evaluate instructional plans and progress.

Technology has changed the way statisticians work and has therefore been changing what and how statistics is taught. Interactive data visualizations allow for the creation of novel representations of data. It opens up innovative possibilities for students to make sense of data but also place new demands on teachers to assess the validity of the arguments that students are making with these representations and to facilitate conversations in productive ways. Several types of technological tools are currently used in statistics education to help students understand and reason about important statistical ideas. However, using technological tools and how to avoid common pitfalls are challenging open issues (Biehler et al. 2013).

These changes in the learning goals of statistics have led to a corresponding rethinking of how to assess students. It is becoming more common to use alternative assessments such as student projects, reports, and oral presentations than in the past. Much attention has been paid to assess student learning, examine outcomes of courses, align assessment with learning goals, and alternative methods of assessment.

## For Further Research

Research in statistics education has made significant progress in understanding students' difficulties in learning statistics and in offering and evaluating a variety of useful instructional strategies, learning environments, and tools. However, many challenges are still ahead of statistics education, mostly in transforming research results to practice, evaluating new programs, planning and disseminating high-quality assessments, and providing attractive and effective professional development to mathematics teachers (Garfield and Ben-Zvi 2007). The ongoing efforts to reform statistics instruction and content have the potential to both make the learning of statistics more engaging and prepare a generation of future citizens that deeply understand the rationale, perspective, and key ideas of statistics. These are skills and knowledge that are crucial in the current information age of data.

## Cross-References

▶ Inquiry-Based Mathematics Education
▶ Mathematical Literacy
▶ Probability Teaching and Learning

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# Deaf Children, Special Needs, and Mathematics Learning

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## Keywords

Deaf children; Special needs; Mathematics difficulty

## Characteristics

The aim of mathematics instruction in primary school is to provide a basis for thinking mathematically about the world. This is as basic a skill as literacy in today's world. Mathematical knowledge is also a means to achieve better employment and to enter higher education. For all these reasons, it is of great importance that deaf children have adequate access to mathematical thinking, but unfortunately most deaf children show a severe delay in mathematics learning. This delay has been persistent over many years. The average score in mathematics achievement tests for deaf children in the age range 8–15 in a study carried out in 1965 showed that they were one standard deviation below the average for hearing children, a result replicated about three decades later. This means that about 50 % of the deaf pupils perform similarly to the weakest 15 % of the hearing pupils. Later results continue to confirm this weak performance. In the UK, deaf students aged 16–17 years, at the end of compulsory school, were found to have a mathematical age between 10 and 12.5 years. In the USA, the mathematical ability of 80 % of the deaf 14-year-olds was described as "below basic" in problem solving and knowledge of mathematical procedures. A recent systematic review confirmed these findings (Gottardis et al. 2011) and analyzed individual differences among deaf children.

This serious and persistent difficulty is not universal among children who are deaf; approximately 15 % perform at age appropriate levels. The successful minority indicates that deafness is not a direct cause of difficulty in mathematics learning (see Nunes 2004, for a discussion). This article considers what is involved in learning mathematics in primary school, why deaf children may be at a disadvantage, and how schools can support their learning of mathematics.

### Learning Mathematics in Primary School

In order to think mathematically, people need to learn to represent quantities, relations, and space using culturally developed and transmitted thinking tools, such as oral and written number systems, graphs, and calculators.

Some researchers argue that numerical concepts have a neurological basis that is independent of language learning, without which learning mathematics is extremely difficult. In view of the pervasiveness of deaf children's mathematical difficulties, it could be hypothesized that they have an inadequate development of such concepts. Basic numerical cognition has been studied in research with young deaf children as well as adults, and the hypothesis has been discarded. Deaf children and adults performed at least as well as their hearing counterparts in such tasks.

The possible consequences of delays in the acquisition of other language-based numerical concepts have also been explored. Two examples are knowledge of counting and understanding of arithmetic operations.

#### Counting

Deaf children lag behind hearing children in learning to count, independently of whether they are learning to count orally or in sign (Leybaert and Van Cutsem 2002). Consequently, they perform less well than hearing children on school-entry numeracy tests, which typically include tasks that require counting (e.g., "show me 5 blocks"; "tell me which number is bigger"). This delay could be related to the well-established finding that deaf people perform less well than hearing people on serial learning tasks, in which words or gestures must be learned in an exact sequence, just as the number string. However, they perform better if the tasks are presented differently and use spatial cues to organize the information. These findings are provocative rather than conclusive. First, they raise the possibility that deaf children could learn to count more easily if appropriate visual and spatial methods were used for teaching rather than serial learning instruction. Second, serial learning is not an appropriate description of counting skills beyond a certain number (about 20 or 30 in English but this may differ depending on the counting system). Research with hearing and deaf children shows that counting is a structured activity: for example, errors are more likely to occur at the boundaries between decades (e.g., . . .38, 39, 50, 51, 52. . .) than within decades. Therefore, in principle deaf children's initial disadvantage in counting could be overcome with appropriate teaching methods and with support for mastery of the structure of the system. However, it is possible that their initial struggle with learning to count lowers adults' expectations about what they can learn in mathematics, resulting in less stimulation on mathematical tasks, and that it also interferes with the children's own discoveries in the domain of mathematical reasoning.

#### Early Mathematical Reasoning and Arithmetic Operations

The development of mathematical reasoning starts before school, when children solve practical problems using actions, which they learn to combine with counting. When most children start school (at age 5 or 6), they can already solve simple addition and subtraction problems by putting together or separating objects and counting, and some can also solve multiplication and division problems. By counting, children use explicit numerical representation both for thinking and communicating. When numbers are small and the children can use objects, deaf children do as well as hearing children in solving these problems, but if the numbers go above 10 or 20, most deaf children fall behind. When they are compared with hearing children of the same counting ability, they are just as competent in solving numerical tasks (Leybaert and Van Cutsem 2002), but their disadvantage in counting is reflected in their problem-solving skills when they are compared to same-age hearing peers. Thus, it is possible that, not knowing number words well enough to support their mathematical reasoning, they do not discover how to use counting to solve simple arithmetic problems or important ideas for their later success, such as the inverse relation between addition and subtraction. However, Nunes and colleagues (2008a, b) have shown that relatively small amounts of teaching can effectively improve young deaf children's performance in the mathematical reasoning and arithmetic tasks, with which they were struggling before the teaching.

### Conclusion

There is little doubt that many deaf children show severe and persistent difficulties in learning mathematics. Evidence suggests that there is no direct connection between deafness and problems with basic number concepts that precede language. However, deaf children lag behind hearing children in learning to count, whether orally or in sign, and at school entry they are behind their hearing counterparts in mathematical knowledge. It is possible that falling behind in counting places deaf children at a disadvantage from the adults' perspective and that they end up receiving less stimulation to solve mathematical problems early on. It is also possible that their own informal mathematical knowledge is limited by their difficulty in representing quantities explicitly with number words. These findings and conclusions suggest that, if parents and preschool teachers could find visual and spatial ways to teach counting to deaf children, one would see positive changes in the average achievement of deaf children in mathematics in the future.

## Cross-References

- ▶ Blind Students, Special Needs, and Mathematics Learning
- ▶ Concept Development in Mathematics Education
- ▶ Inclusive Mathematics Classrooms
- ▶ Language Disorders, Special Needs and Mathematics Learning

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# Deductive Reasoning in Mathematics Education

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## Keywords

Logic; Reasoning; Proof; History of proof; Proof schemes

## Definition

This entry examines the different facets of deductive reasoning with respect to the learning and teaching of mathematical proof. Deductive reasoning may be defined as a formal way of reasoning, usually top-down [from the general to the particular] with adherence to logical consistency.

### Characteristics of Deductive Reasoning

The examinations of the learning and teaching of proof are multifaceted. They address a broad range of factors: mathematical, historical-epistemological, cognitive, sociological, and instructional. Research questions involving these factors include the following:

Mathematical and Historical-Epistemological Factors

1. What is proof and what are its functions?
2. How are proofs constructed, verified, and accepted in the mathematics community?
3. What are some of the critical phases in the development of proof in the history of mathematics?

Cognitive Factors

4. What are students' current conceptions of proof?
5. What are students' difficulties with proof?
6. What accounts for these difficulties?

Instructional-Sociocultural Factors

7. Why teach proof?
8. How should proof be taught?
9. How are proofs constructed, verified, and accepted in the classroom?
10. What are the critical phases in the development of proof with the individual student and within the classroom as a community of learners?
11. What classroom environment is conducive to the development of the concept of proof with students?
12. What form of interactions among the students and between the students and the teacher can foster students' conception of proof?
13. What mathematical activities – possibly with the use of technology – can enhance students' conceptions of proof?
14. How is proof currently being taught?
15. What do teachers need to know in order to teach proof effectively?

Theoretical Factors

16. What theoretical tools seem suitable for investigating and advancing students' conceptions of proof?

One's investigation of these questions is greatly influenced by her or his philosophical orientation to the processes of learning and teaching and would reflect her or his conclusion to questions such as the following: What bearing, if any, does the epistemology of proof in the history of mathematics have on the conceptual development of proof with students? What bearing, if any, does the way mathematicians construct proofs have on instructional treatments of proof? What bearing, if any, does everyday justification and argumentation have on students' proving behaviors in mathematical contexts?

### Historical-Epistemological Developments

Deductive reasoning is a mode of thought commonly characterized as a sequence of propositions where one must accept any of the propositions to be true if he or she has accepted the truth of those that preceded it in the sequence. This mode of thought was conceived by the Greeks more than twentieth centuries ago and is still dominant in the mathematics of our days. So remarkable is the Greeks' achievement that their mathematics became a historical mark to which other kinds of mathematics are compared. The nature of deductive reasoning varies throughout history (Kleiner 1991). Of particular contrast is Greek mathematics versus modern mathematics. In Greek mathematics, the particular entities under investigation are idealizations of experiential spatial realities and so also are the propositions on the relationships among these entities. Logical deduction came to be central in the reasoning process, and it alone necessitated and cemented the geometric edifice they created. In constructing their geometry, as is depicted in Euclid's Elements, the Greeks had only one model in mind – that of imageries of idealized physical reality. From the vantage point of modern mathematics, neither the primitive terms nor the axioms in Greek mathematics were variables, but constants referring to a single spatial model (Klein 1968; Wilder 1967), as is expressed in the ideal world of Plato's philosophy. In modern mathematics, on the other hand, primary terms and axioms are open to different possible realizations. An important manifestation of this revolution is the distinction between Euclid's Elements and Hilbert's Grundlagen. The latter characterizes a structure that fits different models, that is, in an abstraction of numerous models, such as the Euclidean space, the surface of a half-sphere and the ordered pairs and triples of real numbers, including the interpretation that the axioms, are meaningless formulas.

Considerations of historical-epistemological developments led to new research questions with direct bearing on the learning and teaching of proofs. For example, to what extent and in what ways is the nature of the content intertwined with the nature of proving? In geometry, for example, does students' ability to construct an image of a point as a dimensionless geometric entity impact their ability to develop the Greek conception of proof? What is the cognitive or social mechanism by which deductive proving can be necessitated for the students? The Greek's construction of their geometric edifice seems to have been a result of their desire to create a consistent system that was free from paradoxes. Would paradoxes of the same nature create a similar intellectual need with students? Students encounter difficulties in moving empirical reasoning to deductive reasoning, particularly from the Greek's conception of proof to the modern conception of proof. Exactly what are these difficulties? What role does the emphasis on form rather than content in modern mathematics (as opposed to Greek mathematics, where content is more prominent) play in this transition?

### Classifications of Conceptualizations of Proof

Harel and Sowder (1998) call these conceptualizations proof schemes, which they classify into a system of subcategories. Their taxonomy is organized around three main classes of categories: the *external conviction proof schemes* class, the *empirical proof schemes* class, and the *deductive proof schemes* class. A partial description of these classes follows.

#### External Conviction Proof Schemes

Proving within the *external conviction* proof schemes class depends either (a) on an authority such as a teacher or a book, (b) on strictly the appearance of the argument (e.g., proofs in geometry must have a two-column format), or (c) on symbol manipulations, with the symbols or the manipulations having no potential coherent system of referents (e.g., quantitative and spatial) in the eyes of the student. Accordingly, the *external conviction* proof schemes class consists of three categories: the *authoritarian proof scheme* category, the *ritual proof scheme* category, and the *non-referential symbolic proof scheme* category.

### Empirical Proof Schemes

Schemes in the *empirical proof scheme* class are marked by their reliance on either (a) an evidence from examples (sometimes just one example) of direct measurements of quantities, substitutions of specific numbers in algebraic expressions, etc., or (b) perceptions. Accordingly, this class consists of two categories: the *inductive proof scheme* category and the *perceptual proof scheme category*.

### Deductive Proof Schemes

The *deductive proof schemes* class consists of two subcategories, each consisting of various proof schemes: the *transformational proof scheme* category and the *axiomatic proof scheme* category.

## Classifications of Functions of Proof

In general, the empirical proof schemes and the deductive proof schemes categories correspond to what Bell (1976) calls *empirical justification* and *deductive justification* and Balacheff (1988) calls *pragmatic* justifications and *conceptual justifications*, respectively. *Pragmatic justification* is further divided into three categories: *naïve empiricism* (justification by a few random examples), *crucial experiment* (justification by carefully selected examples), and *generic example* (justification by an example representing salient characteristics of a whole class of cases). *Conceptual justification* is divided into two categories: *thought experiment*, where the justification is disassociated from specific examples, and *symbolic calculation*, where the justification is based solely on transformation of symbols.

These taxonomies are not explicit enough about many critical functions of proof within mathematics. There is a need to point to these functions due to their importance in mathematics in general and to their instructional implications in particular. The work by Hanna (1990), Balacheff (1998), Bell (1976), Hersh (1993), and de Villiers (1999) explicitly address these functions. De Villiers, who built on the work of the others scholars mentioned here, raises two important questions about the role of proof: (a) "What different functions does proof have within mathematics itself?" and (b) "how can these functions be effectively utilized in the classroom to make proof a more meaningful activity?" According to de Villiers, mathematical proof has six not mutually exclusive roles: ***Verification*** refers to the role of proof as a means to demonstrate the truth of an assertion according to a predetermined set of rules of logic and premises – the *axiomatic proof scheme*. ***Explanation*** is different from verification in that for a mathematician it is usually insufficient to know only that a statement is true. He or she is likely to seek insight into why the assertion is true. "Mathematicians routinely distinguish proofs that merely demonstrate from proofs which explain" (Steiner 1978, p. 135). For many, the role of mathematical proofs goes beyond achieving certainty – to show that something is true; rather, "they're there to show... why [an assertion] is true," as Gleason, one of solvers of the solver of Hilbert's Fifth Problem (Yandell 2002, p. 150), points out. Two millennia before him, Aristotle, in his *Posterior Analytic*, asserted, "... We suppose ourselves to possess unqualified scientific knowledge of a thing, as opposed to knowing it in the accidental way in which the sophist knows, when we think that we know the cause on which the fact depends as the cause of the fact and of no other" (p. 4). ***Discovery*** refers to the situations where through the process of proving, new results may be discovered. For example, one might realize that some of the statement conditions can be relaxed, thereby generalizing the statement to a larger class of cases. Or, conversely, through the proving process, one might discover counterexamples to the assertion, which, in turn, would lead to a refinement of the assertion by adding necessary restrictions that would eliminate counterexamples. ***Systematization*** refers to the presentation of verifications in organized forms, where each result is derived sequentially from previously established results, definitions, axioms, and primary terms. ***Communication*** refers to the social interaction about the meaning, validity, and importance of the mathematical knowledge offered by the proof produced. ***Intellectual Challenge*** refers to the mental state of self-realization and fulfillment one can derive from constructing a proof.

### Students' Proof Schemes

Status studies on students' conceptualization of proof show the absence of the deductive proof scheme and the pervasiveness of the empirical proof scheme among students. Students base their responses on the appearances in drawings, and mental pictures alone constitute the meaning of geometric terms. They justify mathematical statements by providing specific examples, not able to distinguish between inductive and deductive arguments. Even more able students may not understand that no further examples are needed, once a proof has been given. Students' preference for proof is ritualistically and authoritatively based. For example, when the stated purpose was to get the best mark, they often felt that more formal – e.g., algebraic – arguments might be preferable to their first choices. These studies also show a lack of understanding of the functions of proof in mathematics, often even among students who had taken geometry and among students for whom the curriculum pays special attention to conjecturing and explaining or justifying conclusions in both algebra and geometry. They believe proofs are used only to verify facts that they already know and have no sense of a purpose of proof or of its meaning. Students have difficulty understanding the role of counterexamples; many do not understand that one counterexample is sufficient to disprove a conjecture. Students do not see any need to prove a mathematical proposition, especially those they considered to be intuitively obvious. This is the case even in a country like Japan where the official curriculum emphasizes proof. They view proof as the method to examine and verify a later particular case. Finally, the studies show that students have difficulty writing valid simple proofs and constructing, or even starting, simple proofs. They have difficulty with indirect proofs, and only a few can complete an indirect proof that has been started.

### Impact of Instruction

Students who receive more instructional time on developing analytical reasoning by solving unique problems fare noticeably better on overall test scores. Likewise, students who have been expected to write proofs and who have had classes that emphasized proof were somewhat better than other students. It also seems possible to establish desirable sociomathematical norms relevant to proof, through careful instruction, often featuring the student role in proof-giving. There has been the concern that the ease with which technology can generate a large number of examples naturally could undercut any student-felt need for deductive proof schemes. Several studies have shown that with careful, nontrivial planning and instruction over a period of time, progress toward deductive proof schemes is possible in technology environments, where such desiderata as making conjectures and definitions occur.

## Cross-References

▶ Abstraction in Mathematics Education
▶ Argumentation in Mathematics
▶ Argumentation in Mathematics Education
▶ Logic in Mathematics Education

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## 22q11.2 Deletion Syndrome, Special Needs, and Mathematics Learning

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### Keywords

Genetic disorder; Mathematics difficulties; Cognitive impairment

### Characteristics

Chromosome 22q11.2 deletion syndrome (22q) is the most common genetic deletion syndrome with an estimated prevalence of between one in 3,000 and 6,000 births (e.g., Kobrynski and Sullivan 2007). It has only been detectable with 100 % accuracy since 1992 using techniques such as the FISH test (fluorescence in situ hybridization). Prior to identification of a single associated deletion, the syndrome had been given a number of different labels according to the primary medical condition, for example, velocardiofacial syndrome, DiGeorge syndrome, Cayler syndrome, Shprintzen syndrome, and Catch 22.

The majority of individuals with 22q experience some degree of learning difficulty and generally show a marked imbalance in performance across different subtests within IQ batteries. Verbal IQ scores are usually significantly higher than performance IQ scores (e.g., Moss et al. 1999; Wang et al. 2007). The majority of children will receive some form of support at school although some individuals experience no difficulties at all. Indeed a very wide level of individual differences in attainment in individuals with 22q is noted in all studies to date.

There is consistent evidence that mathematics skills are weaker than literacy skills in the majority children with 22q. This profile is unusual as children with mathematics difficulties are often reported to have comorbid reading difficulties. Typically, performance on standardized tests of reading and spelling is within the normal range, but performance on mathematical reasoning and arithmetic tasks is at least one standard deviation below age norms in children with 22q. Children with 22q specifically selected to have full scale IQ of at least 70 also demonstrate this profile, thereby suggesting that it is associated with 22q per se rather than low general ability.

There are very few studies examining number skills in detail in children with 22q. De Smedt et al. (2006, 2007a, b) tested children, selected to have an IQ of more than 70, on a series of computerized tests assessing performance in number reading and writing, number comparison, counting, and single and multi-digit arithmetic. A mathematical word-solving task was also included and reading ability was measured. Children were individually matched with typically developing children from the same class at school for gender, age, and parental education level. Consistent with their hypotheses, De Smedt et al. (2007a, b) report group differences on multi-digit operations involving a carry, word-solving problems, and speed in judging the relative value of two digits. There was no difference in reading, number reading and writing, single digit addition, or verbal and dot counting accuracy. The difficulties with multi-digit operations are unsurprising given the visuospatial requirements of operations such as borrowing and carrying. Previous researches suggest that multi-digit arithmetic is an area of particular difficulty in children with visuospatial learning disability as well as arithmetic difficulties (Venneri et al. 2003). More research is needed to further uncover the nature of the mathematical difficulties experienced by children with 22q and to aim to uncover best practice methods for teaching number skills in 22q as so far, certainly in the UK, no consensus has been reached.

## Cross-References


- ▶ Autism, Special Needs, and Mathematics Learning
- ▶ Deaf Children, Special Needs, and Mathematics Learning
- ▶ Down Syndrome, Special Needs, and Mathematics Learning
- ▶ Inclusive Mathematics Classrooms
- ▶ Language Disorders, Special Needs and Mathematics Learning
- ▶ Learning Difficulties, Special Needs, and Mathematics Learning


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# Design Research in Mathematics Education


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## Keywords

Engineering research; Design experiments; Design research


## Definition

Design-based research is a formative approach to research, in which a product or process (or "tool") is envisaged, designed, developed, and refined through cycles of enactment, observation, analysis, and redesign, with systematic feedback from end users. In education, such tools might, for example, include innovative teaching methods, materials, professional development programs, and/or assessment tasks. Educational theory is used to inform the design and refinement of the tools and is itself refined during the research process. Its goals are to create innovative tools for others to use, describe and explain how these tools function, account for the range of implementations that occur, and develop principles and theories that may guide future designs. Ultimately, the goal is *transformative*; we seek to create new teaching and learning possibilities and study their impact on teachers, children, and other end users.

## The Origins and Need for Design Research

Educational research may broadly be categorized into three groups: the *humanities* approach, scholarly study that generates fresh insights through critical commentary, the *scientific* approach that analyzes phenomena empirically to better understand how the world works, and the *engineering* approach that not only seeks to understand the status quo but also attempts to use existing knowledge to systematically develop "high-quality solutions to practical problems" (Burkhardt and Schoenfeld 2003). Design research falls into this "engineering" category and, as such, seeks to provide the tools and processes that enable the end users of mathematics education (teachers and students, administrators and politicians) to tackle practical problems in authentic settings.

Design research is an unsettled construct and the field is in its youth. It is only at the beginning of the last two decades that we see design research as an emerging paradigm for the study of learning through the systematic design of teaching strategies and tools. The beginnings of this movement, at least in the USA, are usually attributed to Brown (1992) and Collins (1992), though in a sense, it was an idea waiting to be named (Schoenfeld 2004). In Europe there have long been traditions of principled design-based research under other guises, such as curriculum development and didactical engineering (e.g., Bell 1993; Brousseau 1997; Wittmann 1995).

Prior to the 1990s, much educational and psychological research had relied heavily on quasi-experimental studies that had been developed successfully in other fields such as agriculture. These involved experimental and control treatments to evaluate whether or not particular variables were associated with particular outcomes. In mathematics education, for example, one might design a novel approach to teaching a particular area of content, assign students to an experimental or control group, and assess their performance on some defined measures, using pre- and posttesting. Though sounding straightforward, this practice proved highly problematic (Schoenfeld 2004): the goals of education are more complex than the mastery of specific skills; the control of variables in naturalistic settings is often impossible, undesirable, and sometimes even unethical; and much of the theory is "emergent," only becoming apparent as one engages in the research.

In the early 1990s, a number of researchers began to question the limitations of traditional experimental psychology as a paradigm for educational research. Brown's paper on "design experiments" was seminal (Brown 1992). Brown recounts how her own research moved away from laboratory settings towards naturalistic ones in which she attempted to transform classrooms from "worksites under the management of teachers into communities of learning." She vividly recounts her own struggles in reconceptualizing her focus and methodology, deconstructing methodological criticisms against it (such as the Hawthorne effect). Interestingly Brown still saw the need for lab-based research, both to precede and stimulate work in naturalistic settings and also for the closer study of phenomena that had arisen in those settings. At about the same time, Collins (1992, pp. 290–293) began to argue for a design science in education, distinguishing *analytic* sciences (such as physics or biology) as where research is conducted in order to *explain* phenomena from *design* sciences (such as aeronautics or acoustics) where the goal is to determine how designed artifacts (such as airplanes or concert halls) behave under different conditions. He argued strongly for the need of the latter in education. In mathematics education, such designed artifacts might include, for example, new teaching methods, materials, professional development programs, assessment tasks, or any combination of these.

Since that time, "design research" has become more widespread and respectable in education. However it must be said that not all so-called "design research" studies satisfy the definition described above. Some, for example, do not satisfy the requirement that the designs should be theory-based and develop theory, while others do not move beyond the early stages and test their designs in the hands of others not involved in the development process.

## Characterizing Design-Based Research

There have been many attempts to characterize design-based research (Barab and Squire 2004; Bereiter 2002; Cobb et al. 2003; DBRC 2003, p. 5; Kelly 2003; Lesh and Sriraman 2010;

Swan 2006, 2011; van den Akker et al. 2006). While design research is still in its infancy and its characterization is far from settled, most researchers do seem to agree that design-based research is:

### Creative and Visionary

The researcher identifies a problem in a defined context and, drawing on prior research, envisions a tool that might help end users to tackle it. A draft design is developed, possibly with the assistance of end users. For example, the researcher identifies a particular student learning need and uses research to design a series of lessons. The ultimate aim is to produce an effective design, an account of the theory and principles underpinning the design, and an analysis of the range of ways in which the design functions in the hands of a typical sample of the target population of teachers and students.

### Ecologically Valid

The researcher studies and refines the design in authentic settings, such as classrooms. This precludes the prior manipulation of variables in the study. It is important, therefore, to distinguish those aspects of the design that are being studied from those that are extraneous.

### Interventionist and Iterative

The role of the researcher evolves as the research proceeds. During early iterations, the design is usually sketchy and the researcher needs to intervene to make it work. With teaching materials, for example, this phase may be conducted with small samples of students. Later, as the design evolves, the researcher holds back, in order to see how the design functions in the hands of end users. Early iterations are often conducted in a few favorable contexts. Early drafts of teaching materials, for example, may be tested in carefully chosen classrooms with confident teachers, in order to gain insights into what is possible with faithful implementation. Later iterations aim to study how the design functions in a wider range of authentic contexts, with teachers who have not been involved in the design process. Under these conditions, "design mutations" invariably occur. Rather than viewing these as negative, interfering factors, the designs and theories evolve to explain these mutations. With each cycle of the process, the sample size is increased and becomes more typical of the target population. From time to time, a particular issue may arise that the researcher wants to study closely. In such a case, it is possible to go back to the small-scale study of that isolated issue.

### Theory-Driven

The outputs of design research include developing theories about learning, interventions, and tools. Rather than focusing on learning outcomes, using pre- and posttests, the research seeks to understand *how* designs function under different conditions and in different classroom contexts. The theories that evolve in this way are *local* and *humble* in scope and should not be judged by their claims to "truth" but rather their claims to be useful (Cobb et al. 2003). Theory in design research usually focuses on an explanation of how and why a particular design feature works in a particular way. It is both specific and generative in that it can be used to predict ways in which future designs will function if they embody this feature.

## Some Issues and Challenges

Design research done well requires great skill on the part of researchers. Indeed, the combination of skills required is not usually found in individuals but in teams. A design research team will typically involve people with knowledge of the literature (researchers), an understanding of pedagogy (teachers), creative "care and flair" (designers), and facility with "delivering" the design (publishers IT technicians).

Secondly, design research often takes a great deal longer than other forms of research. There is often a significant "entry fee" in terms of time and energy taken up with producing a prototype before any study of it can begin. This is particularly true if the design involves creating new software. Then, each cycle of design, implementation, analysis, and redesign can each occupy weeks, if not months.

Thirdly, design research is data rich. A mixture of qualitative and quantitative methods is used to develop a rich description of the way the design works as well as the kinds of learning outcomes that may be expected. This often results in a proliferation of data. Brown, for example, found that she "had no room to store all the data, let alone time to score it" (Brown 1992, p. 152). Data may include lesson observations, videos of the designs in use, and questionnaires and interviews with users. In early iterations, observation plays a dominant role. Later, however, more indirect means are also needed as the sample size grows. Reliability may be improved through the use of triangulation from multiple data sources and repetition of analyses across cycles of implementation and through the use of standardized measures.

Fourthly, design research requires discipline. It is all too tempting to turn a "good idea" into a draft design and then ask someone to try it out to "see what happens." Good design-based research is more than formative evaluation, however; it is theory-driven. In preparation for a design-based research study, one must try to articulate the theory and draw clear lines of connection between this and the design itself. This may be done by eliciting "principles" to direct the design. The research involves putting these principles in "harms way" (Cobb et al. 2003). Then, the focus of the research needs to be articulated. For early iterations this may be on the potential impact of the faithful use of the design, while on later iterations, we may be more interested in refining the design by studying end users' interpretations and mutations.

Finally, writing up design research is problematic. Most designs are too extensive to be described and analyzed in traditional journal articles that emphasize methods and results over tools. Recently e-journals have begun to appear that allow for a much clearer articulation of design-based research. These, for example, allow extensive extracts of teaching and professional development materials to be displayed, along with videos of the designs in use (see for example, http://www.educationaldesigner.org).

## Cross-References

▶ Curriculum Resources and Textbooks in Mathematics Education
▶ Mathematics Curriculum Evaluation
▶ Didactic Engineering in Mathematics Education
▶ Didactic Situations in Mathematics Education



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# Dialogic Teaching and Learning in Mathematics Education

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## Keywords

Dialogue; Literacy; Mathemacy; Investigation; Inquiry; Inquiry cooperation model; Critical mathematics education

## Definition

Dialogic teaching and learning refers to certain qualities in the interaction between teachers and students and among students. The qualities concern possibilities for the students' involvement in the educational process, for establishing enquiry processes, and for developing critical competencies.

## Characteristics

### Sources of Inspiration

There are different sources of inspiration for bringing dialogue into the mathematics classroom, and let me just refer to two rather different.

The notion of dialogue plays a particular role in the pedagogy of Paulo Freire. He sees dialogue as crucial for developing *literacy*, which refers to a capacity in reading and writing the world: reading it, in the sense that one can interpret sociopolitical phenomena, and writing it, in the sense that one becomes able to make changes.

With explicit reference to mathematics, the crucial role of dialogue can be argued with allusion to Imre Lakatos' presentation in *Proof and Refutations* (Lakatos 1976). Here Lakatos shows that a process of mathematical discovery is of dialogic nature, characterized by proofs and refutations.

Critical mathematics education and social constructivism have developed dialogic teaching and learning through a range of examples and studies. It has been emphasized that dialogue is principal for establishing critical perspectives on mathematics and for a shared construction of mathematical notions and ideas. In fact dialogic teaching and dialogic learning represents two aspects of the same process.

Marilyn Frankenstein (1983) has emphasized the importance of Freire's ideas for developing critical mathematics education, and Paul Ernest (1998) has opened the broader perspective of social constructivism, also acknowledging the importance of Lakatos work.

### The Inquiry Cooperation Model

The notion of dialogue appears to be completely open. As a consequence, it becomes important to try to characterize what a dialogue could mean. The *Inquiry Cooperation Model* as presented in Alrø and Skovsmose (2002) provides such a specification with particular references to mathematics.

This model characterizes different dialogic acts: *Getting in contact* refers to the act of tuning in at each other. *Locating* and *identifying* refer to forms of grasping perspectives, ideas, and arguments of the other. *Advocating* means providing arguments for a certain point of view – although not necessary one's own. Thinking aloud means making public details of one's thinking, for instance, through gestures and diagrams. *Reformulating* refers to particular attempts in grasping other ideas by rethinking, rephrasing, and reworking them. *Challenging* means questioning certain ideas, which is an important way of sharpening mathematical arguments. *Evaluating* refers to reflexive questioning, like: What insight might we have reached? What new questions have we encountered?

Dialogic teaching and learning can be characterized as a process rich of such dialogic acts.

### New Qualities in Teaching and Learning

The idea of dialogic teaching and learning is to promote an education with new qualities. Let me refer to just a few having to do with the students' interest, making investigations, and developing a mathemacy.

*Students' Interest*. It has been emphasized that dialogic teaching and learning includes a sensitivity to the students' perspectives and possible interests for learning. This sensitivity has not only to do with the dialogic act of "getting in contact" but with all the acts represented by the Inquiry Cooperation Model. A principal point of dialogic teaching is to invite students into the learning process as active learners.

*Making Investigations*. Dialogic teaching and learning can be characterized in terms of investigative approaches, where both teacher and students participate in the same inquiry process. Barbara Jaworski (2006) makes a particular emphasis on establishing communities of inquiry, and in any such communities, dialogue plays a defining role. Landscapes of investigations (Skovsmose 2011) might also provide environments that facilitate dialogic teaching and learning.

Similar to literacy, *mathemacy* refers not only to a capacity in dealing with mathematical notions and ideas but also to a capacity in interpreting sociopolitical phenomena and acting in a mathematized society. Thus, mathemacy combines a capacity in reading and writing *mathematics* with a capacity in reading and writing *the world* (see Gutstein 2006). Dialogue teaching and learning is in hectic development, both in theory and in practice. A range of new studies and new classroom initiatives are being developed. In particular, the very notion of dialogue is in need of further development; see, for instance, Alrø and Johnsen-Høines (2012).

## Cross-References


▶ Critical Mathematics Education
▶ Mathematization as Social Process


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# Didactic Contract in Mathematics Education


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## Keywords

Didactical situations; Mathematical situations; Didactical contract; Didactique; Milieu; Devolution; Institutionalization

## Introduction

Teachers manage *didactical situations* that create and exploit *mathematical situations* where practices are exercised and students' mathematical knowledge is developed. The study of the *didactical contract* concerns the compatibility on this precise subject of the aspirations and requirements of the students, the teachers, the parents, and the society.

## Definition

A "*didactical contract*" is an interpretation of the commitments, the expectations, the beliefs, the means, the results, and the penalties envisaged by one of the protagonists of a *didactical situation* (student, teacher, parents, society) for him- or herself and for each of the others, *à propos of the mathematical knowledge being taught* (Brousseau and Otte 1989; Brousseau 1997). The objective of these interpretations is to account for the actions and reactions of the partners in a didactical situation.

The didactical contract can be broken down into two parts: a contract of devolution – the teacher organizes the mathematical activity (see ► Didactic Situations in Mathematics Education) of the student who in response commits him- or herself to it – and a contract of institutionalization – the students propose their results and the teacher vouches for the part of their results that conforms to reference knowledge.

Customary practices (Balacheff 1988), whether explicit or tacit, leave the hope that divergences are accidental and reducible and that there exist real contracts, whether or not they can be made explicit, that are compatible and satisfactory. This is not so, owing to various paradoxes that became apparent in the course of teaching in a way that is based on mathematical situations. This gave rise to many questions, among them are as follows:

How could students commit themselves to the subject of knowledge that they have not yet learned?

What are the respective roles of what is inexpressible, of what is said, of what is not said or cannot be said to the other in the teaching relationship?

Does there exist knowledge that ought not to be made explicit before being learned?

The study of these questions was the origin of the theory of didactical situations.

## Characteristics

### Background: Illustrative Examples

These questions arose in the course of research at the COREM (Center for Observation and Research on Mathematics Education, entity formed of a laboratory and a school establishment by the IREM of the University of Bordeaux (1973–1999)) on the possibility of assigning to *mathematical situations* the job of managing what the teacher cannot say or the student cannot yet understand from a text, and in the clinical observation of students failing selectively in mathematics:

(a) *The Case of Gaël*. Gaël (8 years old) always responded in the manner of a very young child. It was not a developmental delay, but rather a posture. By replacing some lessons with "games" in which he could take a chance and see the effects of his decisions and by getting him to make bets – without too much risk – on whether his answers were right, the experimenters saw his attitude changes radically and his difficulties disappear. A new "didactical contract" with him had been constructed (Brousseau and Warfield 1998).

(b) *The Age of the Captain*. Researchers at the Institute for Research on the Teaching of Mathematics (IREM of Grenoble) offered students at age 8 the following problem: "On a boat there are 26 sheep and 10 goats. How old is the captain?" 76 of the 97 students answered, "36 years old."

This experiment produced a scandal. Some accused the teachers of stupefying their students; others reproached the researchers for "laying stupid traps for the children." In a letter to the experimenters, G. Brousseau indicated to them that it was a matter of an "effect of the contract" for which neither the students nor the teachers were responsible. So the researchers asked the students: "What do you think of this problem?" The students responded: "It is stupid!" The researchers ask: "Then why did you answer it?" The students answered: "Because the teacher asked for it!" The researchers ask: "And if the captain was 50 years old?" The students made a response: "The teacher didn't give the right numbers." A similar experiment done with established teachers produced the same behavior: for various reasons (such as the hope of an explanation that the teacher wanted to hear) the subjects produce the answer least incompatible with their knowledge, even when they see very well that it is false: the obligation of answering is stronger than that of answering correctly. Despite these explanations, for years the initial observation elicited strong criticisms of the work of the teachers (Sarrazy 1996).

### Didactical and Ethical Responsibility

The teacher has the responsibility of supporting the collective and individual activity of the students, of attesting in the end to the truth of the mathematics that has been done, of confirming it or giving proofs, of organizing it in the standard way, of identifying errors that have been or might be made and passing judgment on them (without passing judgment on their authors), and of providing the students with a moderate amount of individual help (as with the natural learning of a language.) Occasional individual help conforms to the collective process of mathematical communities. If the teacher finds himself acceding to an institutional function, he may be subject to obligations of equity and of means for which the responsibility is shared with the institution. Decisions made about the teacher and the students based on individual and isolated results are a dangerous absurdity. Experts, parents, and society share the responsibility for the effects of such decisions.

### Paradoxes of the Didactical Contract

The teacher wants to teach what she knows to a student who does not know it. This has many consequences, among them are as follows:

(a) Custom can determine pedagogical and psychological relationships, but not those proper to new knowledge, because new knowledge is a specific unexpected adventure that consists of a modification and an augmentation of old knowledge and of its implications. Thus, it cannot be known in advance by the student: the teacher can only commit himself to general procedures, and for her part the student cannot commit herself to a project of which she does not know the main part.

(b) Paradox of devolution: the knowledge and will of the teacher need to become those of the student, but what the student knows or does by the will of the teacher is not done or decided by his own judgment. The didactical contract can only succeed by being broken: the student takes the risk of taking on a responsibility from which he already releases the teacher (a paradox similar to that of Husserl).

(c) Paradox of the said and unsaid (consequence of the preceding): it is in what the teacher does not say that the student finds what she can say herself.

(d) Paradox of the actor: the teacher must pretend to discover with his students knowledge that is well known to him. The lesson is a stage production.

(e) The paradox of uncertainty: knowledge manifests itself and is learned by the reduction of uncertainty that it brings to a given situation. Without uncertainty or with too much uncertainty, there is neither adaptation nor learning. The result is that the optimal progression of normal individual or collective learning is accompanied by a normal optimal rate of errors. Artificially reducing it damages both individual and collective learning. It is useful to arrange things so that it is not always the same students who are condemned to supply the necessary errors.

(f) As in the case of learning, excessive or premature adaptation of complex knowledge to conditions that are too particular leads it to be replaced by a simplified and specific knowledge. This can then constitute an *epistemological* or *didactical obstacle* to its later adaptation to new conditions. (For example, division of natural numbers is associated with a meaning, sharing, which becomes an obstacle to understanding it in the case where a decimal number needs to be divided by a larger decimal number, e.g., 0.3/0.8.)

(g) The paradox of rhetoric and mathematics. To construct the students' mathematical knowledge and its logical organization, the teacher uses various rhetorical means, designed to capture their attention. The culture, pedagogical procedures, and even mathematical discourse (commentaries on mathematics) overflow with metaphors, analogies, metonyms, substitutions, word pictures, etc. The mathematical concepts are often constructed against these procedures (e.g., "correlation is not causation"). The teacher should thus at the same time as an educator teach the culture with its historical mistakes and as a specialist cause the rejection of the parts that science has disqualified.

These paradoxes can only be unraveled by specific situations and processes carefully planned out in the light of well-shared knowledge of mathematical and scientific *didactique* (Brousseau and Otte 1989; Brousseau 2005).

## Observations of Reactions of Teachers to Difficulties

These observations and the experimental and theoretical studies of the *didactical contract* make it possible to understand and predict the cumulative effects of teachers' decisions.

The contract manifests itself essentially in its ruptures. These are revealed by the reactions of the students or by the interventions of the teachers, and they can be classified as follows:

(a) *Abandonment*. The teacher does not react to an error made by the students (e.g., because it would be too complicated to explain it), or she repeats the question identically or she gives the complete solution.

(b) The progressive *reduction* or manipulation of the students' uncertainty, using a great variety of means:
   - Bringing in mathematical, technical, or methodological information
   - Decomposition of the problem into intermediate questions (decomposition of the objectives)
   - Use of various extra-mathematical rhetorical means: analogies, metaphors, metonyms, or mnemonic minders (the "Topaze effect")

(c) *Critical commentary on the errors*, the question, the knowledge, or the material

(d) *A trial of the student* and its consequences: penalties, discrimination, and individualization

In case of failure, the contract obligates the teacher to try again. The new attempt either replaces the preceding one or criticizes and corrects it, making of it a new teaching object (a meta-process).

For each of these types of response, there are conditions under which it is the most appropriate response; *thus there is no universal response*.

For example, Novotná and Hošpesová (2007) identify and classify the behaviors whose systematic repetition generates Topaze effects:

1. Explicitly, the teacher
   (a) Gives the steps of the solution and transforms it into the execution of a sequence of tasks
   (b) Asks questions in a sequence that mandates the procedures of the solution
   (c) Gives warnings about a possible error
   (d) Enumerates previous experiences or knowledge, pointing out analogies with problems that have previously been resolved or are obvious or well known
2. Implicitly, he
   (a) Reformulates students' propositions or his own
   (b) Uses "guide" words
   (c) Pronounces the first syllable of words
   (d) Poses new questions that orient the student towards the solution
   (e) Shows doubt about dubious initiatives

Their research confirms that the resulting Topaze effects go unnoticed but have a high cost. The students, apparently active, become dependent on this aid and lose their confidence in themselves. An error is understood to be a transgression of the didactical contract and proof itself, badly supported, becomes something to be learned rather than understood.

By using jointly the notions of *milieu*, of situation, and of the didactical contract, Perrin-Glorian and Hersant (2003) were able to show in numerous examples on the one hand what the student and the *milieu* are in charge of and thus the occasions for learning that are their responsibility, and on the other hand the help brought in by the teacher.

## Predicting and Explaining Certain Long-Term Effects

The uncontrolled recursive resumption of the same type of response leads to drifting and inevitable failures. For example, for the students studying the procedure for solving problems by the same pedagogical methods, studying theorems is just as costly, less sure, and less useful.

As another example, a sequence of meta-slippages contributed to the failure of the reform of "modern math": the foundations of mathematics were interpreted by "naïve" set theory, which was itself formalized into algebra. This was metaphorically represented by "graphs," which were finally interpreted in vernacular language. Each representation betrayed the preceding one slightly and supported new conventions, and the slippages were ultimately uncontrollable. In the absence of didactical situations and proven epistemological processes, varying the types of response seems to be the best strategy.

Enforcing requirements based on the results of individuals leads to a mincing up of the objectives, to the abandonment of high-level objectives, and to addressing the objectives by painful behaviorist methods. These slow the learning and lead to an individualization that slows it yet further. Each of these tends to destroy the role of provisional knowledge and to augment mechanically the time for teaching and learning without positive impact on the results.

Specifying the means of teaching a subject involves precise and specific protocols for performances that are known and accepted by the population. Specifying required results for the teachers as for the students has absolutely no scientific basis. Its disastrous effects, predicted since 1978, have been observable for 40 years.

The mean rate of success is a "regulated" variable of the system. Otherwise stated, the global progress of *all* the students is less rapid if one requires at every stage a 100 % rate of success. The conception of mathematical activity as an adventure and a collective practice makes it possible to mitigate the effects of difference in rhythms of learning.

It seems that today the requirements of the different partners of teaching towards one another are less and less compatible with each other, perhaps because of the variety of possibilities, of offers, and of perspectives provided by numerous ill-coordinated sciences.

The experiments on teaching rational and decimal numbers (Brousseau 1997) or statistics and probability (Brousseau et al. 2002) prove that it is possible to organize efficient and communicable processes with the help of didactical contracts based on the nature of the knowledge to be acquired.

## Extensions

Sarrazy (1996, 1997) studied the pitfalls of these meta-didactical slippages and more particularly those that are consequences of a teaching that aims at making the contractual expectations explicit, frequently taking the form of the teaching of metacognitive or heuristic procedures – or even of algorithms for solving problems. Complementing the work engaged in by Schubauer-Léoni (1986) in a psychosocial approach to the didactical contract, Sarrazy radicalized the paradox of the consubstantial rule (A rule does not contain in itself its conditions for use) of the contract at the intersection of the theory of situations and Wittgensteinian anthropology (Wittgenstein 1953). Contrary to the psychological or linguistic interpretations of the contract (such as that of "the age of the captain"), he showed how these slippages lead to a veritable dematematization of teaching by a displacement of the goals of the contract. These works also made it possible to establish the primacy of the role of situations and that of school cultures (Sarrazy 2002; Clanché and Sarrazy 2002; Novotná and Sarrazy 2011) and family habits conceived as *backgrounds* (Searle 1979) of the didactical contract. These backgrounds make it possible to explain the differences *in sensitivity to the didactical contract*, that is, the objective differences of the various positions of the students with regard to the implicit elements of the contract and thus of their spontaneous (and not necessarily conscious or thought-out) "representations" of the division of responsibilities in the contract (e.g., some of the students answer that the captain is 36 years old, others refrain from giving an answer, still others finally say that they do not know, and some of them authorize themselves to declare that this problem is absurd). These results reaffirm the importance of the Theory of Situations and notably the explicative power of the contract, but also underline the interest of considering the pedagogical ideologies of the teachers and the cultures of the students in the interpretation of contractual phenomena. These works together lead into a perspective of study baptized "anthropo-didactique," situating the phenomena of the didactical contract in the double perspective mentioned above. This theoretical current has made it possible to reinterpret in a fertile way a certain number of phenomena of teaching (*lato sensu*), as much on the micro-didactical level as the macro-didactical, and of their interactions, such as school inequities (Sarrazy 2002), school difficulties (Clanché and Sarrazy 2002; Sarrazy and Novotna´ 2005) heterogeneities, didactical time and didactical visibility (Chopin 2011), student teacher interactions, and the effects of the *genre*. These themes have traditionally been studied by connected disciplines (psychology, sociology, anthropology, etc.) but independently of the didactical dimensions which in fact are necessarily involved in these phenomena. This approach thus realized the study of what Brousseau designated in 1991 "didactical conversions": "The causes of phenomena of a non-didactical nature can only influence didactical phenomena by the intermediary of elements having their origin in didactical theory." This "reinterpretation" of a non-didactical phenomenon in didactical terms is a didactical conversion (Brousseau and Centeno 1991, p. 186).

## Cross-References

► Didactic Situations in Mathematics Education

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# Didactic Engineering in Mathematics Education

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## Keywords

Didactical engineering; Theory of didactical situations; A priori and a posteriori analysis; Research methodology; Classroom design; Development activities

## Definition

In mathematics education, there exists a tradition of research giving a central role to the design of teaching sessions and their experimentation in classrooms. Didactical engineering, which emerged in the early 1980s and continuously developed since that time, is an important form taken by this tradition. In the educational community, it mainly denotes today a research methodology based on the controlled design and experimentation of teaching sequences and adopting an internal mode of validation based on the comparison between the a priori and a posteriori analyses of these. However, since its emergence, the expression didactical engineering has also been used for denoting development activities, referring to the design of educational resources based on research results or constructions and to the work of didactical engineers.

## History

From its emergence as an academic field of study, mathematics education has been associated with the design and experimentation of innovative teaching practices, in terms of both mathematical content and pedagogy. The importance to be attached to design was early stressed by researchers as Brousseau and Wittman, for instance, who very early considered that mathematics education was a genuine field of research that should develop its own frameworks and practices and not just a field of application for other sciences such as mathematics and psychology.

The idea of didactical engineering (DE), which emerged in French didactics in the early 1980s, contributed to firmly establish the place of design in mathematics education research. Foundational texts regarding DE such as Chevallard (1982) make clear that the ambition of didactic research of understanding and improving the functioning of didactic systems where the teaching and learning of mathematics takes place cannot be achieved without considering these systems in their concrete functioning, paying the necessary attention to the different constraints and forces acting on them. Controlled realizations in classrooms should thus be given a prominent role in research methodologies for identifying, producing, and reproducing didactic phenomena, for testing didactic constructions. As a research methodology, DE emerged with this ambition, relying on the conceptual tools provided by the Theory of Didactical Situations (TDS), and conversely contributing to its consolidation and evolution (Brousseau 1997). It quickly became a well-defined and privileged methodology in the French didactic community, accompanying the development of research from elementary school up to university level as evidenced in the synthesis proposed at the 1989 Summer School of Didactics of Mathematics (Artigue 1990, 1992).

From the 1990s, DE migrated outside its original habitat, being extended to the design of teacher preparation and professional development sessions, used by didacticians from other disciplines, for instance, physical sciences or sports, and also by researchers in mathematics education in different countries. Simultaneously, the progressive shift of research attention towards teachers increased the use of methodologies based on naturalistic observations of classrooms, leading to theoretical developments and results that, in turn, affected DE. Moreover, design-based research perspectives emerged in other contexts, independently of DE (Design-Based Research Collaborative 2003). These evolutions and the resulting challenges are analyzed in Margolinas et al. (2011).

## DE as a Research Methodology

As a research methodology, DE is classically structured into four different phases: preliminary analyses; design and a priori analysis; realization, observation, and data collection; and a posteriori analysis and validation (Artigue 1990, 2009).

Preliminary analyses usually include three main dimensions: an epistemological analysis of the mathematical content at stake, an analysis of the conditions and constraints that the DE will face, and an analysis of what educational research has to offer for supporting the design.

In the second phase, design and a priori analysis, research hypotheses are engaged in the process. Design requires a number of choices, from global to local. They determine *didactic variables*, which condition the interactions between students and knowledge, between students and between students and teachers, thus the opportunities that students have to learn. In line with TDS, in design, particular importance is attached:

To the search for *fundamental situations*, i.e., mathematical situations encapsulating the epistemological essence of the concepts

To the characteristics of the *milieu* with which the students will interact in order to maximize the potential it offers for autonomous action and productive feedback

To the organization of *devolution* and *institutionalization* processes by which the teacher, on the one hand, makes students accept the mathematical responsibility of solving the task and, on the other hand, connects the knowledge they produce to the scholarly knowledge aimed at

The a priori analysis makes clear these choices and their relation to the research hypotheses. Conjectures are made regarding the possible dynamic of the situation, students' interaction with the *milieu*, students' strategies, their evolution and their outcomes, about teacher's necessary input and role. Such conjectures regard not individuals but a *generic and epistemic student* entering the mathematical situation with some supposed knowledge background and accepting to enter the mathematical game proposed to her. The actual realization will involve students with their personal specificities and history, but the goal of the a priori analysis is not to anticipate all these personal behavior; it is to build a reference with which classroom realizations will be contrasted in the a posteriori analysis.

During the phase of realization, data are collected for the analysis a posteriori. The nature of these data depends on the precise goals of the DE, the hypotheses tested, and the conjectures made in the a priori analysis. The realization can lead to some adaptation of the design in *itinere*, especially when the DE is of substantial size. These adaptations are documented and taken into account in the a posteriori analysis.

A posteriori analysis is organized in terms of contrast with the a priori analysis. Up to what point the data collected during the realization support the a priori analysis? What are the significant convergences and divergences and how to interpret them? The hypotheses underlying the design are put to the test in this contrast. There are always differences between the reference provided by the a priori analysis and the contingence analyzed in the a posteriori analysis. The validation of the hypotheses underlying the design does not, thus impose perfect match between the two analyses. Moreover, the validation of the research hypotheses may require the collection of complementary data to those collected during the classroom, especially for appreciating the learning outcomes of the process. Statistical tools can be used, but what is essential is that validation is internal, not in terms of external comparison between control and experimental groups.

These are the characteristics of DE as research methodology when associated with the conception of a sequence of classroom sessions having a precise mathematical aim. However, as shown in Margolinas et al. (2011), this methodology has been extended to other contexts such as teacher education, more open activities such as project work or modeling activities, and even mathematical activities carried out in informal settings. In these last cases, the content of preliminary analyses must be adapted; what the design ambitions to control in terms of learning trajectories and the reference provided by the a priori analysis cannot exactly have the same nature.

## Realizations

The first exemplars of DE research regarded elementary school. Paradigmatic examples are the long-term designs produced by Brousseau, on the one hand, and by Douady, in the other hand, for extending the field of numbers from whole numbers to rational numbers and decimals (Brousseau et al. 2014; Douady 1986). The two constructions were different, but they proved both to be successful in the experimental settings where they were tested, and they significantly contributed to the state of the art regarding the learning and teaching of numbers. Beyond that, they had theoretical implications. The development of the *tool-object dialectics* and the identification of the learning potential offered by the organization of *games between mathematical settings* by Douady are intrinsically linked to her DE for the extension of the number field; the idea of *obsolescence of didactic situations* emerged from the attempts made at reproducing Brousseau's DE year after year. These are only two examples among the many we could mention. DEs were progressively developed at all levels of schooling, covering a diversity of mathematical domains and addressing a diversity of research issues. At university level, for instance, paradigmatic examples remain the construction developed by Artigue and Rogalski for the study of differential equations, combining qualitative, algebraic, and numerical approaches to this topic (Artigue 1993) and that developed by Legrand for the teaching of Riemann integral within the theoretical framework of the scientific debate (Legrand 2001). Both were experimented with first year students and showed their resistance to students' diversity. Constraints met at more advanced levels of schooling contributed to the deepening of the reflection on an optimized organization of the sharing of mathematical responsibilities between students and teacher in DE and to the softening of the conditions and structures often imposed to design at more elementary levels. DE was also enriched by its use in other domains than mathematics and by researchers trained in other cultural traditions. A good example of it is provided by its use in sports, already mentioned, and by the elaboration of DE combining the theoretical support of TDS and that of semiotic approaches (cf. for instance, (Falcade et al. 2007; Maschietto 2008) using such combination for studying the educational potential of digital technologies). More globally, ICT has always been a privileged domain for DE, for exploring and testing the potential of new technologies, and for supporting technological development as well as theoretical advances in that area. Another interesting example is the use of DE within the socio-epistemological framework in mathematics education (Farfán 1997; Cantoral and Farfán 2003).

## Challenges and Perspectives

DE developed as a research methodology, but DE from the beginning had also the ambition of providing a model for productive interaction between fundamental research and action on didactic systems. DEs produced by research were natural candidates for supporting such a productive interaction. Quite soon, researchers however experienced the fact that the DEs they had developed and successfully tested in experimental setting did not resist to the usual dissemination processes. This problem partly motivated the shift of interest towards teachers' representations and practices. Addressing it requires to clearly differentiate research DE (RDE) and development DE (DDE), acknowledging that these cannot obey the same levels of control. In Margolinas (2011), this issue is especially addressed by Perrin-Glorian through the idea of DE of second generation, in which the progressive loss of control that the elaboration of a DDE requires is co-organized in collaboration with teachers and illustrated by an example. Such a strategy implies a renewed conception of dissemination of research results, in line with the current evolution of vision of relationships between researchers and teachers.

Another challenge is the issue of relationships between the tradition of DE described above and the different forms of design which are developing in mathematics education under the umbrella of design-based research, reflecting the increased interest for design in the field, or the vision of design introduced in the Anthropological Theory of Didactics (ATD) in the last decade in terms of Activities of Study and Research (ASR) and Courses of Study and Research (CSR) (Chevallard 2006). Despite de fact that ATD and TSD emerged in the same culture, the visions of design they propose today present substantial differences. Establishing productive connections between the two approaches without losing the coherence proper to each of them is a problem not fully solved but also addressed in Margolinas (2011).

## Cross-References


- ▶ Anthropological Approaches in Mathematics Education, French Perspectives
- ▶ Design Research in Mathematics Education
- ▶ Didactic Situations in Mathematics Education


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# Didactic Situations in Mathematics Education

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## Keywords

Didactical situation; A-didactical situation; Mathematical situation; Acculturation; Didactique

## Didactical Situation

A *didactical situation* in mathematics is a project organized so as to cause one or some students to *appropriate* some piece of mathematical reference knowledge. (The organizer and the student may be individuals, a population, institutions, and so on.)

### Components

Every didactical process is a sequence of situations, each pertaining to one of the following three types:

A "**situation of devolution**" in which the teacher sets the students up:

- to accept boldly and confidently the challenge of an engaging and instructive mathematical situation whose instructions he gives in advance: conditions, rules, goal, and above all the criterion for success
- and to do it without his help, on their own responsibility (Brousseau 1997, pp. 230–235)

A "**mathematical situation**" that supports the students in autonomous mathematical activities, both individual and collective, that represent those in use by mathematicians. Rather than looking to gain credit for themselves, the students are engaged in:

- Producing "new" statements and discussing their validity
- Making decisions, formulating hypotheses, predicting and judging their consequences, attempting to communicate information, producing and organizing models, arguments and proofs, etc., adequate for certain precise projects
- and evaluating and correcting by themselves the consequences of their choices

It is thus not the students who are in question, but some conjectures and some knowledge (Brousseau 1997, pp. 230–235).

A "**situation of institutionalization**" in which the teacher:

- Takes note of the progress of the mathematical situation, of the questions and answers that have been obtained or studied from it, and of those that have emerged, and places them within the perspective of the curriculum
- Distinguishes among the pieces of knowledge (*connaissances*) that have appeared those that have revealed themselves to be false and those that are correct, and among the latter those that will serve as references, presenting in that case the canonical way of formulating them
- And draws conclusions for the organization of further sequences (exercises, problems, etc.) (Brousseau 1997, pp. 235–243).

### Teaching methods

Teaching methods can be distinguished first by the interpretation, the role, and the importance assigned to each of the components. Here are two very different examples of this:

Example 1: In certain methods, devolution consists of a prerequisite teaching of new knowledge (a lecture), followed by examples and exercises, and followed by the presentation of problems whose autonomous solution by the students constitutes the mathematical situation. Institutionalization consists of correction, evaluation, and the conclusions that the teacher draws from them. Sometimes the mathematical situation is considered only as a means of verifying the individual learning produced by the lecture.

Example 2: In other methods, devolution is reduced to the organization, presentation, and staging of an individual or collective *mathematical situation* aimed at provoking activities and processes like those of mathematicians: a search for solutions or proofs but also production of questions, hypotheses or conjectures, reformulations, definitions and study of objects, sorting, debates, challenges, etc. Learning is the means and the product of this activity. Institutionalization then consists of identifying and organizing, among the correct pieces of knowledge produced by the students, those consistent with common usage and with *accepted mathematical knowledge*, and among those the ones that are sufficiently "acquired" by all of the students so that the teacher and students can refer to them with each other in future mathematical situations. The "lecture" consists of a conclusion and of putting things in order. Exercises are a means of training available to the students (Margolinas et al. 2005; Illustrative examples in Warfield 2007).

## Origin and Necessity of the Concept of "Didactical Mathematical Situation"

### The Reform of the Foundations (1907–1980)

The term "didactical situation" appeared in the 1960s with the meaning "mathematical situation for teaching."

The new mathematical concepts on which teaching was to be rebuilt were communicated by formalized texts in a symbolic language unintelligible to students and/or by reformulations, metaphorical representations, and ambiguous commentaries. On the other hand, they referred necessarily to examples taken from the classical mathematics that they were reorganizing. The "fundamental" concepts were thereby postponed to the end of the studies.

The challenge was thus to imagine conditions, situations, that could induce in the students the geneses of fundamental mathematical concepts, in a form and by processes comparable to those put into operation by mathematicians *before* the final presentation of their results, in the process mathematical development. This idea found justification in the work of the period: the acquisition of language does not follow the classic formulation of its grammar, and Piaget identified certain mathematical structures in the genesis of logical thought in children.

Conceiving of similar geneses, and especially imagining conditions capable of inducing them, could only arise from the competence of the mathematical community. It did so through a gigantic effort of its researchers and of its teachers, realizing as it did so the aspirations of pedagogues like Dewey, Montessori, or Freinet. But diffusing these conceptions more widely, *against* the traditional culture of teaching, posed yet more redoubtable problems, which have not at this point been surmounted.

### Learning Mathematics by Doing It Reverses the Classic Pedagogical Order

The teaching of mathematics is based on a text or some texts that express it in a canonical way (i.e., in the order: definitions, properties and theorems, and finally proofs). The classical conception consists of teaching using the texts first, so that a student could never argue that he or she is being required to use a piece of knowledge that was not first revealed and taught. Teaching pieces of knowledge before needing to use them gives the appearance of being a "rational" method, but it introduces a disassociation (learn with metamethods that have no relationship with the object and its use), an inversion (learn terms before understanding them and doing anything with them), and finally teleological requirements: the student is blamed in the course of learning for not having first learned what is in fact the goal of the teaching that is going on. This epistemological error greatly limits the field of application, the age of learning, and the degree of success of the classical method.

Conversely, direct *acculturation* to specific mathematical practices that can produce these texts brings their learning closer to that of vernacular language or natural thought. Everything then rests on the power of the situations to induce in the children the "process of mathematization."

It would be absurd and detrimental to want to exclude some method or to uniformly recommend it over some other. The conditions to which each is best adapted must be scientifically studied and their advantages combined. For example, situations of cooperative discovery and collective adventures create homogeneity and motivation and make it possible to acquire the classical practices by use. Exercises can help in doing well and rapidly what is worthwhile and has been understood (Brousseau 1992).

### The Project of a Mathematical Science: *Didactique*

The organization of these mathematical situations and their succession obey various reasons: mathematical, epistemological, rational, empirical, ideological, etc. Their scientific study combines:

1. The (anthropological) observation and the analysis (semiological) of the practices and conceptions of the teachers and of the students
2. The conception, realization, and experimental study of original mathematical situations appropriate to each of the pieces of mathematical knowledge aimed for (▶ Didactic engineering in mathematics education)
3. The inventory of possible choices, their modeling in the form of situations, the experimental and theoretical study of their conditions and of their properties, and the creation of appropriate instruments of analysis (theory of didactical situations)

The conception of these situations requires prior and specific mathematical study of the *knowledge to be taught*, along with that of its historical genesis, of its epistemological properties, and of its possible didactical geneses and their properties. But the scientific confrontation of these speculations with actual teaching is fundamental.

The theory of situations, its concepts, and its research methods is one of the most ambitious among the numerous scientific approaches to the phenomenon of *didactique*.

But well before being able to offer teachers, in the name of mathematicians, an aid, or some ready-to-implement solutions for teaching mathematics, *didactique* must describe, understand, and explain in a scientific manner mathematical activity and its possible *didactical transpositions*.

*Didactique plays a role* in the reorganization and transformation of mathematical knowledge. Its results are thus first addressed to the community of mathematicians, to whom falls – for good reasons – the responsibility towards society of the reference in teaching materials to the established knowledge of its specialty. *Didactique* of mathematics requires specific concepts and methods of study. It thus joins logic, computer science, epistemology, history of mathematics, and so on as one of the mathematical sciences. It takes charge of the knowledge of everything that is specific to the discovery, the diffusion, or the appropriation of each piece of mathematical knowledge, new or not, that results from the adventures specific to it. It extends, enriches, and puts to the test the general contributions of classical social sciences, which are indispensible but insufficient for clarifying all the facets of this teaching.

## Mathematical Situations

### Definition

Every mathematical concept is the solution of at least one specific system of mathematical conditions, which itself can be interpreted by at least one situation, for example, a game, whose solution (decision, message, argument) is one of the typical manifestations of the concept. A situation is composed of a milieu and a project. The duration of the life of a mathematical situation (the time of studying it) can vary from a few seconds to several centuries for humanity or several months for teaching.

### Examples

Example 1: Children 4–5 years old. From a collection of thirty or so familiar objects, 5 or 6 are hidden in a box by a child in the morning. In the afternoon, she is supposed to enumerate them to another child, who confirms the presence or absence of the objects she names. The solution of this game is the creation, enumeration, and use of lists. Knowing neither how to read nor how to write, the children represent the objects in their own way (pictograms) to distinguish them, first individually and then collectively. The lists of symbols represent sets; belonging or not, conjunctions and disjunctions of characters are used, corrected, understood, and formulated in vernacular language (Pérès 1984; Digneau 1980).

Example 2: Children 10–11 years old. To be certain of the number of white marbles contained in a firmly closed opaque bottle with a known number of marbles, some white and some black, students invent hypothesis testing and the measure of events (33 short sessions) (Brousseau et al. 2002).

A great many researchers have imagined and studied various types of situations destined for all sorts of notions, for all levels of school and even university. See, for example, Bessot (2000), Laborde and Perrin-Glorian (2005), Bloch (2003).

### Types of Mathematical Knowledge, Reference Knowledge (*Savoirs*)

Classical methods forbid the teacher from tolerating without immediate correction, the manifestation of anything contrary to written established mathematics. A genuine mathematical activity necessarily gives rise to all sorts of knowledge. Some is knowledge sought for – these are the references, recognized as correct, true and known: they are professed and expected. But there also necessarily appear pieces of knowledge that are ill made, ill formed, incomplete, doubtful, false, or even inexpressible. They are "knowledge" in the sense of "the trace of an encounter." Their presence, whether or not firmly nailed down, is indispensible to thought. For example, a theorem that the student knows very well (savoir), but about whose usefulness in a situation is unsure, functions provisionally as a simple piece of nonestablished knowledge (connaissance).

The teacher cannot intervene in this flow of activities without blocking its functioning and must therefore delegate the responsibility for exercising a pragmatic penalty to the initiatives of the students that result from their knowledge. He entrusts it to a *milieu* that is clearly stripped of teleological or pedagogical intentions [its reactions depend neither on the intended goal nor on the individuals].

**The *milieu*** of a situation is what the students exercise their actions on and what gives them objective responses. The teacher thus entrusts to the *milieu* the job of showing the students' errors by their effects, without using an argument of authority or revealing any intentions. The milieu may comprise informative texts; material objects; other students, cooperating or concurrent; and so on. To this must be added the established knowledge of the student as well as her memories of relevant previous events, and objective conditions, that may not be known to the student but that intervene in her choices and in the effects of her decisions. The *cognitive variables* of the situation are those whose value has an influence on the issue of the situation or on the knowledge developed. These variables are didactical if their value can be chosen by the teacher (the sex of the students may influence the progress of a situation, but it is not a didactical variable). The *milieu* can be interpreted metaphorically by games that present some states that are permissible and some that are excluded, rules of action, and issues of which one would be the goal sought (Warfield 2007).

### Examples of *Milieux*

1. *Cabri geometry* permits the student to realize, in the context of geometrical objects and transformations, which of her projects are constructible, that is, compatible with the axioms (Laborde et al. 1995). The projects lead the students to gain knowledge of, formulate, and test what the *milieu* permits them to glimpse.
2. Analysis of a situation. The reader will find an example of the analysis of a didactical situation (the Race to 20), of its *milieu*, of the strategies used by students, of the theorems in action that support them, and of the didactical methods that make it possible to lead them to a complete proof and then to extend it so as to have them reinvent an algorithm: the search for the remainder of a Euclidean division, in Brousseau (1997, pp. 3–18). This work also includes numerous other examples.

**The project** is an objective, a final state of the *milieu*, the response to a question, or even a pretext for exploration. It is what explains, justifies, or condemns after the fact the choices that have been chosen or ventured by the subject.

**The resolution** is the occasion to put to trial not the student, but a way of knowing.

**Remarks:** The *milieu* of a situation is not a natural *milieu* and does not turn mathematics into a sort of experimental science. The project is essential, and its goal is to establish the consistency of certain statements.

Different branches of mathematics developed in different *milieux*: geometry in the knowledge of space, probability in the statistics of games, algebra in arithmetic, arithmetic in the measurement of amounts, etc.

In elementary teaching, knowledge of these *milieux* is neither spontaneous nor contained in their mathematical interpretation. For example, the knowledge that is useful for finding one's way around a big city merits specific work that cannot be reduced to some geometry.

## Types of Mathematical Situations Characteristic of Activities, of Pieces of Knowledge, and of Pieces of Mathematical Learning

The mathematical knowledge of a student manifests itself in her interactions with a milieu, as a means of attaining or maintaining a desired state. These interactions are grouped in four types of situations which are, in the order of didactical necessity, inverse to the ordinary chronological order:

1. Situation of *reference*: A person (student or teacher) refers the person asking to a piece of mathematical knowledge (a proof, a theorem, a definition, etc.) that belongs to their common repertoire (Perrin-Glorian 1993).
2. Situation of *argumentation* (of *proof*): A proposer communicates to an opponent an argument, an element of proof. He makes use for that of their common repertoire which his message tends to augment. The argument makes reference to a *milieu* and a (mathematical) project in common that gives it its meaning and its value. The two speakers have the same information, in particular, on the *milieu*, the same rights of refutation, and the same interest in arriving at a consistent agreement (for an action on the *milieu*).
3. Situation of *information (communication)*: The transmitter and receiver cooperate on an action on the *milieu*, in whose success they are interested and which depends on their joint action. Neither of the two has at the same time all of the information and all of the necessary means of action. They exchange messages in order to realize a common mathematical project.
4. Situation of *action*: A subject intervenes on the *milieu* to modify it with a determined aim. She observes the effect of her actions and attempts to anticipate them by constructing pieces of knowledge, conscious and explainable or not. This situation encompasses all of the others, but it extends beyond them by stimulating the existence of inexpressible and possibly even unconscious models of action.

Each of these types of situation creates *distinct typical motivations* (modify a *milieu*, communicate some information, debate the validity of a declaration, establish a reference) that mobilize and expand the *corresponding repertoires* (implicit models of action, semiological or linguistic repertoires, logical repertoires, mathematics or metamathematics, established knowledge and theory) which are themselves acquired according to specific different *modes of learning or acculturation*.

The *actual situations* are, every one of them, specific to a precise piece of knowledge.

This is the level which must be appealed to in order to judge the relevance of the contributions of other scientific domains (pedagogy, psychology, sociology, etc.).

## The Processes

Different modes of composition and articulation of these elementary situations make it possible to create composite situations and sequences of situations that form processes:

1. *Process of mathematization*: A sequence of autonomous mathematical situations that are introduced by didactical interventions of the teacher and that work together towards

the construction of the same complex knowledge (e.g., rational and decimal numbers (Brousseau et al. 2004, 2007, 2008, 2009)).

2. *Genetic situation*: It introduces and without other external intervention generates the sequence of situations that lead to the acquisition of a concept (e.g., how many white marbles [article cited]).

The didactical work of the teacher then consists of maintaining the intensity and the relevance of the exchanges and implementing their progress and their conclusion. Examples of process: on areas, Perrin-Glorian M.J. (1992), and on geometry, Salin M.H., Berthelot R. (1998).

## Some of the Results of Research on Didactical Situation

The notion of didactical situation was used in many research projects. It gave rise to numerous reflections and, with modifications, was expanded in more work than it is possible to summarize or cite here:

1. One of its first results was to establish that adaptation to certain conditions tends to render it more difficult to adapt to others and thus creates the phenomenon of didactical obstacles, then to show that the history of mathematics presents phenomena similar to the epistemological obstacles detected by G. Bachelard, and finally, to take advantage of this phenomenon in teaching by use of situations presenting "jumps in informational complexity" (▶ Epistemological Obstacles in Mathematics Education)
2. Research on situations had the goal of furnishing alternatives to the classical conceptions that showed their limitations in the face of the influx of knowledge to be taught and of the fundamental reorganizations necessitated by that influx. This research showed the importance of the role of the *unsayable* in mathematical situations and of the *unsaid* in the didactical relationship.

Rather than imagining teaching and producing learning of the *texts* that resulted from real mathematical activity by universal, that is nonspecific, nonmathematical teaching methods, it appeared that it would be preferable to have the students themselves produce this knowledge and these texts, thanks to specific mathematical activities that best stimulated the real activity of mathematicians.

The many didactical situations realized showed that this project was realizable. Experiments proved it. Curricula were conceived, experimented with, and reproduced for all the branches of mathematics and for all the basic levels of teaching (3–12 years old) in an establishment conceived for the purpose (the COREM).

Currently they cannot be developed because of the complexity of knowledge necessary for the teachers to conduct them and for the public to accept them.

This research produced counterexamples to most of the "universal principles," explicit or implicit, of classical didactics, for behaviorist methods as well as radical constructivism. It showed, for example, that in the classical conception, errors can have no status other than that of being far from some norm. They are interpreted as a failure of the student and/or the teacher that involves their responsibility and ultimately their guilt for a failure of their will. This absurd process generates very bad working conditions for the students and for the teachers.

Among many other results, The classical conception led to seeking out individualization of teaching, but this individualization did not improve the results, because mathematical knowledge is produced by the cooperation of numerous individuals operating in the same community, and no isolated brain can produce the exact form that history has given it. For a large portion of the students, the real use of communication and mathematical debates is indispensible.

The concept of situation has been the object and has been illustrated in a great deal of research of different types:

1. *Empirical*, so as to identify the observables of a given teaching episode and analyze them a priori and a posteriori
2. *Experimental*, to conceive of either a precise teaching project (engineering) or a teaching design (of cognitive psychology, of sociology, of *didactique*, etc.)

3. *Theoretical*, to study their properties (economic, ergonomic, etc.) on appropriate models, possibly mathematics (automata, games, various systems), or conceive of modes or specific indices for these studies (implicative statistical analysis for the study of dependencies) (Artigue and Perrin-Glorian 1991)

The results of these studies were used in many research projects more particularly centered on students, teachers, or school knowledge and the didactical transposition (Mercier et al. 2000).

## Research Perspectives

1. The study of the optimal conditions for articulation of mathematical situations and of institutionalization is a necessity. Pieces of "knowledge" proposed in mathematical situations, whether erroneous or valid, must evolve sufficiently rapidly to arrive at established knowledge. Making these pieces of provisional knowledge the object of classical teaching, on the pretext that they were produced by the students themselves, is a major error. On the contrary, the reorganization of spontaneous knowledge around established knowledge with a complement of information (a lecture) is a mathematical necessity that offers an indispensible time gain. *Didactique* is a science of dynamic equilibrium of situations.
2. What are the relationships between the teaching of mathematics (*microdidactique)* and the explicit or latent mathematical or didactical conceptions held by the various social, economic, cultural, and scientific components of a society (*macrodidactique*)?
3. What are the factors in the failure of the reform of modern mathematics?

## Cross-References

▶ Didactic Contract in Mathematics Education
▶ Didactic Engineering in Mathematics Education
▶ Epistemological Obstacles in Mathematics Education

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# Didactic Transposition in Mathematics Education

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## Keywords

Anthropological theory of the didactic; Scholarly knowledge; Knowledge to be taught; Institutional transposition; Noosphere; Ecology of knowledge; Reference epistemological models

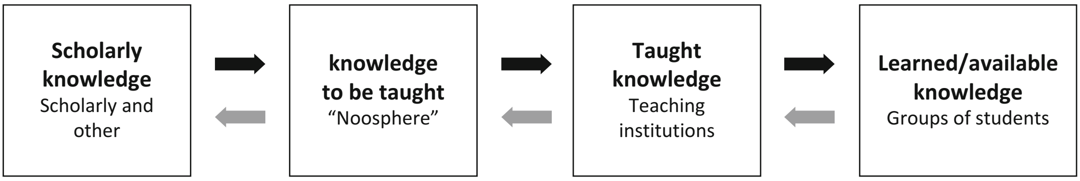
## Definition

The process of didactic transposition refers to the transformations an object or a body of knowledge undergoes from the moment it is produced, put into use, selected, and designed to be taught until it is actually taught in a given educational institution. The notion was introduced in the field of didactics of mathematics by Yves Chevallard (1985, 1992b). It highlights the fact that what is taught at school is originated in other institutions, constructed in concrete practices, and organized in particular sets of objects. In the case of mathematics or any other subject, the *taught knowledge*, the concrete practices and bodies of knowledge proposed to be learned at school, originates from what is called the *scholarly knowledge*, generally produced at universities and other scholarly institutions, also integrating elements taken from a variety of related social practices. When one wishes to "transpose" a body of knowledge from its original habitat to school, specific work should be carried out to rebuild an appropriate environment with activities aimed at making this knowledge "teachable," meaningful, and useful.

Different actors participate in this *transpositive work* (see Fig. 1): producers of knowledge, teachers, curriculum designers, etc. They belong to what is called the *noosphere*, the sphere of those who "think" about teaching, an intermediary between the teaching system and society. Its main role is to negotiate and cope with the demands made by society on the teaching system while preserving the illusion of "authenticity" of the knowledge taught at school, thus possibly denying the existence of the process of didactic transposition itself. It must appear that *taught knowledge* is not an invention of school. Although it cannot be a reproduction of *scholarly knowledge*, it should look like preserving its main elements. For instance, the body of knowledge taught at school under the label of "geometry" (or "mechanics," "music," etc.) has to appear as genuine. It is thus important to understand the choices made in the designation of the *knowledge to be taught* and the construction of the *taught knowledge* to analyze what is transposed and why and what mechanisms explain its final organization and to understand what aspects are omitted and will therefore not be diffused.

## Scope

Besides mathematics, research on didactic transposition processes has been carried out in many other educational fields, such as the natural




**Didactic Transposition in Mathematics Education, Fig. 1** Diagram of the process of didactic transposition



sciences, philosophy, music, language, technology, and physical education. These investigations have spread faster in the French- and Spanish-speaking communities (Arsac 1992; Arsac et al. 1994; Bosch and Gascón 2006) than in the English-speaking ones, although some prominent figures soon contributed to develop the first transpositive analyses (Kang and Kilpatrick 1992). The notion of didactic transposition has been generalized to *institutional transposition* (Chevallard 1989, 1992a; Artaud 1995) when knowledge is transposed from one social institution to another. Because of social needs, bodies of knowledge originated and developed in different "places" or institutions of society need to "live" in other institutions where they should be transposed. They have to be transformed, deconstructed, and reconstructed in order to adapt to their new institutional setting. For instance, the mathematical objects used by economists, geographers, or musicians need to be integrated in other practices commonly ignored by the mathematicians who produced them. It is clear from the history of science that institutional transpositions – including didactic ones – do not necessarily produce degraded versions of the initial bodies of knowledge. Sometimes the transpositive work *improves* the organization of knowledge and makes it more understandable, structured, and accurate to the point that the knowledge originally transposed is itself bettered. The organization of knowledge in fields and disciplines as it exists today is the fruit of complex and changing historical interactional processes of institutional and didactic transpositions that are not well known yet.

## An Emancipatory Tool

In a field of research, new notions are not only introduced to describe reality but to provide new ways of questioning and new possibilities to modify it. The notion of didactic transposition is conceived, first of all, as an analytical instrument to avoid the "illusion of transparency" concerning educational phenomena and, more particularly, the nature of the knowledge involved, that is, to emancipate research from the viewpoint of the scholarly and the teaching institutions about the knowledge involved in educational processes.

Any *taught* field or discipline is the product of an intricate process the singularity of which should never be underrated. As a consequence, one should not take for granted the current, observable organization of a field or discipline taught at school, as if it were the only possible one. Instead one should see it against the (fuzzy) set of organizations that *could* have existed, some of which may someday turn into reality. Considering the "scholarly knowledge" as part of the object of study of research in didactics is part of this emancipatory movement of detachment. Although school teaching has to be legitimized by external entities that guarantee the pertinence and epistemological relevance of the knowledge taught (in a complex process of negotiations which includes crises and disagreements), researchers do not have to consider these institutional perspectives as the true or correct viewpoints or as the wrong ones; they just need to know them and integrate them in the analysis of educational phenomena.

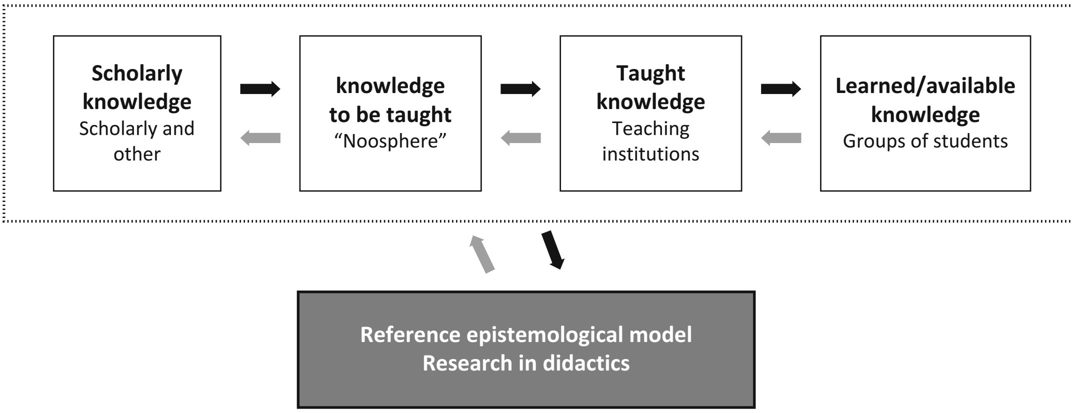
In some cases, the "scholarly legitimation" of school knowledge can be questioned by the noosphere, on behalf of its cultural relevance: "Is this the geometry citizens need?" Such a conflict situation can change significantly the conditions of teaching and learning, by allowing a self-referential, epistemologically confined teaching. Moreover, there are certain teaching processes in which the scholarly body of knowledge is created afterwards because of the need to teach a given content that has to be organized, labeled, and recognized as something relevant (an illustrative example is the case of accounting and its corresponding body of knowledge, accountancy). It is also possible that something that is not even commonly recognized as a proper body of knowledge may appear as "scholarly knowledge" for the role it assumes in a given educational process. For instance, in the teaching of sports, the scholarly knowledge, albeit not academically tailored, includes that of high-level sport players, even if they are a far cry from what we normally consider "scholars" to be!

## Enlargement of the Object of Study

The second consequence of the detachment process introduced by the notion of didactic transposition is the evolution of the object of study of didactics as a research discipline. Besides studying students' learning processes and how to improve them through new teaching strategies, the notion of didactic transposition points at the object of the learning and teaching itself, the "subject matter," as well as its possible different ways of living – its diverse *ecologies* – in the institutions involved in the transposition process.

Let us take an example on negative numbers. Regarding the transpositive process, the first issue is to consider what the *taught knowledge* is made of (what concrete activities that are proposed to the students, their organization, the domain or block of contents they belong to, etc.) and how official guidelines and *noospherian* discourses present and justify these choices (the *knowledge to be taught*). Today, at most schools, negative numbers are officially related to the measure of quantities with opposite directions and introduced in the context of real-life situations. Where does this school organization come from? It results from different scholar ("new mathematics") or social ("back-to-basics") pressures, canalized by the noosphere, that cannot be presented here but that delimit the kind of mathematical practices our students learn (or fail to learn) about this body of knowledge. If we look at scholarly knowledge, the environment is different: negative numbers are defined as an extension of the set of natural numbers **N** and form the ring of integers **Z**, without any specific discussion (http://www.encyclopediaofmath.org/index.php/Integer). This has not always been the case: it is very well known that until the mid-nineteenth century, the possibility of "quantities less than zero" was still denied by many scholars. Their final acceptation was strongly related to the needs of algebraic work, which explains why, for a long time, integers were called "algebraic numbers." It also explains why the introduction of negative numbers was considered one of the main differences between arithmetic and algebra. This relationship to elementary algebraic work has now completely disappeared from the scholar's and school's conception of negative numbers, despite the fact that some practices of calculation – for instance, those involving the product of integers – acquire their full sense when interpreted in this context.

Various other analyses have brought similar results regarding how the transposition process has affect other different mathematical contents (school algebra, linear algebra, limits of functions, proportionality, geometry, irrational numbers, functions, arithmetic, statistics, proof, modeling, etc.): more generally speaking, there is no such thing as an eternal, context-free notion or technique, the matter taught being always shaped by institutional forces that may vary from place to place and time to time. These investigations underline the institutional relativity of knowledge and show to what extend most of the phenomena related to the teaching and learning of mathematics are strongly affected by constraints coming from the different steps of the didactic transposition process. Consequently, the empirical unit of analysis of research in didactics becomes clearly enlarged, far beyond the relationships between teachers and students and their individual characteristics.




**Didactic Transposition in Mathematics Education, Fig. 2** The external position of researchers

## The Need for Researchers' Own Epistemological Models

Taking into consideration *transpositive phenomena* means moving away from the classroom and being provided with notions and elements to describe the bodies of knowledge and practices involved in the different institutions at different moments of time. To do so, the epistemological emancipation from scholarly and school institutions requires researchers to create their own perspective on the different kinds of knowledge intervening in the didactic transposition process, including their own way of describing knowledge and cognitive practices, their own epistemology. In a sense, there is no privileged reference system from which to observe the phenomena occurring in the different institutions involved in the teaching process: the scholarly one, the noosphere, the school, and the classroom. Researchers should build their own *reference epistemological models* (Barbé et al. 2005) concerning the bodies of knowledge involved in the reality they wish to approach (see Fig. 2). The term "model" is used to emphasize the fact that any perspective provided by researchers (what mathematics is, what algebra is, what measuring is, what negative numbers are, etc.) always constitutes a methodological proposal for the analysis; as such, it should constantly be questioned and submitted to empirical confrontation.

## From Didactic Transposition to the Anthropological Approach

When knowledge is considered a changing reality embodied in human practices taking place in social institutions, one cannot think about teaching and learning in individualistic terms. The evolution of the research perspective towards a systematic epistemological analysis of knowledge activities explicitly appears at the foundation of the anthropological theory of the didactic (Chevallard 1992a, 2007; Winslow 2011). It is approached through the study of the conditions enabling and the constraints hindering the production, development, and diffusion of knowledge and, more generally, of any kind of human activity in social institutions.

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▶ Anthropological Approaches in Mathematics Education, French Perspectives
▶ Curriculum Resources and Textbooks in Mathematics Education
▶ Didactic Engineering in Mathematics Education
▶ Didactic Situations in Mathematics Education

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# Didactical Phenomenology (Freudenthal)


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## Keywords

Phenomena in reality; Mathematical thought objects; Didactics; Realistic mathematics education; Analyses of subject matter


## What Is Meant by Didactical Phenomenology?

The term *didactical phenomenology* was coined by Hans Freudenthal. Although his initial ideas for it date from the late 1940s, he likely first used the term in a German article in 1974. A few years later, the term appeared in English in his book *Weeding and Sowing – Preface to a Science of Mathematical Education* (Freudenthal 1978). Understanding the term requires comprehending Freudenthal's notion of a *phenomenology* of mathematics, which refers to describing mathematical concepts, structures, or ideas, as thought objects (*nooumena*) in their relation to the phenomena (*phainomena*) of the physical, social, and mental world that can be organized by these thought objects.

The term *didactical* is used by Freudenthal in the European continental tradition referring to the way we teach students and the organization of teaching processes. This definition of didactics goes back to Comenius' (1592–1670) *Didactica Magna* (Great Didactics) that contains a well-founded view on what and how students should be taught. As such, this meaning of didactics contrasts with the Anglo-Saxon tradition in which it merely has a superficial meaning involving a set of instructional tricks.

Combining the two terms into *didactical phenomenology* implies considering the *phenomenology* of mathematics from a *didactical* perspective.

## Merit of a Didactical Phenomenology for Mathematics Education

In Freudenthal's words (1983, p. ix), a didactical phenomenology of mathematics can "show the teacher the places where the learner might step into the learning process of mankind." In other words, a didactical phenomenology informs us on how to teach mathematics, including how mathematical thought objects can help organizing and structuring phenomena in reality, which phenomena may contribute to the development of particular mathematical concepts, how students can come in contact with these phenomena, how these phenomena beg to be organized by the mathematics intended to be taught, and how students can be brought to higher levels of understanding. As such, Freudenthal's didactical phenomenologies are landmarks for developing teaching outlines.

## Relation with Realistic Mathematics Education

By disclosing the sources of mathematics in reality, a didactical phenomenology is strongly related to Realistic Mathematics Education (RME), the domain-specific instruction theory for mathematics, which has been developed in the Netherlands and in which Freudenthal was heavily involved (Freudenthal 1991). In RME, rich, realistic situations have a prominent position in the learning process. These situations serve as sources for initiating the development of mathematical concepts, tools, and procedures. What situations can serve as contexts for this development is revealed by a didactical phenomenology. By tracing phenomena in reality that can elicit mathematical thoughts, the students are given access to the sources of mathematics in everyday experiences. Building on these sources offers them an orientation basis they experience as real and opens the possibility of personal engagement and solving problems in a way they find meaningful. This attachment of meaning to mathematical constructs students have to develop touches on a main principle of RME.

## Examples of Didactical Phenomenology

In *Weeding and Sowing*, Freudenthal exemplified his idea of a didactical phenomenology by providing an analysis of the topic of ratio and proportion. Furthermore, he announced to deal comprehensively with didactical phenomenology in a following book. That book was *Didactical phenomenology of mathematical structures* (Freudenthal 1983). In this book, he gave more examples of didactical phenomenologies, including those of length, natural numbers, fractions, geometry and topology, negative numbers and directed magnitudes, algebraic language, and functions.

Remarkably, these examples did not just deal with connecting mathematical thought objects to phenomena in reality to find starting points for learning mathematics. In fact, these examples were profoundly scrutinized analyses of subject matter in which the key concepts of a particular mathematical topic were disclosed together with contexts which have a model character and with significant landmarks in students' learning pathways.

## The Method

Unfortunately, in *Didactical phenomenology of mathematical structures*, Freudenthal did not elaborate much on how to establish these didactical phenomenologies. Although the book contains a short chapter titled *The method*, this did not reveal how to generate such phenomenologies. Nevertheless, a corner of the veil was lifted when Freudenthal (1983, p. 29) considered the material he needed to write this book:

> I have profited from my knowledge of mathematics, its application, and its history. I know how mathematical ideas have come or could have come into being. From an analysis of textbooks

> I know how didacticians judge that they can support the development of such ideas in the minds of learners. Finally, by observing learning processes I have succeeded in understanding a bit about the actual process of constitution of mathematical structures and the attainment of mathematical concepts

This statement and the provided examples show how a didactical phenomenology results from a number of analyses, each taking a different perspective: didactical, phenomenological, epistemological, and historical-cultural.

## Mathematics-Related Analyses Constituting the Didactics of Mathematics

These analyses have in common that they all take mathematics as their starting point. Didactical analyses examine the nature of the mathematical content as a basis for teaching this content. By identifying the determining aspects of mathematical concepts and their relationships, knowledge is gathered about didactical models that can help students to understand these concepts. Phenomenological analyses disclose possible manifestations of these mathematical concepts in reality and can suggest contexts for students to meet these concepts. Epistemological analyses focus on students' learning processes and can uncover how the mathematical understanding of students in a classroom interaction may shift. Finally, in historical-cultural analyses, we may encounter current and past approaches to teaching mathematics through which we can gain a better understanding of learning mathematics and how education can contribute to it.

These analyses are all included in Freudenthal's didactical phenomenology and surpass its narrow literal meaning, which would certainly have his approval, as in *Weeding and Sowing* Freudenthal (1978, p. 116) already stated: "[T]he name does not matter; nor is that activity [didactical phenomenology] an invention of mine; more or less consciously it has been practiced by didacticians of mathematics for a long time" (Freudenthal 1978, p. 116). Indeed, the name is not essential, but these analyses are. In Freudenthal's view, they form the heart of researching and developing mathematics education.

## Cross-References

▶ Realistic Mathematics Education

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# Discourse Analytic Approaches in Mathematics Education

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## Keywords

Discourse; Discourse analysis; Foucault; Identity; Language; Linguistics; Methodology; Social practice

## Introduction

The first challenge in addressing this topic is the multiplicity of ways in which the term *discourse* is used and defined – or not defined – within mathematics education (see Ryve 2011). It is frequently found, especially in discussions within the context of curriculum reform, simply to signify student engagement in talk in the classroom. Without denying the value of the development of such engagement, the approaches to discourse and discourse analysis considered in this article all take rather more complex and theoretically shaped views of the nature of discourse – views that influence the focus of research and the analytic methods. An important component of the ways these approaches conceive of discourse is a concern with the relationship between language (and other modes of communication), the social context in which it is used, and the meanings that are produced in this context (Howarth 2000). It is this concern and the fundamental assumption that studying the way language is used can provide insight into the activity or practice (mathematics or mathematics education) in which it is used that leads researchers to adopt discourse analytic approaches. Of course, a very high proportion of the data used in studies across many branches of mathematics education research is primarily linguistic or textual: interviews, written responses to questionnaires, classroom transcripts, written texts produced by students, etc. Increasingly it has also been recognized by researchers using a wide range of approaches that the language produced by students or other research subjects is not a transparent medium through which it is easy to decipher an underlying truth. What distinguishes research that adopts a discourse analytic approach is the assumption that the language is itself an inextricable part (or, for some researchers, even the whole) of the object of study. This assumption is shared with another analytic approach, conversation analysis, and some discourse analysts make use of methods developed in conversation analysis. However, whereas discourse analysis is generally interested in characterizing the practices within which language plays a role, conversation analysis focuses primarily on how linguistic interactions themselves are organized to achieve social actions (see Wooffitt 2005, for an introduction to the two approaches from a conversation analytic perspective).

Gee (1996) makes a useful distinction between *discourse*, defined as instances of communication, and *Discourses*, the conjunctions of ways of speaking, subject positions, values, etc. that characterize and structure particular social practices. The notion of *Discourses* has its origin in the thinking of the French philosopher Foucault (e.g., 1972) whose work includes studies of the construction of "regimes of truth" about notions such as madness or sexuality. Though not all discourse analytic research in mathematics education comes from this tradition, it can generally be characterized as tending either towards analysis of *discourse*, focusing on communication events and the local social practices within which they arise, or towards analysis of *Discourse*, taking larger scale social practices and structures as the object of research. Of course, some approaches move between the two, generating interpretation of specific communication events by applying knowledge of wider social practices and structures or building a picture of a significant social practice through analysis of local communication events. Discourse analytic approaches thus vary in two dimensions: the extent to which they make use of detailed linguistic analysis and the extent of their focus on social practices, structures, and institutions.

The adoption and development of discourse analytic approaches in mathematics education research largely coincided with what Lerman termed the "social turn" (Lerman 2000). Increased recognition of the importance of studying and taking account of the social nature of mathematics education practices as well as of individual cognition demanded the development of theoretical ways of conceiving of social practices and methodological approaches to studying them. Discourse analytic approaches provided one way of addressing this demand. This development within the field of mathematics education reflected a much wider development of theories of discourse and discourse analytic methods within social science and the humanities. As researchers have begun to draw on theories and methods originating outside the field of mathematics education, they have faced the challenge of ensuring that both theory and methods take account of the specialized nature of mathematical communication and practices and that they have the power to illuminate issues of interest to mathematics education. Facing this challenge is a continuing project; notable contributions have come from within mathematics education (e.g., Morgan 1998; Sfard 2008) and from linguistics (e.g., O'Halloran 2005).

With a few exceptions, notably the work of Walkerdine (1988) who used analyses of *Discourses*, including Discourses of gender and of child-centered education, in order to understand how differences between various social groups are constructed in mathematics education practices, early interest in discourse analytic approaches, such as that represented in the Special Issue of *Educational Studies in Mathematics* edited by Kieren et al. (2001), was dominated by analysis of communication events (*discourse*), focusing on understanding classroom interaction and the development of mathematical thinking in interaction. At a time when the majority of research in mathematics education focused on the mathematical thinking of individuals, this application of discourse analysis may be seen as an incremental manifestation of the "social turn," addressing the same interest in mathematical thinking but reconceptualizing it as a phenomenon that is evident (and, for some researchers, produced) in social interaction. More recently, the issues addressed by the mathematics education research community have expanded, incorporating a wider conceptualization of mathematics and mathematics education as social practices. Thus more research has addressed, *inter alia*, identity, power relationships, and social justice – issues that lend themselves to study using approaches that focus on *Discourses*. Some of this research has adopted approaches that may be characterized as structuralist, drawing on sociological accounts of social structures such as the work of Basil Bernstein (e.g., 2000) to describe and interpret discursive phenomena. Others have adopted poststructural approaches, in which the communicative action itself constructs the "reality" of which it speaks. A recent edited book entitled *Equity in Discourse for Mathematics Education* (Herbel-Eisenmann et al. 2012) reflects this range of approaches and interpretations, combining detailed analyses of classroom interactions with concern for how these interactions and broader social practices affect the possibilities for participation in mathematics of students from different social groups.

In this article there is no space to provide a detailed review of the full range of approaches taken to discourse analysis. Instead, we provide a small number of contrasting cases, exemplifying the scope of discourse analytic methods and the problems in mathematics education that they may be used to address.

## Critical Discourse Analysis

Critical discourse analysis (CDA) comprises a group of analytic approaches, all of which make strong analytic connections between forms of language use, social practices, and social structures. The label "critical" indicates a concern of the researchers to make use of the knowledge achieved through the analysis in order to enable critique and transformation of the social practices and/or structures. Research using CDA approaches thus tends to produce analyses that not only describe existing practices but also critique the ways these practices position students and/or teachers and the kinds of mathematics and mathematical identities that are valued and made possible.

CDA studies generally involve detailed analyses of texts, including oral and written texts produced and used by students and teachers in the classroom but also including texts such as the curriculum and policy documents that structure and regulate these educational practices and thus affect the interpretation of classroom texts. Within mathematics education, probably the most widely used type of CDA is based on the approach of Norman Fairclough (2003), using linguistic tools drawn from systemic functional linguistics (SFL). This approach has been used to investigate specific practices such as the assessment of student reports of mathematical investigation (Morgan 1998) or the use of "real-world problems" in an undergraduate mathematics course (Le Roux 2008). Research adopting a CDA approach may also use a range of other methods to address textual data, including corpus analysis of large data sets (e.g., Herbel-Eisenmann et al. 2010).

Whatever the linguistic tools used to describe the data, the interpretative stage of CDA involves considering how the features identified in the data function to construe the "reality" of the practice being studied and the social positionings and relations of the participants. As Fairlough argues, such interpretation requires explicit use of "insider knowledge" of the social practices studied (Fairclough 2003). This means that researchers in mathematics education need to bring knowledge of broader mathematics education practices and knowledge of mathematical practices to bear on their analyses. For example, Morgan's study of teachers' assessment practices is informed by an analysis of the constructs and values found in the associated curriculum documents, policy, and professional literature, while Le Roux draws on Sfard's (2008) characterization of mathematical discourse (discussed further below) to enable her analysis to address the nature of the mathematical activity involved in the use of real-world problems.

## Poststructural Approaches

The approaches to discourse analysis discussed under the heading of postmodern or poststructural share with CDA approaches a concern with issues such as power and subjectivity that arise in considering relationships between individuals and social practices and structures. There are, however, both philosophical and methodological differences between the approaches. There is a range of philosophical positions associated with postmodern and poststructural thought; however, a shared foundation is a rejection of the notions of an objective world and of the fixed subjectivity of a rational knowing subject. These philosophical assumptions are shared by some but certainly not by all those employing CDA approaches, though there is a common interest in characterizing the key entities that play a role in a Discourse and the possibilities for individual subjectivities, identities, or positioning.

The major distinction drawn here between the approaches to discourse analysis discussed in this section and those identified under the heading CDA is methodological. While CDA involves close analysis of specific texts, usually employing analytical tools and methods drawn from linguistics, the starting point for postmodern/poststructural researchers tends to be at the level of the major functions of discourse. For example, Hardy (2004) uses the Foucauldian constructs of *power as production* and *normalization* as her analytical tools for interrogating a teacher training video produced as part of the English National Numeracy Strategy to demonstrate "effective teaching" of mathematics in a primary classroom. Rather than focus on detailed characteristics of the discourse of this video, Hardy uses these constructs to provide an alternative perspective on the data as a whole. This enables her to tell a story of what the Discourse of the National Numeracy Strategy achieves – how it produces assumptions about what is normal and what is desirable – a story that runs counter to the "common sense" stories about effective teaching.

A rather different approach is taken by Epstein et al. (2010), though again founded in Foucauldian theory. They first characterize the ways in which mathematics and mathematicians are represented in popular media – as hard, logical, and ultrarational but also as eccentric or even insane. Having identified different and in some cases apparently contradictory Discourses about mathematics, Epstein et al. then use these to analyze interviews with students, identifying how individual students deploy the various discursive resources in order to produce their own identities as mathematicians or as nonmathematicians and their relationships to mathematics as a field of study.

## Mathematical Discourse, Thinking, and Learning

The main discourse analytic theories mentioned so far have their origins outside mathematics education, drawing on fields such as linguistics, ethnography, sociology, and philosophy. For mathematics education researchers, this raises the important theoretical and methodological problem of the extent to which the specifically mathematical aspects of the practices being studied may be captured and accounted for. In order to address this problem, an increasing number of researchers, including some of those working with CDA or other discourse analytic approaches, are turning to the work of Anna Sfard (2008). While Sfard draws on a number of sources, including Wittgenstein's notion of language game, her own theory of mathematical discourse has been developed within the field of mathematics education and is designed to address the problems arising in this field. Her communicative theory of cognition identifies thinking mathematically as participating in mathematical discursive practices, that is, as communicating (with oneself or with others) using the forms of discourse characteristic of mathematics. Sfard identifies four aspects of mathematical discourse that form the basis for her analytic method: specialized mathematical *vocabulary and syntax* (what may be considered the "language" of mathematics), *visual mediators* (nonlinguistic forms of communication such as algebraic notation, graphs, or diagrams), *routines* (well-defined repetitive patterns, e.g., routines for performing a calculation, solving an equation, or demonstrating the congruence of two triangles), and *endorsed narratives* (the sets of propositions accepted as true within a given mathematical community). Scrutinizing how these four aspects are manifested in discourse provides a means of describing mathematical thinking and hence allows one to address questions such as the following: How does children's thinking about a mathematical topic vary from that expected by their teacher or by an academic mathematical community? How does children's thinking develop (i.e., how does their use of a mathematical form of discourse change over time)? What kinds of mathematical thinking are expected of students taking an examination or using a textbook?

As may be seen from the research topics and questions illustrated in this article, discursive approaches can address a wide range of issues of concern within the field of mathematics education, bridging, as indicated in the title of Kieran et al.'s (2001) Special Issue of *Educational Studies in Mathematics*, the individual and the social. While the various approaches share a basic assumption that language and social practices play a role in the ways that individuals make sense of mathematical activity, they differ in the ways they conceptualize this role (and, indeed, in how they conceptualize language, social practice, and mathematics). Hence they also differ in the research questions they pose and the methodological tools they employ. It can be argued that discourse analytic approaches allow us to see through *what is said* to reveal *what is achieved* by using language. The challenge for researchers and for the readers of research is to clarify how the theoretical and methodological tools employed enable this and to distinguish which kinds of actions and achievements are made visible by the different approaches.

## Cross-References

- ▶ Discursive Approaches to Learning Mathematics
- ▶ Mathematical Language
- ▶ Poststructuralist and Psychoanalytic Approaches in Mathematics Education
- ▶ Sociological Approaches in Mathematics Education

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# Discrete Mathematics Teaching and Learning

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## Keywords

Discrete mathematics; Discrete; Continuous; Reasoning; Proof; Mathematical experience

## Definition

The teaching of "discrete mathematics" is not always clearly delimited in the curricula and can be diffuse. In fact, the meaning of "discrete mathematics teaching and learning" is twofold. Indeed, it includes the teaching and learning of discrete concepts (considered as defined objects inscribed in a mathematical theory), but it also includes skills regarding reasoning, modeling, and proving (such skills are specific to discrete mathematics or transversal to mathematics).

## What Is Discrete Mathematics?

*Discrete mathematics* is a comparatively young branch of mathematics with no agreed-on definition (Maurer 1997): only in the last 30 years did it develop as a specific field in mathematics with new ways of reasoning and generating concepts. Nevertheless, the roots of discrete mathematics are older: some emblematic historical discrete problems are now well known, also in education where they are often introduced as enigma, such as the Four Color Theorem (map coloring problem), the Königsberg's bridges (traveling problem), and other problems coming from the number theory for instance.

There is no exact definition of *discrete mathematics*. The main idea is that *discrete mathematics* is the study of mathematical structures that are "discrete" in contrast with "continuous" ones. Discrete structures are configurations that can be characterized with a finite or *countable* set of relations. (A *countable* set is a set with the same cardinality (number of elements) as some subset of the set of natural numbers. The word "countable" was introduced by Georg Cantor.) And discrete objects are those that can be described by finite or countable elements. It is strongly connected to number theory, graph theory, combinatorics, cryptography, game theory, information theory, algorithmics, discrete probability but also group theory, algebraic structures, topology, and geometry (discrete geometry and modeling of traditional geometry with discrete structures).

Furthermore, discrete mathematics represents a mathematical field that takes on growing importance in our society. For example, discrete mathematics brings with it the mathematical contents of computer science and deals with algorithms, cryptography, and automated theorem proving (with an underlying philosophical and mathematical question: is an automated proof a mathematical proof?).

The aims of discrete mathematics are to explore discrete structures, but also to give a specific modeling of continuous structures, as well as to bring the opportunity to consider mathematical objects in a new manner. Then new mathematical questions can emerge, as well as new ways of reasoning, which implies a challenge for mathematicians.

Some famous problems of discrete mathematics have inspired mathematics educators. That is the case of a combinatorial game: the game of Nim, played since ancient times with many variants. The regular game of Nim is between two players. It is played with three heaps of any number of objects. The two players alternatively take any number of objects from any single one of the heaps. The goal is to be the last one to take an object. Brousseau (1997) explicitly refers to the game theory to conceptualize the theory of didactical situations. The game of Nim is the background of the generic example of Brousseau's theory, "the Race to 20."

## Why Teach and Learn Discrete Mathematics? New Context, Concepts, and Ways of Reasoning: A New Realm of Experience for the Classroom

### To Integrate Discrete Mathematics into the School Curriculum: A Current Challenge

More and more fields of mathematics use results from discrete mathematics (topology, algebraic geometry, statistics, among others). Moreover, discrete mathematics is an active branch of contemporary mathematics. New needs for teaching are identified: they are linked to the evolution of the society and also other disciplines such as computer science, engineering, business, chemistry, biology, and economics, where discrete mathematics appears as a tool as well as an object. Then discrete mathematics should be an integral part of the school curriculum: the concepts and the ways of reasoning that should be taught in a specific field labeled "discrete mathematics" still should be more precisely identified. A dialog between mathematicians and mathematics educators can help for this delimitation.

However, the place of discrete mathematics in curricula is today very variable depending on the countries and on the levels. In a few countries, there has been a long tradition to introduce graph theory in the secondary level among other components of discrete mathematics. This place is strengthened and attested by the contents at the university level. In other countries, only a very small number of discrete mathematics concepts are taught, especially those involved in the fields of combinatorics and number theory. Things are changing; the reader can refer to Rosenstein et al. (1997), and DIMACS (2001) contributions to go into details regarding the challenge of introducing discrete mathematics in curricula (especially the example of the NCTM standards [*National Council of Teachers of Mathematics*] which focus on discrete mathematics as a field of teaching). The following arguments summarize the main ideas of these contributions, emphasizing the interests and the potential ways to implement discrete mathematics in the curricula:

- Proof and abstraction are involved in discrete mathematics (for instance, in number theory, induction, etc.).
- It allows an introduction to modeling and proving processes, but also to optimization and operational research, as well as experimental mathematics.
- Problems are accessible and can be explored without an extensive background in school mathematics.
- The results in discrete mathematics can be applied to real-world situations.
- Discrete mathematics brings a specific work on algorithms and recursion.
- The main problems in discrete mathematics are still unsolved in ongoing mathematical research: a challenge for pupils and students to be involved in a solving process close to the one of mathematicians and to promote cooperative learning (in a specific and suitable context: in particular, teachers should be trained to discrete problems and also to their teaching and management).

## Benefits from Teaching and Learning Discrete Mathematics: Some Examples

Learning discrete mathematics clearly means learning new advantageous concepts but also new ways of reasoning, making room for a mathematical experience.

Many variants exist of the following famous problems that are developed below. Some of them are presented and analyzed for instance in Rosenstein et al. (1997) and on the website http://mathsamodeler.ujf-grenoble.fr/.

### Accessible Problems and Concepts

Discrete concepts are easily graspable, applicable, accessible, and also neutral when not yet included in the curricula: this last argument implies that the way students deal with discrete concepts is quite new and different from the way they usually consider mathematics.

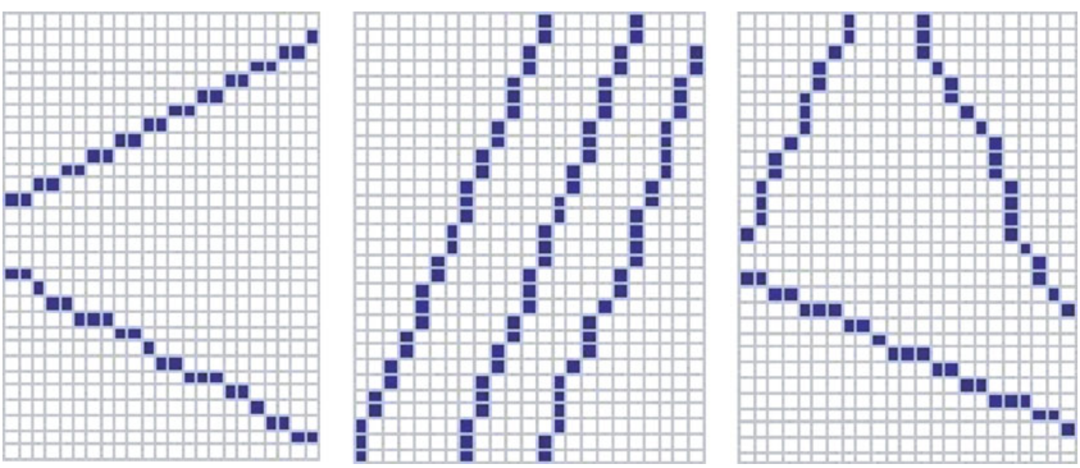
***Traveling salesperson problem***: the problem is to find the best route that a salesperson could take if he/she would begin at the home base, visit each customer, and return to the home base ("best" was defined as minimizing the total distance).

***Map coloring problem*** (combinatorial optimization problem): a map coloring problem consists in discovering the minimum number of colors needed to *properly* color a map (or a graph). A map is *properly* colored if no two countries sharing a border have the same color. The proof of the minimum number of colors is also required. Similar coloring problems exist in graph theory. Such map and graph coloring problems are very useful to explore what discrete mathematical modeling is.

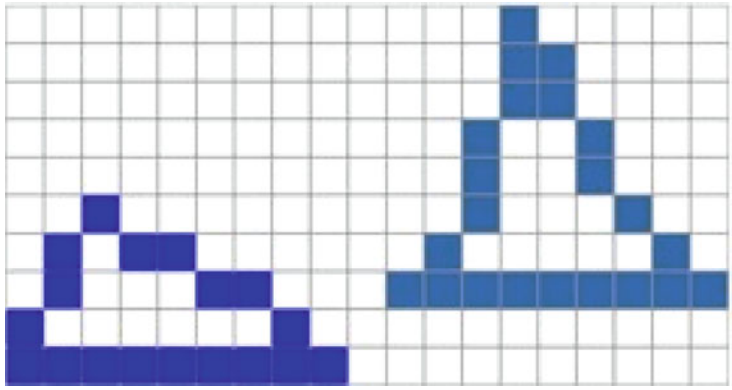
### Richness of Discrete Concepts, A Way to Deal with the Construction of Axiomatic Theory

A certain amount of discrete objects can be defined in several ways, with different characterizations. The modeling of continuous concepts in the discrete case raises the problem of the construction of a mathematical consistent theory from an axiomatic point of view. It is illustrated with the following example of discrete geometry.

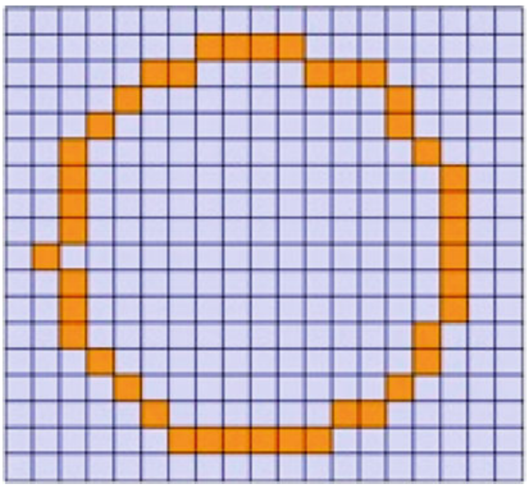
***Discrete Geometry: Example of Discrete Straight Lines*** Discrete straight lines form a concept accessible by its representation. It is noninstitutionalized (an institutionalized concept is a "curriculum" concept, i.e., a concept that has a place in the classic taught content). Delimiting what a straight line can be in a discrete context proves to be quite a challenge. Professional researchers have several definitions of it at their disposal, but the proof of the equivalence of these definitions remains worth considering. Research on a discrete axiomatic theory is still in progress (it implies, for instance, the following questions: what is the intersection of two discrete straight lines? What does it mean to be parallel in the discrete case? etc.): the question of a "good" definition of a discrete straight line is currently an



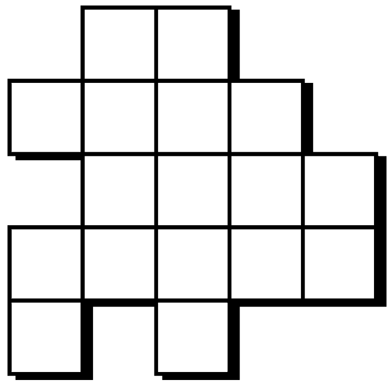
**Discrete Mathematics Teaching and Learning, Fig. 1** Are these lines straight lines?



**Discrete Mathematics Teaching and Learning, Fig. 2** Are these shapes triangles?



**Discrete Mathematics Teaching and Learning, Fig. 3** Is it a circle?



**Discrete Mathematics Teaching and Learning, Fig. 4** A garden

**Discrete Mathematics Teaching and Learning, Fig. 5** A beast (a beast can be rotated or reversed)

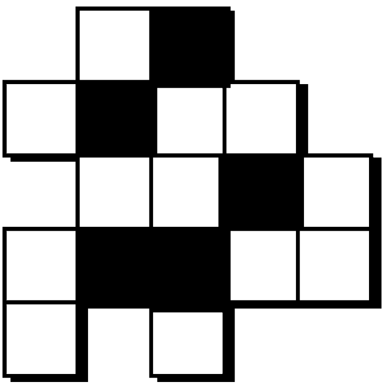
open and interesting problem. So are the questions of the definitions of other discrete geometrical concepts (Figs. 1–3).

## Several Ways of Questioning, Proving, and Modeling

Besides, discrete mathematics arouses interest because it offers a new field for the learning and teaching of proofs (Grenier and Payan 1999; Heinze et al. 2004; http://mathsamodeler.ujf-grenoble.fr/). Some discrete problems fruitfully bring different ways to consider proof and proving processes. How can discrete mathematics contribute to make students acquire the fundamental skills involved in defining, modeling, and proving, at various levels of knowledge? It is still a fundamental question in mathematics education. The following example brings an opportunity to deal with an optimization problem which involves several kinds of reasoning. Besides, this problem is close to the contemporary research in discrete mathematics.

***Hunting the beast***. Your garden is a collection of adjacent squares (see Fig. 4) and a beast is itself a collection of squares (like the one drawn in Fig. 5). Your goal is to prevent a beast from entering your garden. To do this, you can buy traps. A trap is represented by a single black

**Discrete Mathematics Teaching and Learning, Fig. 6** Not a solution



**Discrete Mathematics Teaching and Learning, Fig. 7** A solution with 5 traps

square that can be placed on any square of the garden. The question we ask is the following: what is the minimum number of traps you need to place so that no beast can land on your garden?

On Fig. 6, the disposition of the traps does not provide a solution to the problem, since a beast can be placed. On Fig. 7, a solution with five traps is suggested. Is it an optimal one for this configuration?

In the literature, the problem *Hunting the beast* can be seen as a variation of the *Pentomino Exclusion Problem* introduced by Golomb (1994).

### A Mathematical Experience

Discrete mathematics then brings the opportunity for students to be involved in a mathematical experience. Harel (2009) points out the following principle:

> The ultimate goal of instruction in mathematics is to help students develop ways of **understanding** and ways of **thinking** that are compatible with those practiced by contemporary mathematicians. (p. 91)

The "doing mathematics as a professional" component is clearly a new direction for the educational research in the problem solving area, and discrete mathematics offers promising nonroutine potentialities to develop powerful heuristic processes, as underscored by Goldin (2009).

Bearing in mind the aforesaid arguments, discrete mathematics provides a mathematical experience and is a field of experiments that questions concepts involved in other mathematical branches as well. Nevertheless, if the discrete problems are sometimes (and even often) easier to grasp than the continuous ones, the mathematics behind can be quite advanced. That is the reason why the didactic should analyze both the discrete mathematics for itself and the discrete mathematics helping the teaching of other concepts.

## Interesting Perspectives for Research in Mathematics Education

Discrete mathematics is a relatively young science, still in progress with accessible and graspable concepts and ongoing questionings; hence the questions regarding the introduction of it in the curricula and in the classroom concern both mathematics educators and mathematicians.

Two separated but linked perspectives for the educational research emerge:

- The didactical study of teaching and learning discrete mathematics
- The didactical study of the teaching of concepts and skills (such as proof and modeling) with the help of discrete problems

Besides, discrete mathematics can be introduced either as a mathematical theory or as a set of tools to solve problems. The links between discrete mathematics as a tool and discrete mathematics as an object in teaching and learning should also be analyzed in depth, as well as the proof dimension involved in dealing with discrete concepts and structures. The didactic transposition of discrete concepts and ways of reasoning is still a current problem for mathematics education. It can raise the question of the development of models for teaching and learning discrete mathematics. Some epistemological models do exist (around transversal concepts such as implication, definition, and proof (see, for instance, Ouvrier-Buffet 2006) and specific contents such as the teaching of graph theory (see the work of Cartier 2008)) but the work is still in progress. Note that it involves the same questionings for mathematics education as the introduction of algorithmics in the curricula.

Furthermore, the introduction of discrete mathematics in the curricula clearly offers an opportunity to infuse new instructional techniques. In this perspective, the teacher training should be rethinked.

## Cross-References


- ▶ Algorithmics
- ▶ Argumentation in Mathematics
- ▶ Inquiry-Based Mathematics Education
- ▶ Logic in Mathematics Education
- ▶ Mathematical Games in Learning and Teaching
- ▶ Mathematical Modelling and Applications in Education
- ▶ Mathematical Proof, Argumentation, and Reasoning
- ▶ Mathematics Teachers and Curricula
- ▶ Problem Solving in Mathematics Education
- ▶ Word Problems in Mathematics Education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# Discursive Approaches to Learning Mathematics


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## Keywords

Learning; Mathematics; Communication; Discourse; Language

## Definition

*Discursive approach to learning* is a research framework grounded in the view that learning such subjects as mathematics, physics, or history is a communicational activity and should be studied as such. Learning scientists who adopt this approach treat discourse and its development as the primary object of exploration rather than as mere means to the study of something else (e.g., development of mental schemes). The term *discourse* is to be understood here as referring to a well-defined type of multimodal (not just verbal) communicational activity, which does not have to be audible or synchronous.

## Background

Ever since human learning became an object of systematic study, researchers have been aware of its intimate relationship with language and, more generally, with the activity of communicating. The basic agreement on the importance of discourse notwithstanding, a range of widely differing opinions have been proposed regarding the way these two activities, learning and communicating, are related. At one end of the spectrum, there is the view that language-related activities play only the secondary role of means to learning; the other extreme belongs to those who look upon discourse as the object of learning. It is this latter position, the one that practically equates mathematic with a certain well-defined form of communicational activity, that can be said to fully reflect a discursive approach to learning.

Several interrelated developments in philosophy, sociology, and psychology combined together to produce this approach. It is probably the postmodern rejection of the notion of "absolute truth," the promise of which fuelled the positivist science, that put human studies on the path toward the "discursive turn." Rather than seeing human knowledge as originating in the nature itself, postmodern thinkers began picturing it as "a kind of discourse" (Lyotard 1979, p. 3) or as a collection of narratives gradually evolving in the "conversation of mankind" (Rorty 1979, p. 389).

Following this foundational overhaul, the interest in discourses began crossing disciplinary boundaries and established itself gradually as a unifying motif of all human sciences, from sociology to anthropology, to psychology, and so forth. Throughout human sciences, the discursivity – the fact that all human activities are either purely discursive or imbued with and shaped by discourses – has been recognized as a hallmark of humanity. Nowhere was this realization more evident than in the relatively young brand of psychology known as "discursive" (Edwards 2005) and defined as "one that takes language and other forms of communication as critical in the possibility of an individual becoming a human being" (Lerman 2001, p. 93). As evidenced by the steadily increasing number of studies dealing with interactions in mathematics classroom, the discursive turn has been taking place also in mathematics education research (Ryve 2011).

## Foundations

For many discursively minded researchers, even if not for all, the shift to discourse means that some of those human activities that, so far, were considered as merely "mediated" or "helped" by concomitant discursive actions may now be rethought as being communicational in nature. For example, as an immediate entailment of viewing research as a communicational practice, one can now say that the research discipline known as mathematics is a type of discourse, and thus learning mathematics is a discursive activity as well.

Recognition of the discursive nature of mathematics and its learning, if followed all the way down to its inevitable entailments, inflicts a lethal blow to the famous "Cartesian split," the strict ontological divide between what is going on "inside" the human mind and what is happening "outside." Once thinking, mathematical or any other, is recognized as a discursive activity, mental phenomena lose their ontological distinctiveness and discourse becomes the superordinate category for the "cognitive" and the "communicational." This non-dualist position, which began establishing itself in learning sciences only quite recently, has been implicitly present already in Lev Vygotsky's denial of the separateness of human thought and speech and in Ludwig Wittgenstein's rejection of the idea of "pure thought," the amorphous entity supposed to preserve its identity through a variety of verbal and nonverbal expressions (Wittgenstein 1953).

In spite of the fact that the announcement of the ontological unity of thinking and communicating has been heralded by some writers as the beginning of the "second cognitive revolution" (Harré and Gillett 1995), non-dualism has not become, as yet, a general feature of discursive research. More often than not, discursivist researchers eschew explicit ontological commitments (Ryve 2012), whereas their occasional use of hybrid languages brings confusing messages about the nature of the objects of their study. One can therefore speak about weaker and stronger discursive approaches, with the adjective "strong" signaling the explicit adoption of non-dualist stance (Sfard 2008).

The ontological heterogeneity notwithstanding, all discursively oriented researchers seem to endorse Vygotsky's (1978) famous statement that uniquely human learning originates on the "social plane" rather than directly in the world. Consequently, they also view learning as a collective endeavor and recognize the need to always consider its broad social, historical, cultural, and situational context. Strong discursivists, in addition, are likely to claim that objects of discourses – *numbers* or *functions* in the case of mathematics and *conceptions* or *meanings* in the case of researcher's own discourse – grow out of communication rather than signifying any self-sustained entities preexisting the discourse about them. As a consequence, the researchers always keep in mind that any statement on the existence or nature of these entities is a matter of personal interpretation and must be presented as such. Moreover, since the protagonists of researchers' stories are themselves active storytellers, researchers must always inquire about the status of their own narratives vis-à-vis those offered by the participants of their study.

## Strands

The current discursive research on learning at large and on mathematics learning in particular may be roughly divided into three main strands, according to perspectives adopted, aspects considered, and questions asked. The first two of these distinct lines of research are concerned with different features of the discourse under investigation and can thus be called intra-discursive or inward looking. The third one deals with the question of what happens between discourses or, more precisely, how inter-discursive relations impact learning.

The first intra-discursively oriented strand of research on mathematics learning focuses on learning-teaching interactions, whereas its main interest is in the impact of these interactions on the course and outcomes of learning. Today, when *inquiry learning*, *collaborative learning*, *computer-supported collaborative learning*, and other conversation-intensive pedagogies (also known as "dialogical") become increasingly popular, one of the main questions asked by researchers is that of what features of small group and whole-class interactions make these interactions conducive to high-quality learning. *Participation structure*, *mediation*, *scaffolding*, and *social norms* are among the most frequently used terms in which researchers formulate their responses. Whereas there is no doubt about theoretical and practical importance of this strand of research, some critics warn against the tendency of this kind of studies for being unhelpfully generic, which is what happens when findings regarding patterns of learning-teaching interactions are presented as if they were independent of their topic.

This criticism is no longer in force in the second intra-discursively oriented line of research on mathematics learning, which inquires about the development of mathematical discourse and thus looks on those of its features that make it into distinctly mathematical: the use of specialized mathematical words and visual mediators, specifically mathematical routines, and narratives about mathematical objects that the participants endorse as "true." Comparable in its aims to research conducted within the tradition of conceptual change, this relatively new type of study on learning is made distinct by its use of methods of discourse analysis, and this means, among others, its attention to contextual issues, its sensitivity to the inherent situatedness of learning, and its treatment of the discourse in its entirety as the unit of analysis, rather than restricting the focus to a single concept. Questions asked within this strand include queries about ways in which learners construct *mathematical objects*, develop *sociomathematical norms*, engage in *argumentation*, or cope with uneasy transitions to *incommensurable discourses*. Methods of systemic functional linguistics (Halliday 2003) are often employed in this kind of study. One of the main tasks yet to be dealt with is to forge subject-specific methods of discourse analysis, tailored according to the distinct needs of the discourse under study. Another is to explore the possibility of improving school learning by overcoming its situatedness. Yet another regards the question of how mathematical learning occurring as if of itself while people are dealing with their daily affairs differs from the one that takes place in schools and results from teaching.

Finally, the inter-discursively oriented studies inquire about interactions between discourses and their impact on learning. This type of research is grounded in the recognition of the fact that one's participation in mathematics discourse may be supported or inhibited by other discourses. Of particular significance among these learning-shaping aspects of communication are those that pertain to specific cultural norms and values or to distinct ideologies. Studies belonging to this strand are often concerned with issues of *power*, *oppression*, *equity*, *social justice*, and *race*, whereas the majority of researchers whom this research brings together do not hesitate to openly admit their ideological involvement. The notion of *identity* is frequently used here as the conceptual device with which to describe the way cultural, political, and historical narratives impinge upon individual learning. Methods of critical discourse analysis (Fairclough 2010) are particularly useful in this kind of study.

## Methods

As different as these three lines of research on learning may be in terms of their focus and goals, their methods have some important features in common. In all three cases, the basic type of data is the carefully transcribed communicational event. A number of widely shared principles guide the processes of collection, documentation, and analysis of such data. Above all, researchers need to keep in mind that different people may be using the same linguistic means differently and that in order to be able to interpret other person's communicational actions, the analysts have to alternate between being insiders and outsiders to their own discourse: they must sometimes look "through" the word to what they usually mean by it, and they also must be able to ignore the word's familiar use, trying to consider alternative interpretations. For the same reason, events under study have to be recorded and documented in their entirety, with transcriptions being as accurate and complete records of participants' verbal and nonverbal actions as possible. Finally, to be able to generalize their findings in a cogent way, researchers should try to support qualitative discourse analysis with quantitative data regarding relative frequencies of different discursive phenomena.

The admittedly demanding methods of discourse analysis, when at their best, allow the analyst to see what inevitably escapes one's attention in real-time conversations. The resulting picture of learning is characterized by high resolution: one can now see as different things or situations that, so far, seemed to be identical and is able to perceive as rational those discursive actions that in real-time exchange appeared as nonsensical.

## Cross-References

▶ Argumentation in Mathematics
▶ Collaborative Learning in Mathematics Education
▶ Equity and Access in Mathematics Education
▶ Inquiry-Based Mathematics Education
▶ Mathematics Teacher Identity
▶ Scaffolding in Mathematics Education
▶ Sociomathematical Norms in Mathematics Education

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# Down Syndrome, Special Needs, and Mathematics Learning

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### Keywords

Genetic disorder; Mathematics difficulties; Number difficulties; Cognitive impairment

## Characteristics

Down syndrome is a genetic disorder which has serious consequences for cognitive development. Most children with Down syndrome show mild to moderate cognitive impairments with language skills typically being more severely impaired than nonverbal abilities (Næss et al. 2011). Children with Down syndrome are frequently reported to have problems with short-term and working memory. While a relatively large number of studies have investigated the language and reading skills (Hulme et al. 2011) of children with Down syndrome, much less is known about the development of number skills in this group.

Early case studies and studies using highly selected samples have reported some relatively high levels of arithmetic achievement in individuals with Down syndrome. However, for the majority of individuals with Down syndrome, simple single digit calculations and even counting represent a significant challenge (Gelman and Cohen 1988). Carr (1988) reported that more than half of her sample of 41 individuals aged 21 years could only recognize numbers and count on the Vernon's arithmetic-mathematics test. Buckley and Sacks (1997) surveyed 90 secondary school-age children with Down syndrome in the and found that only 18 % could count beyond 20 and only half of the sample could solve simple addition problems.

Studies conducted on larger samples consistently report low arithmetic achievement in individuals with Down syndrome relative to other scholastic skills such as reading accuracy (Hulme et al. 2010; Buckley and Sacks 1987; Carr 1988). Age equivalents on standardized number tests are typically reported to lag age equivalent reading scores by around 2 years in children with Down syndrome (e.g., Carr 1998).

Arithmetic performance is reported to improve with chronological age in children with Down syndrome, but this varies widely within IQ levels and is not true for all children (e.g., Carr 1988). It seems highly plausible that a relationship might exist between IQ level and arithmetic performance level, but thus far, there is no consensus in the literature. Education has a positive influence on arithmetic performance as might be expected, and individuals in mainstream school are reported to achieve higher levels of mathematical attainment compared to special school (e.g., Carr 1988).

Individual differences in response to intervention are primarily determined by quality and quantity of teaching (Nye et al. 2005). In the UK, Jo Nye has written a book on adapting Numicon for use with children with Down syndrome, "Teaching Number Skills to Children with Down Syndrome Using the Numicon Foundation Kit." In the USA, DeAnna Horstmeier has written a book titled "Teaching Math to People with Down Syndrome and Other Hands-On Learners: Basic Survival Skills." More research is needed to determine the origin of the difficulties that individuals with Down syndrome before a theory driven intervention program can be designed.

## Cross-References


- ▶ 22q11.2 Deletion Syndrome, Special Needs, and Mathematics Learning
- ▶ Autism, Special Needs, and Mathematics Learning
- ▶ Deaf Children, Special Needs, and Mathematics Learning
- ▶ Inclusive Mathematics Classrooms
- ▶ Language Disorders, Special Needs and Mathematics Learning
- ▶ Learning Difficulties, Special Needs, and Mathematics Learning


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## Early Algebra Teaching and Learning

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### What is Early Algebra?

Early algebra refers to a program of research, instructional approaches, and teacher education that highlights the importance of algebraic reasoning throughout K-12 mathematics education. The program stresses that elementary arithmetic rests on ideas and principles of algebra that merit a place in the early curriculum. Early algebra focuses on principles and representations of algebra that can be and presumably need to be mastered by young students as the foundations for later learning.

In some countries, preparation for algebra is implicitly integrated into the early mathematics curriculum. This can be assessed by analysis of curricula implemented in different countries, a task that goes beyond the scope of this account of research on early algebra. For now, it suffices to state that the goal of introducing algebra in elementary school is far from being universally embraced, despite promising results of classroom intervention studies of early algebra.

As early algebra developed as an area of research, different proposals for introducing algebra into the existing K-12 curriculum emerged (see Carraher and Schliemann 2007). Intervention studies based on these perspectives have consistently shown that, well before adolescence, students' demonstrate algebraic reasoning, use conventional algebraic forms for expressing such reasoning, and make mathematical generalizations that have an algebraic character.

### What Is Algebraic Reasoning?

Algebraic reasoning is generally understood as some combination of (a) operating on unknowns; (b) thinking in terms of variables and their relations (where variables have a domain and co-domain containing many, possibly an infinite number of, elements); and (c) acknowledging algebraic structure. Students may be engaged in algebraic reasoning, regardless of whether they are using algebraic notation.

#### Operating on Unknowns

A variable is a symbol or placeholder (typically a letter but sometimes a simple figure or other token) that stands for an element of a set of possible values. The set typically contains numbers or measures (i.e., numbers along with units of measure), but it may be defined over any sorts of objects, mathematical or not.

Although mathematics tends not to distinguish an unknown from a variable, in mathematics education, an unknown is often taken to refer to a fixed number. As a result, the term *unknown* leaves open the issue of whether the variable is

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employed in the former or latter sense. Given this ambiguity, variability (the idea that a variable can take on multiple values) is generally treated as a distinct feature of algebraic reasoning.

Operating on unknowns entails being able to express the relationship among quantities (variable or not) in a novel way. The statement, "Michael had some marbles, then won 8 marbles, finishing with 14 marbles," is a natural language representation of what might be expressed through algebraic notation as "$x + 8 = 14$." A student who realizes that the answer can be found by subtracting 8 from 14 has reconfigured the description of the relationship among known and unknown values such that the answer can be directly calculated from the givens without having to resort to trial and error. This rudimentary form of algebraic reasoning through inverting or "undoing" is significantly different from solving a problem through recall of number facts or adding counting numbers to 8 in order to obtain the sum of 14. Algebraic reasoning is entailed whenever one validly expresses the relationship among givens and unknowns in an alternative form.

Early algebra research (see Kaput et al. 2008; Schliemann et al. 2007) shows that children as young as 8 and 9 years of age can learn to use letters to represent unknown values, to operate on those representations, and to draw new inferences. They can do so without assigning specific values to variables. This brings us to the second characteristic of algebraic reasoning.

#### Thinking About Variables

Algebraic reasoning can take place in the absence of algebraic notation. Variables can be represented through expressions such as *amount of money*, *elapsed time*, *number of children*, *distance* (from school to home), etc. Young students may use simplified drawings to represent variables (e.g., a wallet to represent the amount of money in a wallet). It is important to distinguish such cases from literal drawings depicting one single value or unknown.

Students are engaged in algebraic reasoning whenever they are thinking about variables and relations among variables.

#### Acknowledging Algebraic Structure

Algebraic structure is primarily captured in the Rules of Arithmetic (the field axioms) and in the principles for transforming equations (the original techniques which gave rise to the subject known as algebra).

In the early grades, students can focus on the algebraic structure of simple equations to the extent that they treat the letters as generalized numbers (e.g., when $2n + 2 = 2 \times (n + 1)$, for *all* n in the domain) and, thereby, treat the operations as having validity over a particular set of numbers.

## Approaches to Early Algebra Instruction

Early algebra proponents have adopted three general complementary approaches, each showing some success in developing students' algebraic reasoning. They focus on students' reasoning about (a) physical quantities and measures, (b) the properties of the number system, and/or (c) functions.

#### Reasoning About Physical Quantities and Measures

In this approach, students are encouraged from early on to use letter notation for comparing unknown magnitudes (e.g., a displayed distance or a distance expressed as a magnitude of a unit of measure). For example, they learn to express the length of a line segment, A, as greater than the length of another line segment, B, by the inequality $A > B$ (or $B < A$) or through equations such as $A = B + C$, $B = A - C$. Furthermore, they use multiple forms of representation (diagrams of line segments, tables of values, and algebraic notation) to express relations among givens and unknown magnitudes.

For example, research by Davydov's (1991) group, in the former Soviet Union, shows that quantitative reasoning in concert with multiple forms of representation can support the emergence of algebraic reasoning among second to fourth graders who solve problems like: "In the kindergarten, there were 17 more hard chairs than soft ones. When 43 more hard chairs were added, there were five times more hard chairs than soft ones. How many hard and soft chairs were there?”

### The Properties of Number Systems: Generalized Arithmetic

A generalized arithmetic approach emphasizes algebraic structure early on. For example, the equation $8 + 7 = 9 + \square$ sets the stage for a discussion about the equal sign as meaning something different from the idea of “makes” or “yields”; rewriting the number sentence as $8 + 7 = 8 + (1+ \square)$ may evoke the insight that $1 + \square$ equals 7, making use of the associative property of addition.

Authors whose work falls under this general approach (e.g., Bastable and Schifter 2007; Carpenter et al. 2003) find that elementary school children come to display implicit algebraic reasoning and generalizations supported by intuitive arguments, discuss the truth or falsity of number sentences, and think about the structural relations among the numbers, considering them as placeholders or as variables.

### Functions Approaches to Early Algebra

Functions approaches subordinate many arithmetic topics to more abstract ideas and concepts. Multiplication by 3 is viewed as a subset of the integer function, $3n$, that maps a set of input values to unique output values, thus preparing the ground for the continuous function, $f(x) = 3x$, over the real numbers and its representation in the Cartesian plane. Functions approaches often rely on multiple representations of mathematical functions: descriptions in natural language, function tables, number lines, Cartesian graphs, and algebraic notation. Students are encouraged to treat what might initially appear to be a single value (e.g., “John and Mary each have a box containing the same number of candies. Mary has three additional candies. What can you say about how many candies they each have?”), as a set of possible values.

Results of classroom studies using a functions approach to early algebra are consistently positive. Moss and Beatty (2006) show that, after working with patterns where the position or step is explicitly treated as an independent variable, while the count of some property (e.g., points in a triangular figure) is treated as a dependent variable, students in grades 2–4 can learn to formulate rules that are consistent with a closed form representation of the function such as $3x + 7$. Blanton and Kaput (2005) found that children come to represent additive and multiplicative relations, transitioning from iconic and natural language registers at grades PreK-1 to use of t-charts and algebraic notational systems by grade 3. Students from grades 3 to 5 who participated in a longitudinal study of early algebra, focused on variables, functions, and their multiple representations (Carraher et al. 2008) have been found to perform better than their control peers in the project’s written assessment problems related to algebraic notation, graphs, and equations, as well as in algebra problems included in state mandated tests. The benefits of the intervention persisted 2–3 years later, when treatment students were, again, compared to a control group (Schliemann et al. 2012).

## In Summary

Early algebra highlights the algebraic character of time-honored topics of early mathematics. The successful adoption of early algebra depends upon the fluidity with which teachers are able to move back and forth between algebraic representations and those expressed through natural language, diagrams, tables of values, and Cartesian graphs. There are robust examples of how this can be done in the research literature. The next step is to prepare teachers to interweave these activities into their regular curriculum.

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# Early Childhood Mathematics Education

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## Keywords

Early childhood; Mathematics education; History of early childhood education; Numerical and geometric content domains; Informal knowledge; Play; Picture books; Information and communication technology

## What Is Meant by Early Childhood Mathematics Education?

Early childhood mathematics education includes providing activities or creating learning environments by professionals such as teachers and care takers in order to offer young children experiences aimed at stimulating the development of mathematical skills and concepts. In general, early childhood mathematics education involves children who are 3–6 years old. Depending on the age of the children and the educational system of their country, early childhood education takes place in preschool care centers or in kindergarten classes. Children's mathematical development can also be stimulated by encounters and events that take place outside an educational setting, that is, in the children's home environment, in which, among other things, children can develop some basic notions about number by playing games with their siblings. These family-based activities are highly esteemed as the foundation on which mathematics education in the early years can build.

## History

Teaching mathematics to young children has already a long history. Saracho and Spodek (2009a, b) gave in two articles an overview of it. According to them we can consider the beginning of early mathematics education in 1631 when Comenius, who was at that time a teacher in Poland, published his book *School of Infancy*. In this book, Comenius described the education of children in their first 6 years. By emphasizing the observation and manipulation of objects as the main source for children's learning, Comenius stimulated the creation of mathematics programs for young children which heavily rely on the use of concrete materials. Two centuries later, in the nineteenth century, Comenius' approach was reflected in the educational method of Pestalozzi in Switzerland which also focused on observing and manipulating physical objects.

A further landmark in the development of mathematics education for young children was the foundation of the *Infant School* by Owen in Scotland in 1816. The method of this school for teaching arithmetic was aimed at developing understanding of different arithmetic operations for which, like by Pestalozzi, concrete materials were used. Similarly, in the United States, Goodrich introduced in 1818 in his book *The Children's Arithmetic* the idea that young children can discover arithmetic rules when they manipulate concrete objects such as counters and bead frames. This innovative approach rejected the view that arithmetic is learned through memorization. Later, in the United States, Colburn used Goodrich's and Pestalozzi's work to develop a method which he called "mental arithmetic." The book *First Lessons*, which he published in 1821, was meant for 4- and 5-year-old children and started with simple levels of numerical reasoning elicited by word problems and naturally progressed to more complex levels. Colburn attached much value to children having pleasure in their solutions because this contributes to their learning and the integration of concepts. Moreover, he emphasized the inductive approach, which has many similarities to the constructivist view on learning.

In the second half of the nineteenth century, early childhood mathematics education was influenced by Fröbel who in 1837 established the first kindergarten in Germany and developed an educational program for young children. A central component in this program were the so-called *gifts*, small manipulative materials by which children could be made aware of numerical and geometric relationships and which could provide them experiences with respect to, for example, patterns, symmetry, counting, measurement, addition, division, fractions, and properties of shapes. One of the *gifts* consisted of a series of cubes made out of wood, divided into smaller parts, and followed by square and triangular tablets. The *gifts* were offered to the children in a prescribed sequence, and the children were expected to build precise forms with them. Although children in the Fröbelian kindergarten might have acquired a substantial amount of mathematical knowledge, attained incidentally and instinctively through play, the ultimate goal of Fröbel was not to teach children mathematics, but help 3- to 6-year-olds to understand the relationship between nature, God, and humanity.

At the turn of the twentieth century, many from the kindergarten community began to question the appropriateness of Fröbel's curriculum and his methods. For example, Dewey considered the Fröbelian activities as mindless copying and manipulation of artificial objects. These concerns led to the so-called "child-centered approach," which originated from the eighteenth century philosopher Rousseau. In this approach there was no specific program for mathematics instruction, but children were engaged in activities based on their interests, which would incidentally help children prepare for the later learning of formal mathematics. This approach also applied to the nursery school which was established firstly in England in the beginning of the twentieth century. The educational program was predominantly focused on children's play and ignored academic subjects which would be taught later when the children are older.

A different approach was reflected by Montessori, who at the beginning of the twentieth century introduced a method for teaching young children that was deeply mathematical. Most of the activities she suggested were requiring, for example, working with patterns and exploring the properties of geometric shapes, numbers and operations. Her approach included working with sensory materials and was based on the idea that children use their senses to acquire information about the world. For example, children felt the shape of numerals made of sandpaper before writing these numerals.

Halfway the twentieth century, the ideas of Piaget influenced the teaching of mathematics to young children. He related the construction of number concepts to the development of children's logical thinking and focused on understanding common properties of quantities like conservation, seriation, and class inclusion rather than on counting. Piaget emphasized that there is a relationship between the basic structures of modern mathematics and the mental structures developed in children. Although these and other ideas of Piaget were questioned, Piaget, together with other pioneers since Comenius, has contributed to the present awareness of the importance of mathematics education for young children.

## Recent Interest in Early Childhood Mathematics Education

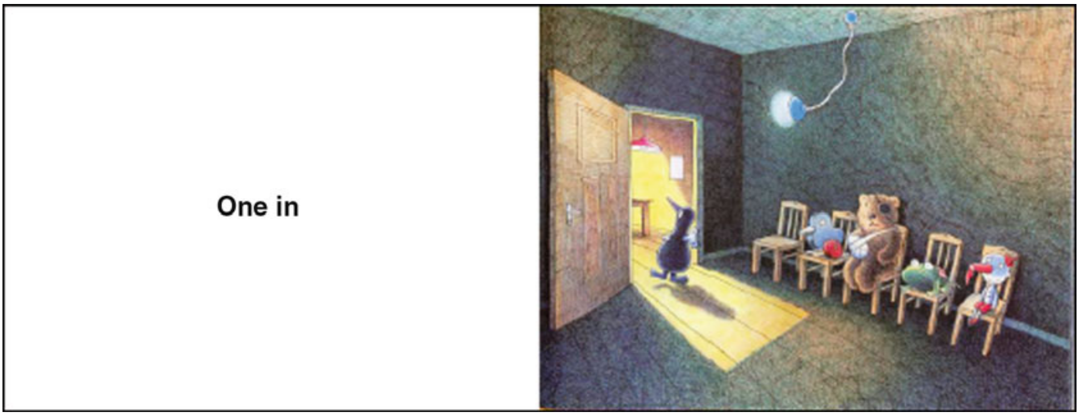
Currently, early childhood education has risen to the top of the national policy agenda with recognition that ensuring educational success and attainment must begin in the earliest years of schooling (National Research Council 2009). An important reason for this is that research has shown that the amount of mathematical knowledge children bring with them when they start in grade 1 has large, long-term consequences for their further learning of mathematics (Duncan et al. 2007).

For example, in the United States, the recent awareness of mathematics as a key aspect of early childhood education was boosted in 2000 when the National Council of Teachers of Mathematics published their revised 1989 standards for elementary and secondary school mathematics and included prekindergarten for the first time in their description of standards. A further step was a joint position statement titled Early Childhood Mathematics: Promoting Good Beginnings by the National Association for the Education of Young Children and the National Council of Teachers of Mathematics (NAEYC and NCTM 2002) that was aimed at achieving high-quality mathematics education in child care and other early education settings. The book resulting from the Conference on Standards for Early Childhood Mathematics Education (Clements et al. 2004) and the Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics (NCTM 2006) were other breakthroughs for early childhood mathematics education. Similar documents for teaching mathematics in the early years of schooling were also released in other countries, for example, in the United Kingdom (Department for Children, Schools and Families 2008), France (Ministère de l'Education Nationale 2002), Australia (Australian Association of Mathematics Teachers and Early Childhood Australia 2006), and the Netherlands (Van den Heuvel-Panhuizen and Buys 2008).

Another indication for the new prominent position of early childhood mathematics education is reflected by the establishment, in 2009, of the working group on Early Years Mathematics in the Congress of the European Society for Research in Mathematics Education (CERME), which focuses into research on learning and teaching mathematics to children aged 3–8. The work of this group in the last two meetings of CERME has shown that investigating mathematics education during the early years is a rather complex and multidimensional endeavor. The specificities of early childhood education in different countries and educational systems, e.g., the differences in the conception of schooling and early years mathematics and in the transition ages from preprimary to primary school and the differences in the education and development of prospective preschool and kindergarten teachers regarding the didactics of mathematics as well as the constraints in the ability of young children to articulate their mathematical thinking and understanding, are only some of the factors that contribute to this complexity.

## Mathematics Taught in Early Childhood

Although in the past, early childhood mathematics education was often restricted to teaching arithmetic, several early pioneers such as Fröbel and Montessori as well as Piaget offered a wider program to children. Presently, there is expert consensus (see National Research Council 2009) that two content areas of mathematics are particularly important for young children to learn, namely, (1) numerical and quantitative ideas and skills and (2) geometric and spatial ideas and skills. Moreover, according to Clements and Sarama (2007), these ideas and skills are permeated by mathematical activities such as dealing with patterns, analyzing data, and sorting and sequencing.




**Early Childhood Mathematics Education, Fig. 1** Page 3 of the picture book Vijfde zijn [Being Fifth], Left side: Text "One in", Right side: Illustration of five broken toys in a doctor's waiting room (Jandl and Junge 2000)



## Ways of Teaching Mathematics to Young Children

There is also general agreement that "teaching" mathematics to young children should have many characteristics of the informal learning as it takes place in the family setting where children come along with mathematics in a natural way and "mathematical ideas permeate children's play" (Ginsburg and Amit 2008, p. 275). Young children develop mathematical ideas and skills primarily in informal ways which make sense to them. Thus a major part of early mathematics education needs to be organized in informal contexts which are meaningful for the young children.

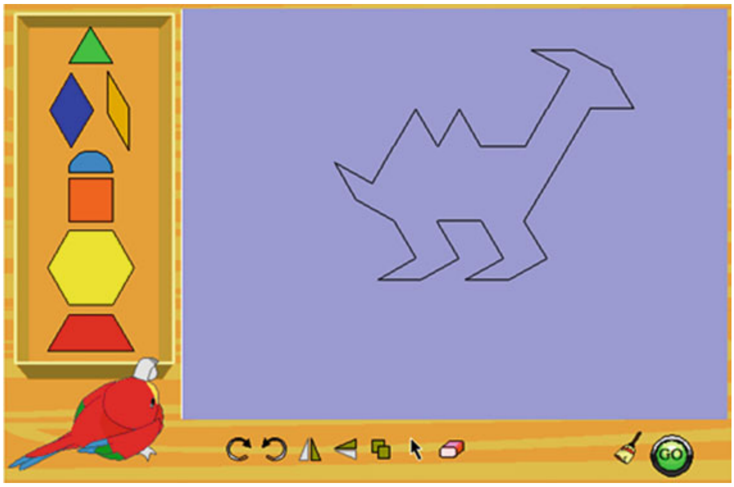
### Play

Such learning opportunities can be provided in kindergarten through play (Pramling-Samuelsson and Fleer 2009). By offering playful activities such as free play, sensorimotor play, making constructions, and role playing, children can know the world mathematically. They can spontaneously deal, for example, with counting up to large numbers, comparing the height of their towers of blocks, creating and extending patterns when jumping up and down, and connecting movements to verbal expressions, investigating shapes, and exploring symmetry and spatial relations.

According to Vygotsky play in early childhood becomes the leading activity of development. The challenges the children encounter during play and the help they receive from more knowledgeable others, such as teachers, who assign mathematical meaning to their play actions, enable the children to move a step forward in their abilities. In this way they enter the zone of proximal development.

### Picture Books

Another way of offering children meaningful contexts in which they can encounter mathematics-related problems, situations, and phenomena that can support the learning of mathematics is by reading them picture books (Van den Heuvel-Panhuizen and Elia 2012). From a Vygotskian and action-psychological approach to learning (Van Oers 1996), picture books can contribute to forming, exchanging, and negotiating all kinds of personal meanings within everyday practices and to acquiring mathematics as an activity involving historically developed and approved meanings. Furthermore, they can offer cognitive hooks to explore mathematical concepts and skills. An example concerns the book *Vijfde zijn* [Being Fifth] (Jandl and Junge 2000), which is about a doctor's waiting room in which five broken toys are waiting for their turn (see Fig. 1). Even though the book was not written for the purpose of teaching mathematics,




**Early Childhood Mathematics Education, Fig. 2** Geometric puzzle in a Building Blocks' software tool

it implicitly touches on counting backwards and spatial orientation as part of the narrative and has the power to offer children a rich environment for eliciting mathematical thinking (Van den Heuvel-Panhuizen and Van den Boogaard 2008).

## Information and Communication Technology

Although there is still debate about whether Information and Communication Technology (ICT) is appropriate for teaching young children, there is ample evidence from research that computer use can be meaningful, motivating, and beneficial for children 3 years of age and above (e.g., Haugland 2000; Clements et al. 2004). The use of computers in early years' mathematics can support young children's mathematical thinking in various ways. One of the most powerful affordances of the use of computers in early childhood mathematics education is that they embody the processes children need to develop and mentally use. Computers can also help children connect concrete and symbolic representations of the same mathematical concept, e.g., by providing a dynamic link between base-ten blocks and numerical symbols. Using mathematical computer games enables children to explore mathematical concepts, such as geometric figures, in ways that they cannot with physical manipulatives. For example, they can modify the size of geometric shapes, without changing their critical attributes. Furthermore, the use of computers can support children in bringing mathematical processes and ideas, such as shape transformations, in an explicit level of awareness. The Building Blocks program (Clements et al. 2004), for example, uses computer software tools (see Fig. 2) to help preschoolers acquire geometric and numerical ideas and skills.

In sum, the computers can provide valuable opportunities for learning in early childhood mathematics education. However, realizing the full potential of technology requires comprehensive, meaningful, and well-planned instructional settings. The development and organization of such settings strongly depends on the curriculum and the teacher (Clements 2002). Thus, effectively integrating technology in the early childhood mathematics curriculum and appropriate professional development of kindergarten teachers should be vitally important concerns in relation to computer use in mathematics education in the early years.

## Future Perspectives in Early Childhood Mathematics Education

Presently there is broad diversity of theories of learning mathematics ranging from cognitivist theories including a Piagetian approach, situated cognition, and semiotic approaches to various constructivist theories and social-cultural theories. A recent research direction in mathematics education is the theory of embodied learning in mathematics which claims on the basis of knowledge from neuroscience that cognition and concepts are strongly founded on bodily experiences. Although this new approach to learning is closely related to how young children explore and make sense of their environment, not much research has been carried out in how ideas from embodiment theory can be used to acquire a better understanding of young children's mathematical development and how early childhood education can contribute to this development.

## Cross-References


- ▶ Constructivism in Mathematics Education
- ▶ Informal Learning in Mathematics Education
- ▶ Information and Communication Technology (ICT) Affordances in Mathematics Education
- ▶ Mathematical Games in Learning and Teaching
- ▶ Semiotics in Mathematics Education
- ▶ Situated Cognition in Mathematics Education
- ▶ Technology and Curricula in Mathematics Education
- ▶ Theories of Learning Mathematics
- ▶ Zone of Proximal Development in Mathematics Education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## Education of Facilitators (for Educators of Practicing Teachers)

► Education of Mathematics Teacher Educators

## Education of Mathematics Teacher Educators

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### Keywords

Education of teacher educators; Knowledge of teacher educators; Professional development; Reflection; Formal education

### Synonyms

Education of facilitators (for educators of practicing teachers)

### Definition

Education of teacher educators refers to the preparation, professional development, teaching, or facilitating of teacher educators. It is understood as a goal-directed intervention in order to promote teacher educators' learning and further development of beliefs, knowledge, and practice, including formal as well as informal activities. Nowadays, the term "teacher educators" commonly refers to both those who educate prospective teachers and those who educate practicing teachers, that is, to those who initiate, guide, and support teacher learning across the lifespan (Even 2008; Krainer and Llinares 2010). Yet, sometimes the term "teacher educators" refers only to educators of prospective teachers, that is, to those who teach future teachers and not to those who provide professional development for practicing teachers.

### Theoretical Background

There is general recognition and agreement today that the education and professional development of teachers is key to improving students' opportunities to learn (Even and Ball 2009; Krainer 2011). Accordingly, the focus and nature of the education of prospective and practicing teachers have received immense international attention in recent years, and the past decades have seen substantial increase in scholarship on mathematics teacher education. A significant issue identified recently as crucial for improving the education and professional development of mathematics teachers is the education and development of teacher educators and related research (Adler et al. 2005; Even and Ball 2009; Jaworski and Wood 2008).

In different countries around the world, various professionals are responsible for initiating, guiding, and supporting teachers' learning: university faculty with disciplinary expertise and those who specialize in education; school teachers, teacher mentors, and staff of curriculum implementation projects; educators whose major occupation is to work with teachers and those who do it only as an add-on part-time temporary activity; those who work with both prospective and practicing teachers; and those whose role is to educate solely prospective or practicing teachers, but not both. Yet, this vast range of teacher educators has little formal preparation for their work. Most become teacher educators through practice with little institutional and professional support. With the expanding current interest in the issue of professional education and development of teacher educators in different countries, pioneering formal programs to prepare educators to educate teachers started to emerge. These include, for example, the Pedagogy and Subject-Didactics for Teachers (PFL) Program in Austria, the MANOR Program in Israel for educating educators of practicing mathematics teachers, the School for Research and Development of Education Programs for Teacher College

Faculty (MOFET Institute) in Israel, a special M.Ed. program in Pakistan, and the Leadership Curriculum for Mathematics Professional Development (LCMPD) Project in the USA.

### Important Scientific Research and Open Questions

The education of teacher educators has only recently become of interest to the international community. Thus, not much is known about the development of teacher educators and about effective ways to educate educators to initiate, guide, and support teacher learning (Even 2008).

Research studies that center on issues pertaining to professional education and development of teacher educators in a specific subject area are rare. Mathematics is among the subjects where efforts in investigating the education of teacher educators have become visible recently (Even 2005; Jaworski and Wood 2008; Nardi 2008; Oikkonen 2009). Most research on the professional education and development of mathematics teacher educators includes reflections of teacher educators on their own personal development (e.g., Cochran-Smith 2003; Jaworski and Wood 2008). This research suggests that reflective inquiry has a central role in learning to teach teachers and in developing as teacher educators. Yet, this line of research provides information mainly on the professional development of university-based teacher educators with research interest in teacher education, but not on that of the wide range of professionals responsible for supporting prospective and practicing teachers' learning.

Because formal preparation for mathematics teacher educators scarcely exists, research that examines formal programs and activities intended to educate mathematics teacher educators is sparse. Pioneering work in this direction addresses various aspects of curriculum (What should teacher educators learn?) and pedagogy (How should teacher educators be taught?). It suggests several areas of professional knowledge base for mathematics teacher educators (Jaworski and Wood 2008); two relate to knowledge shared by teacher educators and teachers: pedagogical knowledge and disciplinary knowledge. A third area of professional knowledge base for educating teacher educators relates to knowledge specific to the mathematics teacher educator: knowledge of teaching teachers and of teachers' learning. In addition to professional knowledge base, research suggests the need to purposely teach practices of educating teachers, giving explicit attention to the nature of work in which mathematics teacher educators engage. These practices may be general, such as teaching courses, supervising student teachers, and facilitating seminars (Cochran-Smith 2003), or subject matter specific, such as planning, conducting, and assessing activities, workshops, and courses for mathematics teachers (Even 2005). This line of research also suggests that inquiry is central to learning to teach teachers and to developing as mathematics teacher educators. Additionally, it shows the importance of attending to the relationships of knowledge and practice.

Thus far, it is not known whether, or in what ways, formal education of mathematics teacher educators needs to be responsive to the wide range of professionals responsible for supporting teachers' learning or may be common to all, for example, whether the professional education of educators of practicing mathematics teachers needs to be different from the education of educators of prospective mathematics teachers, as the education of prospective and that of practicing mathematics teachers are commonly of different nature, often occurring in different settings, and not necessarily conducted by the same people.

## Cross-References

▶ Inquiry-Based Mathematics Education
▶ Mathematics Teacher Educator as Learner

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# Elkonin and Davydov Curriculum in Mathematics Education

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## Keywords

Developmental learning; Activity theory; Sociocultural theory; Vygotsky; Curriculum; Measurement

## Definition

The Elkonin-Davydov mathematics curriculum was an elementary mathematics curriculum developed in Russia based on Russian activity theory. In recent years, the original Russian curriculum has been expanded to include grades K–8 and has been refined into several different curricula. In addition, research projects in other countries (e.g., USA) have investigated applications with local populations.

## Characteristics and Origin

In 1959, Daniil Borissowitsch Elkonin (1904–1984) and Vasily Vasil'evich Davydov (1930–1998), Russian psychologists and students of Lev Vygotsky, developed an elementary mathematics curriculum. Their work was initially situated in experimental school #91 in Moscow where their team functioned as researchers and teachers. The project was grounded in Russian activity theory, which grew out of the cultural-historical theory of Vygotsky.

Davydov was critical of the existing schooling system and argued that traditional pedagogy failed to develop a general concept of number that could support the learning of numbers of all types. Students were forced to learn a new concept of number each time they focused on a different number domain (e.g., integers, rational numbers, irrational numbers, imaginary numbers). Elkonin and Davydov believed that developmental learning coupled with Vygotsky's description of the development of scientific concepts (Vygotsky 1987) could overcome the restrictions of a traditional approach.

The E-D approach is characterized by two essential principles within developmental learning. The first is dialectical logic, which can be thought of as diametrically opposed to empirical thinking in which learning is based on accumulation of cases (Davydov 1990). To support dialectical logic, the E-D approach aims at the learning of more general ideas and then builds on those general ideas to develop advanced concepts that incorporate those ideas. Thus, in the E-D curriculum a general concept of number is developed and then built on as different number domains are explored.

Elkonin and Davydov believed that thinking about conceptual and abstract ideas should lead to a child's ability to analyze, reflect, and plan. Explicitly, analysis is the child's ability to isolate the critical and essential relation in a problem. Reflection is the child's understanding of the bases of his/her own activity. Planning is the child's ability to construct ways to solve a problem based on systems of activities.

The second principle of developmental learning is learning through one's own activity (Leont'ev 1978). In the E-D approach, this is characterized by students' activities in which they reconstruct mathematical ideas from their origin. That is, the mathematics is presented so that students see how ideas build, one on another. There is a specific learning goal toward which the instructional tasks are structured. In their work on the tasks, students interact with specific tools that help them see the mathematics in particular ways during the learning process.

In order to foster a general understanding of number that can support learning related to all types of number, the E-D curriculum (Davydov et. al 1999) starts with a prenumeric stage rather than counting and builds on a foundation of measurement concepts. In the prenumeric stage, children first identify the attributes of objects that can be compared and engage in direct comparison. For example, two bottles can be compared in multiple ways such as their height, the area of their bases, the volume of water they can hold, and their masses. These four attributes are considered to be generalized, nonspecific continuous quantities. Continuous quantities, in contrast with discrete quantities, can be subdivided a limitless number of times and each part of the subdivisions is of the same type. The quantities are generalized and nonspecific because they have no number (as determined by measure or count) associated with them.

By using the attributes of length, area, volume, and mass, children explore equality and inequality including creating an equal relationship from one that is unequal (by adding or subtracting the difference). The fundamental properties of arithmetic (such as commutativity and associativity) naturally arise from these explorations – all without numbers. Reasoning about generalized quantities is supported by introducing letters to represent the quantities and arrow diagrams and equations to represent the relationship between quantities.

The prenumeric work, in which students examine relationships among physical quantities, forms the basis for the E-D curriculum. Number is not a primitive idea as it is in curricula that begin with counting. Number is the result of measuring a quantity with a unit. The need for measurement is introduced in order to compare quantities that cannot be compared directly (e.g., two lengths that cannot be laid side by side). To measure a quantity, one needs to determine a unit that can measure the quantity. If a quantity and a unit exist, then to find the count, the unit is iterated until the quantity has been fully measured. The counting of the iterations drives the introduction of number. Thus a *number* is defined as the result of measuring a quantity with a unit. Note that neither the quantity nor the unit has numbers associated with them. Numbers are produced through measuring one with the other.

In each of the grades, however, the E-D curriculum consistently begins a topic of study with learning problems that lead to a system of activities. Learning problems are situations that significantly change students' thinking. The change occurs within children's activity and thus the material chosen for the learning problem is ultimately an important consideration. It must support the acquisition of constructing a general way to view the activity itself.

For example, initially in grade 1, students use direct comparison to find the relationship between two quantities. In a new learning problem, students are then given the challenge to determine how two quantities compare when they cannot be moved to perform a direct comparison. This motivates students to consider how the direct comparison method can be changed so that it will fit the new parameters of the problem.

In the above example, the inability to perform a direct comparison requires children to consider the use of a tool that mediates the situation. From this task, the need for a portable representation of

$$E \xrightarrow{n} Q \qquad \frac{Q}{E} = n$$

**Elkonin and Davydov Curriculum in Mathematics Education, Fig. 1** Example of two ways to express relationship of quantity, unit, and count

at least one of the quantities is created. Children must now negotiate a tool and find a systematic way to use it. Additionally, if they construct the tool to be only some part of the whole quantity, it becomes the introduction to counting as they measure the quantity through iterations. By changing the task ever so slightly, children are beginning the generalization of the process of measuring. Since the task represented above can occur in any of the four continuous quantities, children come to view this as a generalized model for any measurement, even those associated with discrete sets.

The outcome of this approach is that children see "unit" as the basis of all number. The relationship of the unit to a quantity and its measure is critical in determining how each component relates one to another. The relationship is expressed in multiple forms that reflect the action used to determine the count and show the relationship across the unit ($E$), the quantity ($Q$), and the count ($n$) (See Fig. 1).

From these representations, children generalize that as the unit ($E$) gets larger, the count ($n$) gets smaller. Even though this is introduced in grade 1, it is an important concept for the development of rational number. Subsequent instruction builds on these initial concepts of quantities, units, measurement, and number. Place value is taught as relationship between different size units in a system of units in which each larger unit is n times larger than the prior unit. Multiplication is taught as the use of an intermediate unit to find the number of units in a quantity. For example, a meter could be used as an intermediate unit to find out how many centimeters are in a quantity. Multiplication is the relationship between the number of centimeters in a meter and the number of meters in the quantity that gives the number of centimeters in the quantity. Fractions are taught by introducing partial units, initially by reversing the process that created larger place values.

## Implementation and Adaptation

The E-D elementary mathematics curriculum has been implemented in about 10 % of elementary schools throughout the Russian Federation since the collapse of the Soviet educational system in 1991. Evaluation studies consistently demonstrate that students in E-D elementary classrooms do better overall than students in other elementary classrooms (Nezhnov et al. 2009; Vysotskaia and Pavlova 2007; Zuckerman 2005). In a comparative study of E-D (Davydov et. al 1999) and six other curricula in Russia, Vysotskaia and Pavlova (2007) found that the E-D students were better able to solve a variety of problems than those in other curricula. Similarly, Zuckerman (2005) compared the E-D curriculum to two other curricula using selected problems from the PISA international mathematics tests. She found that 15-year-old students who had been taught through the E-D curriculum demonstrated a higher ability to use diagrams, graphs, and other representations for solving problems.

There are at least two significant adaptations of the E-D curriculum outside of Russia. One adaptation focused on grades 1–3 only in one school in the USA. The results, however, are compelling in that the findings from multiyear implementations indicate the use of E-D curriculum supported computational competency as well as the development of algebraic concepts (Schmittau 2005).

On a larger scale, in 2001, the Curriculum Research & Development Group, University of Hawaii, entered into a collaborative arrangement with the Elkonin-Davydov group to create an adaptation of the E-D curriculum for grades 1–5. The adapted curriculum, Measure Up (Dougherty 2008), closely followed the E-D approach but revised the instructional approaches to include significant language components (reading, writing, speaking, and critical listening). Additionally, some contents, such as fractions, were introduced in a slightly different way even though the focus on quantitative reasoning and measurement was maintained. The resulting curriculum (Dougherty et al. 2004) was implemented and tested in two sites in Hawaii with significant results. A study (Slovin and Venenciano 2008) used the Chelsea Diagnostic Mathematics Test: Algebra (Hart et al. 1985) (originally designed for 13–15-year-old students) to determine how well 5th and 6th grade students who had engaged in the Measure Up curriculum were prepared for algebra. Measure Up students performed disproportionately better than students who had not experienced Measure Up on a subset of items focused on concept of variable.

Even though studies both in the USA and Russia have indicated that students learn significant mathematics, the issue of broader dissemination remains problematic for at least three reasons. First, the approach to mathematics is unique in that it does not follow the conventional approach we have come to expect in elementary mathematics where we begin with counting and number. Second, content knowledge that is expected in teacher preparation courses is not sufficient for teaching the E-D or Measure Up curricula. Finally, high-stakes assessments are often based on a conventional approach and sequence to elementary mathematics. Thus children are learning concepts and skills in a different sequence.

## Cross-References


▶ Activity Theory in Mathematics Education
▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes
▶ Number Teaching and Learning


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# Embodied Cognition


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## Keywords

Abstraction; Cognition; Dienes; Embodiment; Situated cognition


## Definition

Embodied cognition is a subdomain of cognitive psychology that focuses on the interaction between an individual and the environment (social, environmental, instructional). It moves beyond the traditional distinctions between mind and body in the sense that actions or interactions embody projections of the mind and vice versa.

### Some Definitional Differences in Mathematics Education

Many embodied ideas eventually are represented symbolically in mathematics. Examples of these are enumeration systems which are abstractions of human gestures for counting, pointing, and measuring. Freudenthal (1973) claimed that geometry is based upon our experiences with our bodies in the world. This suggests that the only mathematics we are able to know is the mathematics that our bodies and brains allow us to know (Lakoff and Nunez 2000; Fyhn 2010). Freudenthal (1973) also claimed that geometry is about grasping space. Fyhn (2010) interprets "Space" according to this definition as that "in which the child lives, breathes and moves" (p. 296). The idea of a "grounding metaphor" is used to connect different mathematical ideas such as arithmetic, the Cartesian coordinate system, functions (Bazzini 2001), and even calculus (Lakoff and Nunez 2000) to everyday activities. One should note that there is a difference between micro-embodied experiences such as gestures and macro-embodied experiences such as throwing an object, climbing stairs, or climbing a wall.

## Embodied Cognition in Mathematics Education

Nunez et al. (1999) claim that learning and using mathematics are closely associated with the social, cultural, historical, and contextual factors (p. 45). These have also been labeled as "situated" learning (Lave 1988). Mathematics is conceived as a product of human activities in the process of adapting to the external environment and needs, and shared and made meaningful through language and other means, but based ultimately on biological and bodily experiences. The creation of mathematics through "situated" cognition and sensemaking is not arbitrary, rather is bodily grounded (Lakoff and Nunez 2000). From an embodied cognition perspective, the learning of mathematical knowledge occurs in naturally situated, often unconscious, everyday thoughts. The implication of embodied cognition in the pedagogy of mathematics education is that rather than teaching students to learn "rigorous" definitions/theorems of the pre-given mathematical ideas, one needs to focus on the understanding and sensemaking that students need to develop. It is daily experiences that provide the initial grounds for the abstractions that constitute mathematics. This view has been suggested earlier since the early 1960s by Zoltan Paul Dienes (Sriraman and Lesh 2007).

## Cognitive Science of Embodied Cognition

Lakoff and Nunez (2000) discussed the cognitive science of mathematics based on the key concept of embodied cognition. The basic assumption is that mathematics is not mind-free. There are claims such as newborn babies aged 3 or 4 days old having the innate arithmetic abilities to discriminate between collections of two and three items (Antell and Keating 1983) which are supported by other studies beyond the scope of this entry. Basic arithmetic uses various capacities of our brain such as subitizing, perception of simple arithmetic relationships, estimate and approximation, and the ability to use symbols (Dehaene 1997). Mathematical cognition often occurs unconsciously (Lakoff and Nunez 2000). This is because the general cognitive mechanisms that use everyday nonmathematical thoughts can create mathematical understanding and structure mathematical ideas (p. 29). Again Lakoff and Nunez (2000) claim that there are two types of conceptual metaphors that play an important role in the development of mathematical ideas, i.e., grounding metaphors and linking metaphors. The interested reader should examine chapters from *Where Mathematics Comes From* that focus on these ideas. In a nutshell a grounding metaphor refers to basic, direct mathematical ideas. For example, multiplication as repeated addition sets as containers and elements of a set as objects in a container. Linking metaphor refers to abstraction, which produces sophisticated ideas. For instance, geometric figures as algebraic equations (Lakoff and Nunez 2000, p. 53).

### Dienes' Contributions to Embodied Mathematics

Based on a survey of prior studies in mathematics education, Sriraman and Lesh (2007) claimed that Dienes not only studied a phenomenon that later cognitive scientists have come to call embodied knowledge and situated cognition but he also emphasized the *multiple embodiment principle* whereby students need to make predictions from one structured situation to another. And he also emphasized the fact that, when conceptual systems are partly off-loaded from the mind using a variety of interacting representational systems (including not only spoken language written symbols, and diagrams but also manipulatives and stories based on experience-based metaphors), every such model is, at best, a useful oversimplification of both the underlying conceptual systems being expressed and the external systems that are being described or explained. Thus, Dienes' notion of *embodied knowledge* presaged other cognitive scientists who eventually came to recognize the importance of *embodied knowledge* and *situated cognition* – where knowledge and abilities are organized around experience as much as they are organized around abstractions (as Piaget, e.g., would have led us to believe) and where knowledge is distributed across a variety of tools and communities of practice.

## Cross-References

▶ Early Childhood Mathematics Education
▶ Enactivist Theories

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# Enactivist Theories

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## Keywords

Autopoiesis; Structural coupling; Structural determination; Triggering; Co-emergence

## Definition

Enactivist theories assert that cognition is a process that occurs through feedback loops within the interaction of complex dynamical organisms/systems.

## Characteristics

Closely related and often conflated with enactivist theory is embodied cognition. The distinction taken here is made on the basis of the roots of the two theories. Enactivism has biological roots, for example, in the writing of Maturana and Varela (1992) and others, whereas embodied mathematics has linguistic roots (see Embodied cognition).

Enactivist theory is a development of biological and evolutionary science and complexity theory and addresses, among other things, the critique of Cartesian dualistic notions of object/subject. In enactivist theory it is argued that cognition is a process that occurs through the interaction between the living organism and its environment (*autopoiesis*).

We propose as a name the term *enactive* to emphasize the growing conviction that cognition is not the representation of a pregiven world by a pregiven mind but is rather the enactment of a world and a mind on the basis of a history of the variety of actions that a being in the world performs (Varela et al. 1991, p. 9).

From an enactivist perspective learning is seen as a process of restructuring that is *triggered* by interaction that occurs within the complex dynamic system of coupling (*structural coupling*) between person and environment.

We speak of structural coupling whenever there is a history of recurrent interactions leading to the structural congruence between two (or more) systems (Maturana and Varela 1992, p. 75).

Restructuring within the person, however, is determined by the (biological) structural properties of the person (*structural determination*), not by the properties of the environment within which the restructuring occurs. The interaction also triggers changes in the environment, which is also consequently determined by the structure of the environment; this is referred to as coevolution/coadaptation, or *co-emergence*. As can be deduced from the above quotation from Varela et al. enactivism also challenges theories that require some form of mental knowledge representation structures in which perception and reflection are actions upon mental representations of the world constructed independently by the perceiving subject. Cognition and knowing are explained within enactivist theory as active processes that occur directly through the interaction between the cognizing subject and the environment, rather than as a construction of representations of the environment by the cognizing subject.

Knowing is effective action, that is, operating effectively in the domain of existence of living beings (Maturana and Varela 1992, p. 29).

Enactivist theories have roots in biological sciences (Maturana and Varela 1992; Varela et al. 1991) and Darwinian theory of evolution and thus might be viewed as a development of Piaget's constructivism. However, Proulx (2008a) draws attention to some ontological and epistemological differences between enactivism and constructivism. Philosophical antecedents of enactivist theories are shared with closely related "embodied" theory, and more generally situated cognition, these theories refer to seminal philosophical contributions by Edmund Husserl, Maurice Merleau-Ponty, and Ludwig Wittgenstein (Reid 1996).

Autopoiesis: Complex dynamic systems can be defined at many levels, from complex molecular structures within a single cell to solar systems within a galaxy. Autopoiesis is asserted by Maturana and Varela to be the process that distinguishes living beings.

Our proposition is that living beings are characterized in that, literally, they are continually self-producing. We indicate this process when we call the organization that defines them an *autopoietic organization* (Maturana and Varela 1992, p. 43).

Cognition and knowing is one part of autopoietic organization.

Thus a learner within a mathematics classroom constitutes a dynamic system; alternatively one, or a group of, teacher(s) within a professional development setting constitute a system. The learner is a *distinct unity* (Maturana and Varela 1992, p. 40) within the environment of a mathematics class comprising other learners, teacher, and resources. The learner is *structurally coupled* with the classroom environment. Disturbances within the environment *trigger* changes within the learner as she/he adapts herself/himself to the environment. However, the adaptation of the learner is determined by the "structure" (prior experiences and learning and affective characteristics) of the learner, not by the interaction with the environment. The interaction merely "triggers" the change. Thus enactivist theory asserts that cognition is *structurally determined* by the organization of the learner (Maturana and Varela 1992, p. 96).

Enactivist theories began to emerge within the research field of mathematics education in the 1980s, especially following the publication of Maturana and Varela's book *Tree of Knowledge* (1992). A group of Canadian mathematics education researchers established themselves as a center of interest in enactivist theories forming an "Enactivist Research Group" (Reid 1996). However, research within enactivist theories as a framework and methodology is now actively pursued throughout the world, as can be seen from the account below. The account indicates how enactivist theories have entered into the discourse of mathematics over three decades, 1982–2012, thematically, geographically, and through publication in the major scientific journals and conferences in the field.

Tom Kieren and Daiyo Sawada (Canada) became interested in the work of Maturana and Varela in 1982, and later Kieren and Sawada introduced enactivist theory to the mathematics education group at the University of Alberta, Canada (Proulx et al. 2009). The first edition of Humberto Maturana and Francisco Varela's book *The Tree of Knowledge* was published in 1987 (Maturana and Varela 1992). Then around 1993 The Enactivist Research Group was established in Canada (Reid 1996).

Maturana and Varela's theory entered the international discourse of mathematics education through the annual conferences of the International Group for the Psychology of Mathematics Education (PME) during the period 1994–1996. In 1994 at the 18th PME conference held in Lisbon, John Mason (UK) made reference to Maturana and Varela's work in his plenary lecture "Researching from the inside in mathematics education." One year later at the 19th PME conference in 1995 held in Recife, Rafael Núñez and Laurie Edwards (USA) convened a discussion group that focused on embodied cognition; the participants included David Reid (Canada) and Laurinda Brown (UK) who later became significant contributors to the development and application of enactivist theory within mathematics education research and practice. At the same PME conference Edwards and Núñez presented a theoretical paper in which enactivism was identified as one of the several nonobjectivist theories within the compass of new paradigms in cognitive science. A year later David Reid presented a research report at the 20th PME conference held in Valencia in 1996; in this Reid set out enactivism as a methodology. He described research from an enactivist perspective in terms of autopoietic relationships, between researcher and data: between researchers as they engage with each other and the co-emergence of ideas between researchers and the "coemergent autopoetic (sic) ideas which live in the medium of our minds and of which we are emergent phenomena (as the herd is of the antelope)" (Reid 1996, p. 205). The report included a brief review of enactivist theory and its roots.

Also in 1995 Brent Davis (Canada) published a paper in the journal *For the Learning of Mathematics* that set out an enactivist rationale for learning mathematics; the paper included a brief account of the nature of mathematical activity from an enactivist perspective. In this paper Davis applies an enactivist argument to emphasize the inseparability of process and product in mathematical activity (Davis 1995). In 1997 Davis suggested that enactivism provides "a framework for interpreting the phenomenon of mathematics teaching . . . that might allow us to embrace the insights of constructivism without losing the substance of the social critics' arguments," in a report published by *Journal for Research in Mathematics Education* (Davis 1997, p. 355).

During the following decade (1998–2007) interest in enactivist theory developed internationally and in its application to various domains of research within mathematics education. In 1998 Markku Hannula (Finland) applied enactivist theory to research into affect and learning mathematics. He later published more extensively, for example, in the journals *Educational Studies in Mathematics* and *Research in Mathematics Education* (see Hannula 2012 for references). A year later in 1999, Andy Begg

(New Zealand) presented a paper introducing enactivist theory at the annual conference of the Mathematics Education Research Group of Australasia (MERGA-22) (Begg 1999). In the same year, Laurinda Brown and Alf Coles (UK) explained how enactivism informs their research at the November day conference of the British Society for Research into the Learning of Mathematics.

In 2000 the journal *Mathematics Thinking and Learning* published a paper by Edward Drodge and David Reid (Canada) that considers emotional orientation through the lens of embodied cognition. Drodge and Reid take an enactivist perspective to explore the role of decision making in learning mathematics and use illustrations from an episode in which a group of boys engaged in a geometry problem solving task (Drodge and Reid 2000). Later, David Reid, in 2002, adopted an enactivist perspective of learning to describe "clearly one pattern of reasoning observed in the mathematical activity of students in a Grade 5 class" and explore and clarify the characteristics of mathematical reasoning. Reports from this study are published in *Journal for Research in Mathematics Education* and *Journal of Mathematical Behavior* (Reid 2002).

In 2003 Davis and Simmt (Canada) focused on the application of complexity science and how this might contribute "to discussions of mathematics learning and teaching" (Davis and Simmt 2003, p. 138); complexity theory is deeply embedded in the notion of autopoiesis.

In 2005 Elena Nardi, Barbara Jaworski, and Stephen Hegedus (UK) published enactivist framed research into teaching mathematics at university level in *Journal for Research in Mathematics Education* (Nardi et al. 2005). The following year, 2006 Laurinda Brown and David Reid (UK & Canada) applied enactivist theory to explore learner's "non-conscious" decision making processes that occur prior to conscious awareness of making choices and how emotions subsequently structure events (Brown and Reid 2006). The first, nonconscious decisions might be explained as a feature of "structural determinism," and the latter, restructuring of events, explained as "coemergence" as the environment is shaped by the learner.

Maria Trigueros and Maria-Dolores Lozano (Mexico) reported in 2007 on the use of an enactivist approach in the design of resources for teaching and learning mathematics with digital technologies in the journal *For the Learning of Mathematics* (Trigueros and Lozano 2007). A year later, 2008, Lozano reported an enactivist analysis and interpretation of students algebra learning from a longitudinal study of grade 6 (elementary school) through grades 7 and 8 (first years at secondary school) (Lozano 2008). In the same year Jérôme Proulx (Canada) published his use of the enactivist notion of structural determinism to explain characteristics of mathematics teachers' learning (Proulx 2008b). Proulx (2008a) also argues that there are ontological and epistemological differences between constructivist and enactivist theories of cognition, such that enactivism "should not be (mis) interpreted as another form of constructivism" (p. 24).

The period 2009–2012 reveals both consolidation of international effort and maturation of research conducted within enactivist theory. In 2009 the 33rd annual conference of PME held in Mexico included a Research Forum on enactivist theory of cognition (Proulx et al. 2009). The "forum" included brief papers by many researchers and groups (from Canada, Emirates, New Zealand, Mexico, the UK, the USA) that were applying enactivist theory in their research. The report offered a "state of the art" (in 2009) account of enactivism in mathematics education from an international perspective. Proulx concludes the report by suggesting a number of outstanding questions related to learning and teaching mathematics that might focus further research from an enactivist perspective. In 2010 Duncan Samson (South Africa) reported at MERGA-33 the application of enactivism as a theoretical framework and research methodology to inquire into the sense students make of the visual clues held within the figural patterns of algebraic generalization tasks (Samson 2010). Then in 2011 Brown and Coles (UK) reported their application of enactivist theory to teacher learning in professional development settings, and they draw links with the notion of co-learning of teachers and researchers/developers in communities of inquiry. In a paper published in *ZDM*, they explain how an enactive approach is taken to "reframe" teacher education at the University of Bristol. Attention is given to the links between perception and action emphasized with enactivist theory and how this is worked out in terms of experience as the basis of working approaches, discussions, and focusing attention in teacher education (Brown and Coles 2011). In 2012 Hannula (Finland) reported in the journal *Research in Mathematics Education* how enactivist theory can be used to explain a dimension of a "metatheoretical foundation for relating different branches of research on mathematics-related affect to each other" (Hannula 2012). In the same year Brown and Coles (2012) published research in the journal *Educational Studies in Mathematics* that takes an enactivist stance to analyze "how we do reflection" (p. 222) in the processes of learning to teach mathematics.

Enactivist theories have been used within mathematics education including theoretical reflections and studies about the nature of mathematics and the rationale for learning mathematics (Davis 1995), issues of learning topics within mathematics (geometry, Drodge and Reid 2000; reasoning, Reid 2002; algebra, Lozano 2008; and algebraic generalization, Samson 2010), teacher knowledge and teacher learning (Proulx 2008b), teacher education (Brown and Coles 2011), mathematics teaching at university level (Nardi et al. 2005), affective issues in teaching and learning mathematics (Brown and Reid 2006; Hannula 2012), design research (Trigueros and Lozano 2007), and as a research methodology (Reid 1996).

## Cross-References


▶ Complexity in Mathematics Education
▶ Constructivism in Mathematics Education
▶ Embodied Cognition
▶ Situated Cognition in Mathematics Education


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## Epistemological Obstacles in Mathematics Education

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The concept of epistemological obstacle emerges in philosophy of science in the works of Bachelard (1938) who is the first to interpret the genesis of scientific knowledge with the support of this concept: "It is in terms of obstacles that one must pose the problem of scientific knowledge [. . .] it is in the very act of knowing that we will show causes of stagnation and even of regression, this is where we will distinguish causes of inertia that we will call epistemological obstacles."

The examples given by Bachelard are typical of the prescientific thinking and connect to what he calls the obstacle of primary experience. In this, the substantialist obstacle consists in referring to a substance equipped with quasi magic properties in order to explain the observed phenomena: as an example, the attraction of dust by an electrically charged surface will be explained by the existence of an electric fluid. Bachelard rightly explains that the obstacle arises from the fact that this is not a metaphor but indeed an explanation of the situation created by what our senses tell us: "We think as we see, we think what we see: dust sticks to the electrically charged surface, so electricity is an adhesive, is a glue. One is then taking a wrong way where false problems will generate worthless experiments, the negative result of which will fail in their role of warning, so blinding is the first image [. . .]."

Brousseau (1976, 1983) is the first to transpose the concept of epistemological obstacle to the didactics of mathematics by highlighting the change in status for the error, that this notion generates: it is not a "result of ignorance [. . .] or chance" but rather an "effect of prior knowledge that was relevant and had its success, but which now proves to be false, or simply inadequate" (Brousseau 1983). Among the obstacles to learning, Brousseau distinguishes indeed the ontogenic obstacles, related to the genetic development of intelligence, the didactical obstacles, that seem to only depend on the choice of a didactic system, and the epistemological obstacles from which there is no escape due to the fact that they play a constitutive role in the construction of knowledge. At one and the same time, the concept of epistemological obstacle extends to the didactics of experimental science (Giordan et al. 1983).

The pioneering works in didactics deal with, among others, obstacles related to extensions to sets of numbers – relative numbers in Glaeser (1981), rational and decimal numbers in Brousseau (1983) – with obstacles related to the absolute value in the research from Duroux (1983), with those that tend to hide the concept of limit, as studied by Cornu (1983) and Sierpinska (1985), with obstacles related to learning the laws of classical mechanics according to Viennot (1979) and with those arising from a sequential reasoning in solving electrical circuits, of which Closset (1983) shows the excessive strength. From these works and others, Artigue (1991) conducts an analysis in which several questions arise, that are subject to debate when trying to characterize the concept of epistemological obstacle: can we talk about epistemological obstacles when there is no identification of errors and but simply of difficulties? Should we look for their appearance and their resistance in the history of mathematics? Look for their unavoidable character in the students' learning process? What does their status of knowledge consist of, having its domain of validity? Can we talk, in certain cases, about a reinforcement of epistemological obstacles due to didactical obstacles?

Other studies also ask the question of the scale at which it is appropriate to look at the epistemological obstacles, as well as that of their cultural character. The works of Schneider (1988) raise these two questions in an articulated manner by showing that the same epistemological position, namely, empirical positivism, can account for multiple difficulties in the learning of calculus: errors when calculating areas and volumes in relation with misleading subdivisions of surfaces into lines and of solid surfaces into surface slices, a "geometric" conception of limits leading students to think of segments as being "limits" of rectangles, and of the tangent line as being "limit" of secants without reference to any topology whatsoever, and their reluctance to accept that the concept of derivative will provide the exact value for an instantaneous velocity. This empirical positivism which, *mutatis mutandis*, converges with the primary experience from Bachelard in the sense of "experience placed before and above criticism" goes well beyond learning calculus (Schneider 2011). This example illustrates indeed, on the one hand, an obstacle considered at a large scale, with its interpretive scope covering errors or multiple difficulties and, on the other hand, its cultural aspect which can be considered as a pure product of Western modernity. It also shows that, despite the opinion of Bachelard, the notion of epistemological obstacle applies to mathematical thinking, at least on a first level.

The debate on the scope and cultural character of epistemological obstacles, of which the examples above illustrate the probable dependence, is animated and most probably not closed. Regarding the first aspect, Artigue insists on the interest in considering what she calls "obstacle-generating processes," including "undue formal regularization" that, as an example, leads students to the misapplication of linearization processes such as "distributing" an exponent on the terms of a sum, or "fixing on a familiar contextualization or modeling," such as the excessive attachment to the additive model of losses and gains when considering relative numbers. About the second aspect, Sierpinska (1989) puts back in a theory of culture some sayings of Bachelard who thinks that, if empirical knowledge of reality is an obstacle to scientific knowledge, it is because the first acts as an unquestioned "preconception" or as an "opinion" based on the authority of the person who professes it. Johsua (1996) continues to believe that some spontaneous reasonings, like those transgressing the laws of classical mechanics, have a cross-cultural character, while Radford (1997) argues that the so-called epistemological obstacle refers more to local and cultural conceptions that one develops on mathematics and science in general. And presumably, we cannot settle this debate without specifying it, example after example, as cautiously proposed by Brousseau 20 years earlier: "The notion of obstacle itself is beginning to diversify: it is not that easy to propose relevant generalizations on this topic, it is better to perform studies on a case by case basis." All this without yielding to the temptation of qualifying as epistemological obstacle whatever is related to recurring errors for which we did not analyze the origins (Schneider 2011).

The identification of epistemological obstacles brings forward the question of their didactical treatment: should we have students to bypass them or, on the contrary, should we let them clear the obstacle and what does that mean? Let us first turn to "educator" Bachelard (as described by Fabre 1995). It is the intellectual distancing that Bachelard emphasizes as major learning issue, when he writes that "an educator will always think of detaching the observer from his object, to defend the student against the mass of affectivity which focuses on certain phenomena being too quickly symbolized [. . .]" (1949). An echo hereof is the psychological shift of perspective ("décentration") of Piaget that, among children, the interpretation of an experience assumes: as such, it "does not obviously make sense" that sugar dissolved in water has disappeared on the account that one cannot see it anymore! One of the primary goals of education would thus be to promote, among students, the detachment from "false empirical objects" born from the illusion that the facts and observations are given things, and not constructed, that is to say to get them to pass from world 1 of physical realities, in the sense of Popper (1973), to world 2 of states of consciousness and to world 3 of concepts that contain "more than what we did put in them." It is presumably those connections that lead Astolfi and Develay (1989) to place Piaget, Bachelard, and Vygotski at the origin of the constructivist movement in didactics of science, the first explaining "how it works,", the second "why it resists," and the third pointing out "how far one can go." Brousseau (1983), as for him, provides clear-cut answers to the questions above: "an epistemological obstacle is constitutive of achieved knowledge in the sense that its rejection must ultimately be mandatorily justified." There resides, according to him, the interest of "adidactical situations" whose fundamental nature with respect to the target knowledge will allow invalidating an old knowledge that proves to be an obstacle to new knowledge, by highlighting the limits of the scope of operation of the former. Martinand (1986) goes further by making obstacles – be these from the works of Bachelard, Piaget, or Wallon – a selection mode for objectives: the concept of "objective-obstacle" appears then in opposition to the usual idea of blocking point. One can think today, together with Sierpinska (1997), that an equivalent coupling may have been too systematic or even normative at a given time in didactics of mathematics, but it is probably advisable that the teacher should manage, at least by a vigorous heuristic discourse, the epistemological obstacles identified on a large scale (Schneider 2011).

The notion of epistemological obstacle has some kinship with that of conception or more precisely that of misconception, but also with that of cognitive or socio-cognitive conflict as illustrated in the acts of an international symposium on knowledge construction (Bednarz and Garnier 1989). The concept of misconception itself may be related to the mental object from Freudenthal (1973) or to the image-concept in Tall and Vinner (1981) who, despite some differences, indicate that the mind of students being taught is not in a virgin state but is a host of intuitions keen to facilitate learning but also to hinder it. In some examples, misconceptions converge with epistemological obstacles in an obvious manner. As such, some of the probabilistic misconceptions identified by Lecoutre and Fischbein (1988) are explained by causal and chronologist conceptions of the notion of conditional probability which, according to Gras and Totohasina (1995), are obstacles of epistemological nature. As for the concepts of cognitive or socio-cognitive conflicts that underpin the Piagetian and Vygotskian theories, they also rely on the assumption that learning is motivated, on the one hand, by an imbalance between the reality and the image that an individual makes up of it and, on the other hand, by confronting his opinion with that of others or with a contradictory social representation. The transfer of the concept of epistemological obstacle to the didactics of mathematics is then bringing a new contribution to the theories mentioned above, in terms of close dependency between the evolution of conceptions among students and the didactical situations they are confronted with: "[. . .] the crossing of an obstacle barrier requires work of the same nature as the setting up of knowledge, that is to say, repeated interactions, dialectics of the student with the object of his knowledge. This remark is fundamental to distinguish what a real problem is; it is a situation that allows this dialectic and that motivates it" (Brousseau 1983). And this is indeed what makes the link between didactics and epistemology to be so tight.

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# Equity and Access in Mathematics Education

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## Keywords

Equity; Social justice; Bourdieu; Social class; SES (Socioeconomic status); Gender inequality; Group placement

## Definition

*Social class background* – social class is best understood through a Marxist orientation as the groupings people fall into as a result of explicit economic forces within society. These groupings are a direct result of similarities with and differences between people, particularly through the resources to which they have access, but also to their tastes and dispositions, which ultimately position them within educational systems.

## Characteristics

Usually an encyclopedia entry will begin with some definitions. With both "equity" and "access," that's not possible. Each of these terms is politically loaded and reflects political and ideological dispositions both in the pedagogical arena of the classroom and in the intellectual arena of the academy. One problem of defining equity is due to it being assumed to be a universal good; surely everyone wants equity? Actually that's far from the case, and at least there will be little agreement on how we define and more importantly operationalize the terms. Equity is not a key driving force for those who sit on the political right. There, meritocracy and individual endeavor are markers of a democratic society, providing a way out of poverty for those who work hard. For those on the political left, the economic superstructure itself, and the education system which serves that system, hides structural inequality and merely perpetuates that structural inequality based on accumulated wealth. For the left, equity itself is a key feature of a democratic society.

One cannot therefore assume a single perspective on equity and access but needs to look for the relationship to political orientation (Gates and Jorgensen 2009). A first, moderate or conservative, stance on equity focuses on individual responsibility. Here there is a recognition of unfairness but a rejection of the social structural underpinnings of that unfairness. A second, more liberal, stance does recognize structural inequalities but locates itself largely within the classroom looking at what classroom practices might alleviate the disparities between pupils. Finally there is a radical stance that recognizes structural inequality but goes further and examines how social inequality is built into existing classroom practices. This stance sees groups of individuals as subject to vastly different sets of experiences and opportunities such that many choices are restricted. But furthermore, these arbitrary barriers become internalized through school and subject cultures. Consequently pupils develop identities which accept their location in the hierarchy.

Mathematics therefore plays a significant, if often hidden part in the politics of education as the sociologist Pierre Bourdieu claims:

> Often with a psychological brutality that nothing can *attenuate*, the school institution lays down its final judgements and its verdicts, from which there is no appeal, ranking all students in a unique hierarchy of all forms of *excellence*, nowadays dominated by a single discipline, mathematics. (Bourdieu 1998, p. 28)

Indeed if equity was not an important issue, this encyclopedia entry would not have been written. The philosopher of mathematics education Paul Ernest takes this a step further by suggesting mathematics as a social filter:

> Mathematics has been remarked upon as playing a special role in *sorting out* students and preparing them for and directing them to different social stations. . . .. Thus, the teaching and learning of mathematics seems to occupy a special place in the provision of social justice—or its obstruction—within the education system. (Ernest 2007, p. 3)

Here is the argument that if mathematics serves a purpose of filtering and directing people into diverse levels in society, equity – how it does this – ought to be a key concern for those charged with teaching mathematics, the schools. The first question then is can schools help foster equity or can they only perpetuate existing inequality. This is a central consideration and one which differentiates academics.

In order to understand the place of equity in mathematics education, one has to grasp the divergence between individual accounts and collective accounts; meritocracy and individual endeavor contrasted with social influences and restricted opportunities.

Of course it is not a coincidence, as evidence from around the world indicates, that achievement and engagement in mathematics vary according to the social class background of the learners. One argument would suggest that social class is the largest influence in pupil underachievement, whereas others would argue schools can make a difference. Evidence for these claims can be found in every school around the world. Whereas it is well known that individual pupils can succeed against the odds, the reality of many mathematics classrooms is reflected in the following comment from a teacher:

> You know, a lot of my bottom group really struggle with maths – and I've noticed they all come from the same part of town, and they have got similar family backgrounds. Surely that can't be a coincidence? (Cited in Gates 2006, p. 367)

There is now widespread focus in the academic literature on the systematic traditional failure to educate students from disenfranchised groups (Secada 1989), and attempts to understand the "systematic" nature of the patterns of achievement have looked at the schools themselves as playing a fundamental role in the furtherance of structured inequality.

> The vast majority of schooling for children . . . of poor and working class, girls and boys of colour and so many others is not neutral, not its means and certainly not its outcomes . . . but who controls the economic, social and educational conditions that make it so? Whose vision of schooling, whose vision of what counts as real knowledge organises the lives in classrooms? (Apple 1995, p. 330)

Historically, a focus on equity in mathematics education developed out of concerns over the achievement of girls (Burton 1990). While early thinking looked at biological differences, this approach soon became discredited, with a recognition that "girls and boys make choices throughout their education and professional careers, and there are *systematic differences* in these choices" (Herman et al. 2010, p. 3). The previous relative underachievement of girls in mathematics is structurally similar to achievement differences resulting from other social characteristics. For example, both ethnicity and social class have a substantial research literature testifying to the unrepresentative levels of underachievement of young people from disadvantaged and working class backgrounds and from ethnic minority groups, including young people from black, Caribbean, indigenous, and Latino communities.

One of the arguments for a systemic underachievement by certain groups of young people in mathematics is that they do not share the advantages of dominant, more affluent groups. Their culture and histories can be different, their languages and relationships are different, and their economic conditions force a rather different set of priorities to those experienced by more comfortable middle-class communities (see Zevenbergen 2000). As a result, choices are forced on families because they do not have credible alternatives and as a result "*the social world of school operates by different rules or norms than the social world these children live in*" (Pellino 2007). The literature on the effects of poverty draws our attention to some of the characteristics of children in poverty. They experience high mobility, hunger, repeated failure, low expectations, undeveloped language, clinical depression, poor health, emotional insecurity, low self-esteem, poor relationships, difficult home environment, and a focus on survival. A strand of research, often termed critical mathematics education, has examined the conditions of such pupils whose backgrounds are obscured and ignored by both schools and the academic research community. For example, the hungry, the homeless, and those children in care all have particular needs – yet because they do not fit the ideal are placed outside the norms (Skovsmose 2011).

To claim there are systematic differences in the choices individuals can make is fairly controversial on two counts. First it assumes that we are free to make choices. Second, there is the assumption that schools, through the energizing of these choices, can make a difference to outcomes. The first of these assumptions overlooks the structural accumulated history that young people carry with them: expectations, identity, self-efficacy, language fluency, etc., all of which place learners at different starting points. One strand in the literature here assumes that if choices are influenced and limited by misinformation and low expectations, then it is entirely possible for schools to overcome these barriers by providing an environment that redresses those limitations – the second assumption.

Between 1980 and 2010, research in mathematics education has seen a noticeable shift in what some have seen as a sociocultural turn in research agendas (Lerman 2000), placing an emphasis on an understanding through the exploration of sociocultural factors – recognizing the importance of the social context upon one's action and choices. But this has also recognized that we need to look and think beyond the individual level of cognition to see how different responses to mathematics might be explained. How do we explain, for example, that earlier comment by a teacher, that achievement at mathematics is very highly correlated to the pupils' home background? Do we believe it is because some people are not as intelligent as others? Or do we believe some children are held back in order for some others to progress? Where one stands on that will largely influence how you personally think about equity.

One way in which children can be held back is through restriction of the curriculum and a further strand in equity and access to mathematics education is the access afforded by the school curriculum to mathematics itself – and to the powerful ideas it allows us to use. In mathematics education in some – but not all – countries, access to the curriculum is organized around structured grouping usually claimed to be on some measure of ability. In some countries (UK, USA, Australia, etc.), it is an almost universal practice, and teachers seem to be unable to conceive of how it might be otherwise given a claimed hierarchical nature of mathematics. However, in other countries (Denmark, Finland, etc.) the practice of ability discrimination is outlawed.

In the literature, group placement is a highly controversial and contested practice, and much research has indicated the effect it has upon young people who do not fit an ideal model of successful learner – usually pupils from working class homes and some ethnic minorities. Such pupils are systematically more likely to be placed in lower groups than others even when performance is taken into account (Zevenbergen 2003). Various studies have shown "*that placement in ability groups increases the gap between students at different group levels*" (Cahan et al. 1996, p. 37). In other words, the very placement of pupils in a group influences their outcomes.

A lack of equitable practices leads to restricted access by schools and teachers through the provision of a restricted curriculum to lower achieving pupils. The pedagogical jump here made by teachers is to assume that pupils who are doing less well are not (cap)able of *higher-order thinking*. In a series of studies, this has been explored (Zohar 1999; Zohar et al. 2001; Zohar and Dori 2003) with the conclusion that teachers do not really believe weak pupils (invariably pupils from poor backgrounds) can think in higher-order ways.

Studies of pupils' mathematical experiences that take account of social backgrounds (Lubienski 2000a, b, 2007) have found very specific differences in two main areas – *whole class discussion* and *open-ended problem solving* – and these can throw some light onto the way in which equitable practices are compromised and access to big ideas is restricted. These are two well-researched pedagogical strategies and classroom practices which at least in professional discourse are held in some esteem. Discussion-based activities were perceived differently by pupils from different social backgrounds. Pupils from high socioeconomic status (SES) backgrounds thought discussion activities were for them to analyze different ideas while those pupils from lower social groups thought it was about getting right answers. The two groups had different levels of confidence in their own type of contributions with the low SES pupils wanting more teacher direction. Higher SES pupils felt they could sort things out for themselves – as their parents do in life presumably.

The second area was that of *open-ended problem solving* – a mainstay of recent reform agendas in mathematics. The high level of ambiguity in such problems caused frustration in low SES pupils which in turn caused them to give up. High-SES pupils just thought harder and engaged more deeply. It is well known that middle-class pupils come to school armed with a set of dispositions and forms of language which gives them an advantage because these dispositions and language use are exactly the behaviors that schools and teachers are expecting and prioritize (Zevenbergen 2000). High-SES pupils have a level of self-confidence very common in middle-class discourses, while working class discourses tend to be located in more subservient dependency modes, accepting conformity and obedience (Jorgensen et al. 2013).

Equity and access then are both key issues in the provision of mathematics education but are both controversial and deeply political.

## Cross-References

- ▶ Cultural Diversity in Mathematics Education
- ▶ Gender in Mathematics Education
- ▶ Immigrant Students in Mathematics Education
- ▶ Inclusive Mathematics Classrooms
- ▶ Indigenous Students in Mathematics Education
- ▶ Language Background in Mathematics Education
- ▶ Political Perspectives in Mathematics Education
- ▶ Socioeconomic Class in Mathematics Education

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# Ethnomathematics


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## Introduction

In this entry, I will refer mainly to my views and my participation in the emergence of this field as a research area and the benefits acquired in understanding and interpreting the cultural, political, material, and even economic forces recognized in building up these strategies. A basic reference is my 1985 basilar paper in *For the Learning of Mathematics*, which has been republished since then in various handbooks (D'Ambrosio 1985).

I will discuss mainly the theoretical basis of Ethnomathematics and its values as part of a culture. I present Ethnomathematics as a research program in the History and Philosophy of Mathematics with societal and pedagogical implications (D'Ambrosio 1992). The program depends on theories that explain human knowledge and behavior.

In considering Ethnomathematics a research program, it is recognized as a broader focus than simply the recognition of mathematical ideas and practices of different cultural groups. Of course, the Ethnomathematics of different cultural groups is the main source for this research program. But the major objective of the Program Ethnomathematics is to propose a broader vision of knowledge and of human behavior, by making sense of how different communities, societies, and civilizations faced their struggle for survival and transcendence in their environmental, cultural, economic, and social contexts.

The concern with other cultures and with other forms of knowledge has been always present in the History of Ideas and goes back in history to all civilizations. Others may have a different approach and base their reflection on other scenarios, thus showing another picture of the field. Throughout this entry, there are traces of many different approaches to the theme, but there are few explicit references to them.

A basilar question is the reason to look into non-Western cultures and civilizations for a research into the History and Philosophy of Science and Mathematics, which are Western constructs. I paraphrase Brian Fay in the introductory essay in the issue of *History and Theory* devoted to *Unconventional History*, and claim that learning about other cultures and civilizations is, at the very same time, learning about our civilization, its strengths, and limitations (Fay 2002).

## Definition

The word Ethnomathematics may be misleading. It is easily confused with ethnic-mathematics. Although ethnic groups are contemplated, I consider Ethnomathematics a much broader concept, focusing on cultural and environmental identities. The name also suggests Mathematics. Again, I use it in a much broader concept than Mathematics, which is a late Western concept. Indeed, in the sense we use the word "mathematics" today, it goes back to about the fifteenth century. Former uses of the word mathematics have a different meaning. Today, historians opt for using the word "mathematics" also when they refer to some practices and theories of the Antiquity and the Middle Ages, which bare some common objectives, concepts, and techniques with Mathematics. This option is convenient for historical narratives. But it is misleading. A similar, also misleading convenience is adopted by ethnographers and cultural anthropologists, when describing and analyzing other cultures.

There is a very natural question: "Why to use the word Ethnomathematics for my research on the strategies developed by different communities, societies, and civilizations to face the struggle for survival and transcendence in their environmental, cultural, economic, social contexts?" I will try to explain my choice, which is indeed an etymological construction. The word Ethnomathematics is obviously, not new, and it has been used mainly with an ethnographical focus for decades.

The main concern that guides my research is to identify the ways, modes, styles, arts, and techniques, generated and organized by different cultural groups for learning, explaining, understanding, doing, and coping with their natural, social, cultural, and imaginary environment. This is a long explanation, and I tried to synthesize it with the resource of an etymological exercise. I looked for words with meanings that convey this long explanation and I found Greek roots that can do it. The root *techne* means, roughly, the arts and techniques, the ways and modes, the styles; *mathema* is a difficult root, which generally means learning, explaining, understanding, doing, and coping with some reality; and *ethno* means a natural, social, cultural, and imaginary environment. Thus, I may synthesize the long phrase "ways, modes, styles, arts and techniques to learn, explain, understand, doing and coping with distinct natural, social, cultural, imaginary environment" as the *technes* of *mathema* in distinct *ethnos*. Thus, using *tics* as a simplified spelling for *techne*, the long phrase became *tics* of *mathema* in distinct *ethnos*, or making an obvious rearrangement, *ethno* ± *mathema* ± *tics* or *ethnomathematics*. Thus, I started to use the word *Ethnomathematics* as a result of this etymological exercise (D'Ambrosio 1998, 2006).

It is noticeable that Mathematics diverted from the concept of the *mathema*. In the words of Oswald Spengler "The present-day sign-language of mathematics perverts its real content" (Spengler 1962). Ethnomathematics is particularly concerned with real contents. For educational purpose, the restoration of this concept is the major support of my proposal for a modern *trivium* in education: literacy, matheracy, and technoracy (D'Ambrosio 1999).

It should not be surprising at all that Mathematics, as we know it, is a special Ethnomathematics, the same as are the theories and practices of Pharmacology, of Cardio-Surgery, of Dance, of Algebra, and, indeed, any form of knowledge. All these disciplines are the concern of specific professional groups [*ethno*] to develop ways, modes, styles, arts, and techniques [*tics*] for learning, explaining, understanding, doing, and coping with [*mathema*] with specific and related facts, phenomena, and problems. They rely on their natural, social, cultural, and imaginary environments.

It is not surprising that the word Ethnomathematics suggests Mathematics. After all, Mathematics is the dorsal spine of Modern Civilization. Indeed, throughout history, Mathematics has been well integrated into the technological, industrial, military, economic, and political systems and Mathematics has been relying on these systems for the material bases of its continuing progress. The same for Science and Technology and Philosophy as well. Hence for models of society.

The issues are essentially political. There has been reluctance among mathematicians, to a certain extent among scientists in general, to recognize the symbiotic development of mathematical ideas and models of society. Mathematics has grown parallel to the elaboration of what we call Modern Civilization. Historians amply recognize this.

Modern World Civilization sprang out of Europe as the result of 500 years of conquest and colonization. Modern Civilization is a body supported by a dorsal spine, recognized by philosophers, historians, scientists, and just about everyone, as Mathematics.

Mathematics as the dorsal spine of Modern Civilization, is beautiful, rigorous, and perfect, so respected by everyone, even feared, particularly by children and students. But, paradoxically, Modern Civilization, is ugly, plagued with inequity, arrogance, and bigotry.

What went wrong with Modern Civilization? How is it possible that a perfect dorsal spine supports such an ugly body?

To understand this paradoxical discord has been a guiding quest in my research and in proposing the Program Ethnomathematics.

## Knowledge, Behavior, and Culture

How did everything begin? The myths of creation are present in every civilization. The founding myths and traditions of Western civilization leads to the history of monotheistic religions (Judaism, Christianity, Islamism) and the emergence of techniques and the arts and links to understanding how Mathematics permeates all this. A great insight is gained in trying to identify and to understand what happened in the founding myths and traditions of non-Western civilizations.

The main difficulty I encounter, and this is true for every one doing cultural studies, is the difficulty of understanding and interpreting other cultures with the categories and analytic instruments other than those that are part of my cultural heritage. I have been trying to avoid, at least to minimize, this difficulty. We rely on informants, and there is a difficulty in building up trust.

The goal is to develop a generic comprehensive theory of knowledge and behavior. I base my research on universal forms of knowledge (communications, languages, religions, arts, techniques, explanations, or sciences) and in a theoretical/methodological model of knowledge and behavior that I call the "cycle of knowledge."

The aim of research in the Program Ethnomathematics is the recognition of practices and its relation to theories. Thus, I focus history of science (and, of course, of mathematics) trying to understand the role of technology as a consequence of science, but also as an essential element for furthering scientific ideas and theories. I guide my investigation on three basic questions:

1. How do ad hoc practices and solution of problems develop into methods?
2. How do methods develop into theories?
3. How do theories develop into scientific invention?

## Current Work in Ethnomathematics

The Program Ethnomathematics was initially inspired by recognizing ideas and ways of doing that reminds us of Western mathematics. What we call mathematics in the academia is a Western construct. Although dealing with space, time, classifying, and comparing, which are proper to the human species, the codes and techniques to express and communicate the reflections on these behaviors are undeniably contextual. Thus came my approach to Cultural Anthropology (curiously, my first book on Ethnomathematics was placed by the publishers in a collection of Anthropology).

Much work is going on in many countries. Many national, regional, and international meetings are held. An overall account of the progress of the field is seen in the site of the *International Study group on Ethnomathematics/ISGEm*, with links to the most relevant works in the area. Access the links at http://isgem.rpi.edu/pl/ethnomathematics-web.

Although a new field, there are important publications revealing the strength of the area of Ethnomathematics. It would be difficult to produce a bibliography. There are innumerous pioneers and active researchers in this field. In attempting to give a full bibliography I would surely leave important references. I mention three basic works:

- *Native American Mathematics*, Michael Closs editor, University of Texas Press, Austin, 1986.
- *Ethnomathematics. Challenging Eurocentrism in Mathematics Education*, Arthur B. Powell and Marilyn Frankenstein, editors, State University of New York Press, Albany, 1997.
- *Mathematics Across Cultures*. *The History of Non-Western Mathematics*, Helaine Selin editor, Kluwer Academic Publishing, Dordrecht, 2000.

Besides many references, after having put together the bibliographies of each chapter, we have a comprehensive relevant bibliography for the area.

The *International Conferences on Ethnomathematics/ICEm* are well attended events. The *Fourth ICEm* took place in Towson, Maryland, in 2011. Most of the papers presented in this conference are published in the *Journal of Mathematics and Culture* volume 6 Number 1 Focus Issue ICEM4, a free access publication linked to the site of the *ISGEm* indicated above.

It is appropriate to say that the Program Ethnomathematics is a promising emerging research field.

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## External Assessment in Mathematics Education

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### Keywords

Assessment; Assessment design; Assessment for learning; Assessment in education; Assessment of complex systems; Complex thinking; Design-based assessment; Evaluation; Higher-order thinking

### Characteristics

Much of the discussion about measurement in education in the past half century has revolved around the need to move beyond the application of psychometrics to a broader model of educational assessment that supports learning (Flanagan 1951; Ebel 1962; Glaser 1963). Historically, significant attention has been given to the differences between norm-referenced and criterion-referenced tests (Glaser 1963; Hambleton 1994), focusing on relative versus absolute standards of quality that are more or less appropriate to measure abilities or achievements. However, briefly, we will describe why neither of these approaches to assessment allows us to assess higher-order understandings in mathematics that the field is mostly interested in studying nor do they consider latest advancements in what we now know about how students learn mathematics, as they interact with teachers, schools, and curricular innovations. Furthermore, we propose a new challenge and purpose for assessment: *How can assessments of complex mathematical achievements be achieved in a way that provides useful information for relevant decision makers?* After presenting an overview of the dichotomy between norm-referenced and criterion-referenced approaches to assessment, we describe the characteristics of assessment designs that are needed to assess the complexity in the continually adaptive development of high-order mathematical thinking that mostly interests the field of mathematics education.

Norm-referenced tests grow from the psychometric tradition, based on the measurement of general intelligence (g) as an inheritable characteristic of an individual that is fixed over time. This psychometric tradition has its roots in the mid-1800s with the work of Galton and Pearson and Spearman's contributions in the beginning of the 1900s (Gipps 1999; Gardner et al. 1996). Usually associated with the measurement of aptitude (as opposed to achievement), norm-referenced tests are constructed with the purpose of comparing respondents on attributes which presumably (although seldom in reality) do not depend on instruction. Thus, each item is assumed to have a difficulty level relative to other items; again, this level of difficulty is assumed to be independent of individual's experiences. So, items are selected "to discriminate among those tested in order to spread scores along the normal distribution" (Gipps 2012, p. 70), and items that have a low discrimination index are discarded from the test (e.g., items in which most students score correctly and items in which most students score incorrectly). However, items were selected to be those that are not influenced by learning experiences are not likely to provide important information about what students learned or didn't learn. Consequently, their elimination from the test leads to one of the most noted limitations of norm-referenced tests, which is their insensitivity to instruction (Popham 1987; Carmona et al. 2011).

Rather than focusing on relative measures, leading psychometricians have argued that criterion referenced should be used which are dependent upon an "absolute standard of quality"

(Glaser 1963, p. 519) in relation to specific objectives (Popham 1987). Thus, criterion-referenced tests are considered to be more appropriate to measure achievement and determine current levels of student performance. These tests assume a continuum of knowledge acquisition from no proficiency to perfect performance, and the reference criteria are expected to include a representative sample of important achievements in relevant domains, regardless of their discrimination index. So, scores are determined by calculating the proportion of these tasks to determine mastery or nonmastery for an individual.

Supported in behaviorism (e.g., Skinner 1968), and as a rational approach to evaluation through determining individual's learning gains after instruction, Glaser (1963) associates criterion-referenced tests with measuring student attainment of explicit criteria as indicators of behavioral objectives (Popham 1987; Gardner et al. 1996). This learning perspective views the mind as inaccessible and, therefore, studies learning as the way behaviors, which are observable, are acquired. All behaviors are considered to be a result of chained reactions to events in the environments called stimuli, and mental activity is defined in terms of observable and measurable stimuli-response patterns. Learning of complex ideas is formulated as a partitioning into smaller behaviors, or pieces, that are organized along a one-dimensional continuum of increasing level of difficulty, assuming mastery of a lower-level behavior as a prerequisite to achieve higher-level understanding. Behavioral objectives are generally stated in the form of statements as follows: *Given situation **S**, the student will be able to do **D**, to level of proficiency **P***. However, in recently developed curriculum standard documents, it is clear that in fields such as mathematics education, many of the most important goals of instruction cannot be reduced to lists of declarative statements (i.e., facts) or condition-action rules (i.e., skills). To address these shortcomings, Lesh and Clarke (2000) present another type of instructional goal defined as *cognitive objectives*, which are found more relevant in mathematics and science education than their counterparts, because cognitive objectives focus on students' *interpretations* of situations, rather than on their *actions* in these situations. Examples of relevant cognitive objectives in mathematics and science education include models, metaphors, and complex conceptual systems, to mention a few. In order to operationally define what it means to "understand" such cognitive objectives, it is important to include (a) **situations** that optimize the probability that the targeted construct will be elicited in an observable form, (b) **observation tools** that allow observers to identify the construct from other irrelevant information that might also be elicited, and (c) **quality assessment criteria** that allow for meaningful comparisons to be made among alternative possible solutions.

Lesh, Lamon, Lester, and Behr (1992) argue the need for an entire paradigm shift to rethink assessment issues in mathematics education. Rather than focusing on behavioral (or other types of) objectives, they identify conceptual objectives as those we are mostly interested in assessing and which cannot be examined neither from a norm reference nor criterion reference perspectives. Lesh and Lamon (1992) highlight the need to provide well-articulated operational definitions that focus less on value judgments about students (good/bad) and instead focus on providing useful documentation for the decision makers to be able to make a better-informed decision based on specific purposes (Carmona 2012).

This paradigm shift evidences significant changes on assessment-related topics such as data collection, data interpretation, data analysis, and the nature of reports. It involves "new decision makers, new decision-making issues, new sources of assessment information and new understandings about the nature of mathematics, mathematics instruction, and mathematics learning and problem solving" (p. 380). In addition, this new perspective requires a revision on what it means for assessments to be valid, reliable, and generalizable (Pellegrino, Chudowsky, and Glaser 2001), focusing assessment on an increased authenticity of tests and an increase on the credibility and fairness of the inferences made based on test results (Messick 1994). Consistent with these views, Chudowsky and Pellegrino (2003) emphasize the need to generate new situations in a way in which assessments are designed to support and measure learning and elicit student thinking in its complexity (Lesh et al. 2000). The following section provides an overview outlining the main components of this new perspective into assessment design we call *design-based assessment*.

### Design-Based Assessment

During the past 30 years, mathematics educators have pioneered a new class of research methodologies, which have become known as *design research studies*. These design research studies have been proposed to encourage the relevance of research to practice (Brown 1992) and to highlight the importance of incorporating practitioners' wisdom to theory development (Collins 1992; Collins et al. 2004). But, most of all, in mathematics education, where most researchers are also practitioners (e.g., teachers, teacher educators, curriculum developers), the main reasons why design research methodologies have been useful are because (a) like engineers, mathematics education researchers tend to be trying to design and develop the same "subjects" that they are trying to understand and explain and (b) like engineers, the kinds of complex and continually adapting subjects that mathematics educators are trying to understand usually cannot be explained by drawing on only a single theory. Instead, it should be expected that useful conceptual frameworks (or models) will need to integrate ideas and procedures drawn from a variety of relevant theories (and disciplines). One reason why single-theory ways of thinking seldom work is that solutions to realistically complex problems usually involve competing and partly conflicting factors and trade-offs – such as those involving high quality and low costs.

When *design research methodologies* emphasize the measurement of complex and continually adapting subjects, they can be called *design-based assessment methodologies*. And, assessing curriculum innovations can be thought of as being similar to the methodologies that are needed to assess complex artifacts such as space shuttles or transportation systems. Some relevant assumptions include the following.

- For the kinds of complex and continually adapting systems and situations that need to be understood and explained, it generally must be assumed that no two situations are ever exactly alike – and that the exact same thing never happens twice. Furthermore, for most such systems, many of their most important attributes can only be "observed" by documenting their effects on other things, and (like neutrinos or other subatomic particles in physics) to measure them often involves changing them.
- In general, complex systems and complex achievements cannot be understood by breaking them into tiny pieces – and additively combining measurements of the pieces. For example, even if it is true that developing some higher-order *conceptual understanding* (C) implies that a list of lower-order *behavioral objectives* ($B_1, B_2, B_3, \ldots B_n$) should have been mastered, it does not follow that mastering each, $B_1, B_2, B_3, \ldots B_n$, implies that C has been achieved. Yet, this *fragmentation fallacy* is an assumption underlying psychometric conceptions of knowledge development. One of the many things that mathematics educators can learn from engineers and other design scientists is that as the complexity of designed constructs (such as space shuttles) increases, a far greater percentage of assessment activities need to focus on relationships and connections among parts and relatively less time focuses on assessments of isolated pieces.
- Why is it impossible to assess most conceptual understandings using tests that are based on psychometric theory? As stated above, psychometric theory was developed originally to measure aptitude (i.e., general intelligence – where performance is not influenced by teaching and learning). Whereas, tests that are designed to measure the results of learning and instruction are called achievement tests. In particular, in intelligence testing, items are discarded as being "unreliable" if student performance increases in the course of

responding to them. That is, to be reliable, a students' performance should not change for a sequence of tasks which are all designed to test the same attribute.

*Design-based assessment* focuses on three interacting and continually adapting "subjects" of assessment studies – students, teachers, and curriculum innovations (i.e., programs). Space limitations preclude considering other important "subjects" – such as administrators, home environments, or classroom learning environments – even though it is well known that these latter factors often strongly influence the ways that students and teachers interact and adapt in response to curriculum innovations. For example, the impact of a curriculum innovation may vary significantly if the classroom norms that govern student-teacher and student-student discussions emphasize the practice of requiring students to accept procedures and claims based on appeals to authority – rather than requiring them to justify and explain things based on students' mathematical sense making. By focusing on students, teachers, and programs, we hope that readers will find it easy to generalize to other relevant subjects.

Notice that, in our descriptions of assessment practices, we also emphasize the importance of documenting and assessing two-way interactions among "subjects" – rather than restricting attention to one-way/cause-and-effect relationships. For example, teachers don't just influence students' thinking about the meanings of the mathematical concepts and processes that they are expected to develop, but, students also influence teachers' thinking about what it means to "understand" these concepts and processes. So, even in situations where a single teacher teaches two groups of students with comparable abilities, the personae that an excellent teacher adopts for one group of students may need to be significantly different than for another group of students. This is because groups as a whole often develop significantly different group personalities.

Next, notice that our descriptions of assessment practices also emphasize developmental perspectives about "subjects" who are assumed to be complex and dynamically adaptive systems – not at all like widgets being created using machine-like processes. Consequently, regardless whether attention focuses on the continually adapting conceptual systems that are developed by students or teachers or whether attention focuses on the systems of learning experiences that are intended to promote student and teacher development, we recognize that when these systems are acted on, they react. Furthermore, based on results from research involving very simple aptitude-treatment interaction studies, we know that, when such feedback loops occur, second- and third-order effects are often far more significant that first-order effects. So, for realistically large and complex curriculum innovations, entry-level teachers' first-year implementations generally should be expected to be significantly different than second-, third-, or fourth-year implementations (when increasingly more experienced teachers are likely to be available).

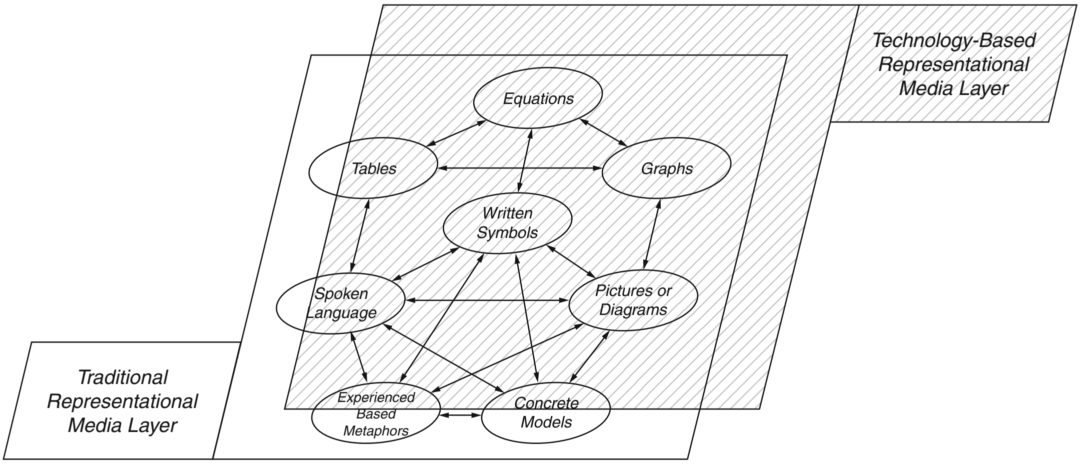
Finally, notice that our attention focuses on *assessment* rather than simply *evaluation*. Whereas evaluation only involves assigning a value to various subjects, assessment involves generating useful descriptions of where various "subjects" are, and where they need to develop in some landscape of possibilities. In general, both assessment and evaluation are intended to provide useful information for decision makers – who may range from students, to teachers, to administrators. So, to assess the quality of a given assessment or evaluation, it is important to consider the following questions: *Who are the intended decision makers?* (because the information that is useful to a teacher may be quite different than the information that is useful to an administrator or politician). *What decisions are priorities for these decision makers to make? What kind of information is most useful for these decision-making purposes?*

For example, low-stakes-but-rapid-turn-around assessments that are intended to help teachers provide individualized attention to students tend to be quite different than high-stakes-and-slow-turn-around assessments that are intended to screen students or limit future opportunities. Sometimes, the former types of assessments are referred to as summative assessments, and the latter are referred to as formative assessments. But, these summative and formative functions often get muddled when (a) summative assessments are used explicitly to change the nature of what is taught and how it is taught and (b) modern statistical procedures often make it possible to use patterns or trends to generate highly reliable summaries of achievement based on collections of documentation.

When analyses of assessment practices begin by asking *who the decision makers are and what decisions are priorities for them to make*, then it tends to become clear that in modern technology-based societies, most decision makers tend to have ready access to computer-based tools which are capable of easily generating interactive graphics-based displays of information that are both simple to understand and easy to customize to fit the purposes and prejudices of individual decision makers and decision-making issues. For nearly any of the "subjects" that are important in educational decision making, single number characterizations are virtually useless and essentially remove decision makers from the decision-making process – by proclaiming, for example, that subject #1 is better than subject #2 regardless of what decisions are being made or what factors are important to consider. Answers may be different for different decision makers.

In educational research and assessment, there is no such thing as a tool or methodology that is "most scientific" (for all subjects, for all decision makers, and for all decision-making issues). Every assessment tool is based on assumptions which may or may not be appropriate for the subjects or purposes of a given study. And, a "scientific methodology" or a "scientific tool" is one whose assumptions are, insofar as possible, consistent with those associated with the subjects, decision makers, and decision-making purposes of the study. For example, when assessing the achievements of students, teachers, or curriculum innovations, the following kinds of questions are important to ask:

- *Do the tools or methodologies emphasize achievements that are well-aligned with the goals of the project, teacher, or students?* For example, even the most recently developed curriculum standards documents, such as the USA's *Common Core State Standards*, none of the higher-order achievements are operationally defined in ways that are measurable. Furthermore, when tests such as the Educational Testing Services' Scholastic Achievement Test were originally designed to be *Scholastic Aptitude Tests*, then the entire *psychometric theory*, which was created to provide development standards, can be expected to emphasize student attributes intended to be unchangeable due to instruction. *Can tests which are explicitly being used to change what is taught and how it is taught be thought of as not being among the most powerful parts of the educational "treatments" being assessed?*
- *Do methodologies which claim to randomly assign students to "treatment groups" and "control groups" really succeed in creating situations which factor out the influences of all but a small number of variables?* (Notice that similar methods have failed even in the case of very small and simple aptitude-treatment interaction studies.) *Can the most important factors really be thought of as being "controlled" when the parallel development of students, teachers, and program implementations interact in ways that usually lead to second-order effects which are as powerful as first-order effects – and when influences due to factors such as administrators, classroom learning environments, and students' home environments tend to be ignored?*
- *Are mixed-methods methodologies adequate to assess students' and teachers' knowledge or content of curriculum innovations?* Quantitative research produces quantitative statements or quantitative answers to questions, whereas qualitative research produces qualitative statements or qualitative answers to questions. But, design-based assessment research is about knowledge development, and very little of what we are studying consists of declarative statements (i.e., facts) or answered questions (i.e., rules). For example, some of the most important kinds of




**External Assessment in Mathematics Education, Fig. 1** A merged Kaput-Lesh diagram for thinking about representational fluency

knowledge that we develop consist of models for describing, explaining, designing, or developing complex systems. So, models (often embedded in purposeful artifacts or tools) are among the most important kinds of knowledge that we need to develop and assess. Consequently, the question we must ask is as follows: *How do we validate models?* And, the answer is that both qualitative and quantitative methods are useful for validating models. But the product isn't simply a quantitative or qualitative claim. It's a validated model – and trends and patterns involving development.

- *Is the unbiased objectivity of an assessment really assured by using "outside" specialists whose only familiarity with the relevant subjects come from pre-fabricated off-the-shelf tests, questionnaires, interviews, and observation protocols which are not modified to emphasize the distinctive characteristics of the subjects and their interactions?* And, if these "outside measures" are used for purposes of accountability, *can they really avoid having powerful influences on the treatments themselves?*
- *Can comparability of treatments really be guaranteed by taking strong steps aimed at trying to ensure that all teachers and all students do exactly the same things, in exactly the same ways, and at exactly the same times?* Notice that, in the literature on the diffusion of innovations, complex systems tend to evolve best when measurable goals are clear to all relevant subjects – and when strong steps are taken to encourage diversity (of interactions), selection (of successful interactions), communication (about successful interactions), and accumulation (of successful interactions).

In mathematics education, many of the most important and powerful types of conceptual understandings occur in one of two closely related forms. The first focuses on students' abilities to mathematize (e.g., quantify, dimensionalize, coordinate) situations which do not occur in pre-mathematized forms and the second focuses on representational fluency – or abilities that are needed to translate from one type of description to another. For example, in the case of representational fluency, Kaput's (1989) research on early algebra and calculus concepts emphasized the importance of translations within and among the three types of representations which are designated in the three ovals shown at the top of Fig. 1 (i.e., equations, tables, and graphs), and in a series of research studies known collectively as *The Rational Number Project*, Lesh, Post, and Behr (1987) emphasized the importance of translations within and among the five types of representations which are designated in the five ovals shown at the bottom of Fig. 1 (i.e., written symbols, spoken language, pictures or diagrams, concrete models, and experience-based metaphors).

From the perspective of psychometric theory, two of the main difficulties with test items that involve representational fluency result from the fact that when tasks involve description of situations (a) there always exist a variety of different levels and types of descriptions and (b) responding to one such task often leads to improvements of similar tasks. So, according to psychometric theory, where tasks are considered to have a single-level of difficulty which is unaffected by instruction and where the relative difficulty of two tasks also is considered to be unaffected by instruction, such tasks are discarded as being unreliable. Similarly, when tasks focus on students' abilities to conceptualize situations mathematically, there once again exist a variety of different levels and types of mathematical descriptions, explanations, or interpretations that can be given. So, once again, the same two difficulties occur as for representational fluency.

Especially when tests are used for accountability purposes and teachers are pressured to teach to these tests, it is important for such tests to include tasks that involve actual work samples of desired outcomes of learning – instead of restricting attention to indirect indicators of desired achievements. For example, if the development of a given concept implies that a student should be able to do skill-level tasks $T_1, T_2, \ldots T_n$, then tasks $T_1, T_2, \ldots T_n$ tend to be *indicators* similar to wrist watches or thermometers – in the sense that it is possible to change the readings on wrist watches or thermometers without in any way influencing the time or the weather. But, *how can assessments of complex achievements be achieved inexpensively, during brief periods of time, and in a timely fashion that provides useful information for relevant decision makers?* In modern businesses where continuous adaption is necessary, and especially in knowledge industries or in academic institutions, decision makers seldom use multiple-choice tests or questionnaires to assess the quality of the kinds of complex work that constitute the most important activities of their employees. So, how do specialists (or teams of specialists) get recognized and rewarded for the quality of their work? For example, how do professors validate their work? Or, how do doctoral students validate the work on their Ph.D. dissertations? Answers to these questions should provide guidelines for the assessment of development related to students, teachers, curriculum innovations, and other "subjects" in mathematics education research. Space limitations do not allow detailed answers to such questions to be given here. But, when attention focuses on the systems of knowledge being developed by students, teachers, and curriculum innovations, (a) it's important to focus on the half-dozen-to-a-dozen "big ideas" which the subjects are intended to develop, (b) it's often useful to recognize that a large part of what it means to "understand" these big ideas tends to involve the development of models (or interpretation systems) for making sense of relevant experiences, (c) these models often are embodied and function within purposeful tools and artifacts, and (d) these tools and artifacts often can be assessed in ways that simultaneously allow the underlying models to be assessed. Procedures for achieving these goals have been described in a variety of recent publications about design research (e.g., Lesh and Kelly 2000; Lesh et al. 2007; Kelly et al. 2008), and it is straightforward to adapt most of these procedures to apply to assessment purposes.

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## Fieldwork/Practicum in Mathematics Education

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### Keywords

Practicum; Field experience; School experience; Teaching practice; Block school experience; Initial teacher education; Preservice teacher education

### Definition

The practicum, teaching practice, or field experience refer to that component of those preservice (or initial) teacher education programs which place student teachers in schools for a stipulated period of time, for the purposes of classroom observation and/or the teaching of lessons, usually under supervision.

### Features

Preservice mathematics teacher education programs offered by high education institutions internationally vary greatly in composition across countries (see Comiti and Loewenberg Ball 1996; Guyton and McIntyre 1990) but commonly comprise three components:

- A university or college-based curriculum, usually involving theoretical foundation courses (educational psychology, philosophy and sociology of education, historical approaches and policy etc.)
- "Method" or "didactics" courses (devoted specifically to the teaching of a specific subject, such as mathematics)
- A school teaching experience, termed "teaching practice," "field experience," or "practicum," during which student teachers are placed in schools

The organization of the field experience component varies considerably, (McIntyre et al. 1996; Knowles and Cole 1996) including in the following ways:

- The contractual arrangement with schools – in some countries universities are required to pay schools to provide for field experience, in others this is not the case; in some countries schools are obliged by regulation to accept student teachers, in others not
- Who undertakes the supervision of student teachers in schools (school teachers, inspectors, educational advisors)
- The length of the field experience, which can range from a few weeks to a whole year and can be organized in discrete in blocks or continuously throughout the year
- The nature of the partnership between the university-based and school-based supervisors

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- The choice of school settings and the degree to which classroom practice in these schools aligns with "good practice" as espoused by the teacher education provider
- The degree of teaching responsibility given to student teachers
- How explicitly requirements for the field experience are set out in advance and how decisions about these requirements are made
- The degree of alignment between the vision and values of the teacher education provider and the schools in which student teachers are located

### Research on the Field Experience

As indicated, initial teacher education involves two distinct sites of learning and practice, each with specialized identities, practices, forms of knowledge and relationships, and preferred modes of pedagogy: the university or college teacher education provider on the one hand and the schools involved in the practicum on the other. (These in turn are oriented towards a third site: the schools into which student teachers will move after graduation to take on their duties as beginning teachers.) The practicum constitutes a potential bridge between these two sites.

Research on the practicum within mathematics education and within education studies more generally is not extensive and focuses in the main on issues such as the degree of change in student teachers' knowledge, beliefs, decision-making strategies, reflectiveness, and teaching practices as a result of the practicum experience (Bergsten et al. 2009). Some research evaluates interventions aimed at reducing the insulation between teacher educator provider and school, in order to align school experiences more closely with the goals of initial teacher education.

Three research areas which contribute towards inquiry in initial teacher education, and the practicum in particular, are:

- Teacher socialization (and in particular, the degree to which the field experience reinforces or alters the predispositions towards teaching of student teachers) (Zeichner and Gore 1990).
- The issue of the relationship between "theory and practice" which informs the study of the interaction between foundational disciplines and "methods" courses within initial teacher education programs, between these programs and the practicum, and between initial teacher education provision and the classroom practice of beginning teachers (see Dewey 1904, Hirst 1990, and McIntyre 1995). Jaworski and Gellert (2003) suggest a four-model continuum to describe the level of integration or insulation of the theoretical and practical aspects of initial teacher education.
- The tacit, or craft dimension in the professional development of teachers, or "professional craft knowledge" (forms of knowledge which are not realizable in language and which are acquired via modeling and mentoring in the site of practice (see Polanyi 1983 and Shön 1983)).

## Cross-References

► Mathematics Teacher Education Organization, Curriculum, and Outcomes

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# Frameworks for Conceptualizing Mathematics Teacher Knowledge

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## Keywords

Mathematics teacher knowledge; Subject matter knowledge; Pedagogical content knowledge; Mathematical knowledge for teaching; Knowledge Quartet

## Introduction

Discussion of the relationship between knowledge and the profession of teaching is particularly convoluted because knowledge is itself the commodity at the heart of education and the very goal of teaching. For a starting point in theorizing knowledge and teaching, one can turn to Aristotle's (384–322 BC) aphorism "it is a sign of the man who knows, that he can teach" (Metaphysics, Book 1). This can be interpreted that "really" knowing something is best evidenced in the performance of teaching. The Oxford philosopher John Wilson (1975) endorsed and extended Aristotle's position on teacher knowledge with the argument that comprehension of the logic of concepts offered guidance on how to teach them. In other words, not only do we need to know what we teach in the sense of understanding it, but such a profound quality of knowing actually acts as a guide to the pedagogy, i.e., the "how to teach," of subjects such as mathematics. This position has recently been developed by Watson and Barton (2011) in terms of pedagogical application of "mathematical modes of inquiry." However, the seminal work of Lee Shulman and his colleagues in the 1980s underpins the dominant frameworks currently in use for conceptualizing mathematics teacher knowledge.

## Lee Shulman

In a presidential address to the American Educational Research Association, Shulman argued that in recent (American) research on teaching, insufficient emphasis had been placed on the subject matter under consideration: he called this omission "the missing paradigm." Shulman's highly influential perspective on teacher knowledge arose from empirical research, the *Knowledge Growth in a Profession* project, conducted at Stanford University in the mid-1980s. His tripartite conception of teachers' knowledge of the *content* that they teach includes not only knowledge of *subject matter* but also *pedagogical* content knowledge, as well as knowledge of *curriculum*. Subject matter knowledge (SMK) refers to the "amount and organization of the knowledge per se in the mind of the teacher" (Shulman 1986, p. 9) and is later (Grossman et al. 1989) further analyzed into substantive knowledge (the key facts, concepts, principles, and explanatory frameworks in a discipline) and syntactic knowledge. The latter is knowledge about the nature of *inquiry* in the field and the mechanisms through which new knowledge is introduced and accepted in that community; in the case of mathematics, it includes knowledge about inductive and deductive reasoning, the affordances and limitations of exemplification, and problem-solving heuristics and proof. Pedagogical content knowledge consists of "ways of representing the subject which makes it comprehensible to others…[it] also includes an understanding of what makes the learning of specific topics easy or difficult …" (Shulman 1986, p. 9).

In addition to his taxonomy of *kinds* of teacher knowledge, Shulman (1986) also draws out three *forms* of such knowledge. These are propositional knowledge, consisting of statements about what is known about teaching and learning; *case* knowledge, being salient instances of theoretical constructs which serve to illuminate them; and *strategic* knowledge, where propositional and case knowledge are applied in the exercise of judgment and wise action. Shulman's analysis remains the starting point for most subsequent analyses of, and further investigation into, the professional knowledge base of mathematics teachers, particularly in the Anglo-American research orbit.

## Mathematical Knowledge for Teaching

Deborah Ball entered the research field on the cusp of Shulman's work at Stanford, and her contribution to research in the field of mathematics teacher knowledge has been extensive and far reaching. Videotapes and other records of her own elementary classroom teaching have been an important source of data in the investigations of her research group at the University of Michigan. The "practice-based theory of knowledge for teaching" (Ball and Bass 2003) that emerges from the Michigan studies unpicks, refines, and reconfigures the three kinds of content knowledge – subject matter, pedagogical, and curricular – identified by Shulman (1986). This *Mathematical Knowledge for Teaching* (MKfT) framework (Ball et al. 2008) has been adopted by many researchers as a theoretical framework for interpreting their own classroom data, as well as a language for articulating their findings.

In the MKfT deconstruction of Shulman, SMK is separated into "common content knowledge" (CCK), "specialized content knowledge" (SCK), and "horizon content knowledge" (HCK). CCK is essentially "school mathematics," applicable in a range of everyday and professional contexts demanding the ability to calculate and to solve mathematics problems. SCK, on the other hand, is knowledge of mathematics content that mathematics teachers need in their work, but others do not. This would include, for example, knowing *why* standard calculation routines work, such as "invert and multiply" for fraction division. Examples of SCK offered by Ball et al. (2008) include the evaluation of various student responses to column subtraction problems, claiming that the kinds of knowledge required to diagnose incorrect strategies or to understand correct but nonstandard ones are essentially *mathematical* rather than pedagogical. On the other hand, they suggest that knowing about typical errors in advance, thereby enabling them to be anticipated, is a type of *pedagogical* content knowledge which they call "knowledge of content and students" (KCS). Thus, the argument goes as follows: SCK is accessible to the competent mathematician, by reference to their knowledge of mathematics (see also Watson and Barton 2011). KCS, on the other hand, is conceived as a body of knowledge deriving from empirical research in the behavioral and social sciences, including mathematics education.

Note that the MKfT model is not a simple elaboration of Shulman's three content categories, since curriculum knowledge is no longer a separate category. In effect, it has been partitioned into two: *horizon* content knowledge, which becomes the third component of SMK, and knowledge of *content and curriculum*, which is now one of three components of PCK. In fact, Ball et al. (2008, p. 391) draw out two aspects of curriculum knowledge, as conceived by Shulman, that are often overlooked. The first, *lateral* curriculum knowledge, relates to cross-curricular mathematical connections, invoking conceptions and applications that enrich students' experience and appreciation. The second, *vertical* curriculum knowledge, entails knowing what mathematical experiences precede those in a given grade level and what will follow in the next, and subsequent, grades. Ball et al. then relabel vertical knowledge as horizon content knowledge and include it within SMK. The importance of this Janus-like quality in mathematics teachers is clear. On the one hand, they need to know what knowledge their students can be expected to bring with them as a result of previous instruction, including restricted conceptions and even misconceptions. On the other hand, Dewey (1903, p. 217) cautioned teachers against fostering "mental habits and preconceptions which have later on to be bodily displaced or rooted up in order to secure a proper comprehension of the subject," thereby impeding progress in the later grades.

## The Knowledge Quartet

In a study of London-based graduate trainee primary teachers, Rowland, Martyn, Barber, and Heal (2000) found a positive statistical connection between scores on a 16-item audit of content knowledge and competence in mathematics teaching on school-based placements. A team at the University of Cambridge then surmised that if superior content knowledge really does make a difference when teaching elementary mathematics, it ought somehow to be observable in the practice of the knowledgeable teacher. The Cambridge team therefore set out to identify, and to understand better, the ways in which elementary teachers' mathematics content knowledge, or the lack of it, is made visible in their teaching.

The Knowledge Quartet (KQ) was the outcome of research in which 24 lessons taught by elementary school trainee teachers were videotaped and scrutinized. The research team identified aspects of trainees' actions in the classroom that could be construed as being informed by their mathematics subject matter knowledge or pedagogical content knowledge. This inductive process initially generated a set of 18 codes (later expanded to 20), subsequently grouped into four broad, superordinate categories or dimensions – the "Quartet."

The first dimension of the Knowledge Quartet, *foundation*, consists of teachers' mathematics-related knowledge, beliefs, and understanding, incorporating Shulman's classic 3-way taxonomy of *kinds* of knowledge without undue concern for the boundaries between them. The second dimension, *transformation*, concerns knowledge in action as demonstrated both in planning to teach and in the act of teaching itself. A central focus is on the representation of ideas to learners in the form of analogies, examples, explanations, and demonstrations. The third dimension, *connection*, concerns the ways by which the teacher achieves coherence within and between lessons: it includes the sequencing of material for instruction and an awareness of the relative cognitive demands of different topics and tasks. The final dimension, *contingency*, is witnessed in classroom events that were not envisaged in the teachers' planning. In commonplace language, it is the ability to "think on one's feet." Rowland, Huckstep, and Thwaites (2005) include a more detailed conceptual account of these four dimensions and of the "grounded theory" approach to analyzing the video recordings of the 24 lessons.

The Knowledge Quartet is a lens through which the observer "sees" classroom mathematics instruction. It is a theoretical tool for observing, analyzing, and reflecting on actual mathematics teaching. Devised first with researchers in mind, it has subsequently been applied to support and facilitate the improvement of mathematics teaching. In particular, it offers a four-dimensional framework against which mathematics lessons can be discussed, with a focus on their subject matter content and the teacher's related knowledge and beliefs. A book aimed at mathematics teachers and teacher educators (Rowland et al. 2009) explains how to analyze and give feedback on mathematics teaching, using the Knowledge Quartet.

Both the Mathematical Knowledge for Teaching framework and the Knowledge Quartet are practice-based theories of knowledge for teaching. However, while parallels can be drawn between the origins of the two frameworks, the two theories look very different. In particular, the theory that emerges from the Michigan studies aims to unpick and clarify the formerly somewhat elusive and theoretically undeveloped notions of SMK and PCK. In the Knowledge Quartet, however, the distinction between different *kinds* of mathematical knowledge is of lesser significance than the classification of the situations in which mathematical knowledge surfaces in teaching. In this sense, the two theories are complementary, so that each has useful perspectives to offer to the other. A more extended comparison and critique is to be found in the chapter by Petrou and Goulding in Rowland and Ruthven (2011).

## Culture: A Caveat

Notwithstanding the influence of the frameworks for mathematics teachers outlined in this article, it is important to bear in mind that they represent perspectives on the topic originating in Anglo-American culture. This is not to say that they cannot, indeed have not, been found relevant and useful far beyond their geographical origins. However, other cultural influences and emphases can be seen, especially in parts of Europe and in the Far East (in particular China and Japan). While these influences do not usually address mathematics *teacher knowledge* explicitly, they significantly shape ways of thinking about how teachers develop as professionals. In France, for example, the *didactique* which draws upon fundamental theoretical approaches due to Brousseau (didactical situations), Chevallard (didactical transposition), and Vergnaud (conceptual fields) is the mold in which thinking about mathematics teaching is set. The German *stoffdidactik* is an approach to analyzing mathematical content with a view to making it accessible to learners – an endeavor at the heart of Shulman's PCK, in fact. The Chinese didactical method of *bianshi* focuses on subtle but significant shifts to achieve variation in problem types. According to Ma (1999), Chinese elementary teachers demonstrate not only personal mathematical competence but a high level of what the MKfT framework would call specialized content knowledge. As yet, however, the teacher-knowledge discourse of each culture tends to have its own vocabulary, although conceptual connections between these separate discourses can be discerned.

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# Functions Learning and Teaching

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## Definition and Brief History

The notion of function has three different, yet interrelated, aspects. Firstly, *a function* is a purely mathematical entity in its own right. Depending on the level of abstraction, that entity can be introduced, for example, as either a correspondence that links every element in a given domain to one and only one element in another domain, called the co-domain, or as a certain kind of relation, i.e., a class of ordered pairs (in a Cartesian product of two classes), which may be represented as a graph, or as a process – sometimes expressed by way of an explicit formula – that specifies how the dependent (output) variable is determined, given an independent (input) variable, or as defined implicitly as a parametrized solution to some equation (algebraic, transcendental, differential). Secondly, functions have crucial roles as lenses through which *other mathematical objects or theories* can be viewed or connected, for instance, when perceiving arithmetic operations as functions of two variables; when a sequence can be viewed as a function whose domain is the set of natural numbers; when maximizing the area of a rectangle given a constant perimeter or perceiving reflections, rotations, and similarities of plane geometrical figures as resulting from transformations of the plane; or when Euler's φ-function (for a natural number n, φ(n) is the number of natural numbers 1,2,. . ., n that are co-prime with n) allows us to capture and state fundamental results in number theory and cryptography, etc. Thirdly, functions play crucial parts in the application of mathematics to and modelling of *extra-mathematical situations and contexts*, e.g., when the development of a biological population is phrased in terms of a nonnegative function of time, when competing coach company tariff schemes are compared by way of their functional representations,, or when the best straight line approximating a set of experimental data points is determined by minimizing the sum-of-squares function.

These aspects of the notion of function make this notion one of the most fundamental and significant ones in mathematics, and hence in mathematics education. This is reflected both in the history of mathematics (the term "function" going back at least to Leibniz (Boyer (1985/1698), p. 444) and in the history of mathematics education, where the notion of function as a unifying concept in mathematics was introduced in the curricula of many countries from the late nineteenth century onwards, following the reform program proposed by Felix Klein (NCTM 1970/2002), p. 41; Schubring 1989, p. 188). Today, some version of the notion of function permeates mathematics curricula in most countries. However, the different aspects of the notion of function also make it highly diverse, multifaceted, and complex, which introduces challenges to the conceptualization as well as to the teaching and learning of functions.

Against this background, the concept of function in mathematics education has given rise to a huge body of research. The origins of this research seem to date back to debates in the 1960s about the right (or wrong) way to define a function. Thus, Nicholas (1966, p. 763) compares and contrasts three definitions (which he labels "variable," "set," and "rule"), which, in his view, generate a dilemma, because they are not logically equivalent. The first empirical studies also seem to stem from the late 1960s. Empirical studies focused on the formation of the concept of function, which has also preoccupied the far majority of subsequent research, as is reflected by the seminal volume on this topic edited by Dubinsky and Harel (1992) and in the relatively recent overview of significant research offered by Carlson and Oehrtman (2005).

### Challenges to the Teaching and Learning of Function

The reason why the concept of function itself has attracted massive attention from researchers is that students (and many pre- or in-service teachers, see Even 1993) have experienced, and continue to experience, severe difficulties at coming to grips with the most significant aspect of this concept in both intra- and extra-mathematical contexts. More specifically, researchers have focused on identifying and analyzing the learning difficulties encountered with the concept of function; on explaining these difficulties in historical, philosophical, and cognitive terms; and on proposing effective means to counteract them in teaching. In so doing, researchers have introduced a number of terms and distinctions (e.g., between "action" and "process" (Dubinsky and Harel 1992b)).

One important issue that arises in this context is the fact that functions can be given several different *representations* (e.g., verbal, formal, symbolic (including algebraic), diagrammatic, graphic, tabular), each of which captures certain, but usually not all, aspects of the concept. This may obscure the underlying commonality – the core – of the concept across its different representations, especially as translating from one representation to another may imply loss of information. If, as often happens in teaching, learners equate the concept of function with just one or two of its representations (e.g., a graph or a formula), they miss fundamental features of the concept itself. This is also true of the many different equivalent *symbolic notations* for the very same function (e.g., y = $x^2$ − 1/x, f: x ®$x^2$ − 1/x, in both cases provided x ≠ 0; f: Â\{0} ®Â defined by f(x) = $x^2$ − 1/x; f(x) = (x − 1)($x^2$ + x + 1)/x, x ÎÂ\ {0}; f = {(x, $x^2$ − 1/x)| x ÎÂ\ {0}}; (x,y) Îf Ûy = $x^2$ − 1/x Ùx ÎÂÙx ≠ 0; x = $y^2$ − 1/y, y ÎÂ\ {0}, just to indicate a few). Interpreting and translating between function representations in intra- or extra-mathematical settings proves to be demanding for learners. Of particular significance here is the translation between visual and formal representations of the same function, which for some learners are difficult to reconcile.

Functions come in a huge variety of sorts, types, and cases, ranging from familiar ones (such as linear or quadratic functions of one variable) to abstract and complex ones (such as the integral as a real-valued functional operating on the space of Riemann-integrable functions of *n* variables). The plethora of functions of very different kinds means that students' concept of function is also delineated by the set of function *specimens and examples* of which the students have gained experience. This is an instance of the well-known distinction between concept definition and *concept image* playing out in a very manifest manner in the context of function (Vinner 1983), in particular in teaching and learning that focuses on abstract functions. This distinction also proves important when zooming in on special classes of functions (such as linear or affine functions, exponential functions, recursively defined functions, and above all the real and complex functions that appear in calculus and analysis), which have been the subject of study in an immense body of research.

Another demanding facet of the concept of function is the *process-object duality* (cf., e.g., several chapters in Dubinsky and Harel (1992a)) that is characteristic of many functions, especially the ones that students encounter in secondary and undergraduate mathematics teaching. In its process aspect, a function yields outputs as a result of inputs. In its object aspect, a function is just a mathematical entity which may engage in relationships with other objects, or be subjected to various sorts of treatment (e.g., differentiation at a point). Oftentimes the transition from a process view to an object view of function is a severe challenge to students.

### Overcoming Learning Difficulties

In response to the observed learning difficulties attached to functions and analyses of these difficulties, mathematics educators have invested efforts in proposing, designing, and implementing intervention measures so as to address and counteract these difficulties specifically. The overarching result is that it is possible to counteract the learning difficulties at issue, but this requires intentional and focused work on designing rich and multifaceted learning environments and teaching-learning activities that are typically extensive and time-consuming. In other words, the desired outcomes are not likely to occur by default with most students, they have to be aimed at, and they come at a price: time and effort.

A few examples: One focal point has been to help students develop a process conception of function (in contrast to an action conception), by way of technology (Goldenberg et al. 1992). Technology has also been used to consolidate students' concept images so as not to "overgeneralize" the prototypical function examples that initially underpinned their conception. Helping students to develop an object conception of function (by way of reification) has preoccupied many researchers, e.g., Sfard (1992).

## Future Research

While research in this area in the past has focused on the learning (and teaching) of the concept of function in contexts when functions are already meant to be present, or presented to students, very little – if any – research has dealt with situations in which students are requested or encouraged to uncover or introduce, themselves, functions or functional thinking into an intra- or extra-mathematical context. Furthermore, there is a need for future research that focuses on designing teaching-learning environments that help generate transfer of the notion of function from one setting (e.g., real functions of one variable) to another (e.g., functions defined on sets of functions).

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## Gender in Mathematics Education

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Gender differences; Sex differences; Equity; Women's movement; Feminism; Social justice; Explanatory models; Technology; Neuroscience

### Issues of Definition

According to Haig (2004), it was the feminist scholars of the 1970s who adopted gender "as a way of distinguishing 'socially constructed' aspects of male–female differences (gender) from 'biologically determined' aspects (sex)" (p. 87). In the mathematics education literature, the gradual shift from "sex differences" to "gender differences" occurred during the period from the late 1970s into the 1980s. Fennema's (1974) seminal work in the field was reported as "sex" differences in mathematics achievement, and in the renowned Fennema and Sherman studies on affective factors (e.g., Fennema and Sherman 1977), the findings were also described as "sex" differences. As noted by Haig (2004), in more recent times, the "distinction is now only fitfully respected and gender is often used as a simple synonym of sex" (p. 97); this is also evident in the mathematics education literature.

In this encyclopedia entry, the term "gender" is used in the sense that Leder (1992) clarified it with respect to mathematics learning. Gender is considered a social construct, and gender differences are considered to be contextually bound and not fixed, that is, they are not genetically determined. Sex differences are only described in this entry with respect to issues associated with biological distinctions.

### Historical Overview of Gender and Mathematics Education Research

Research on gender issues in mathematics education began in earnest during the 1970s. This work was mainly situated in the English-speaking, developed world (USA, UK, Australia), as well as in some European countries. The common research findings were the following: (i) on average, females' achievement levels were lower than males', particularly when it came to challenging problems (it should be noted that it was recognized that the gender difference was small compared to within sex variations), (ii) females' participation rates in mathematics were lower than males' when mathematics was no longer compulsory, and (iii) on a range of affective/attitudinal measures with respect to mathematics or to themselves as mathematics learners, females' views were less "functional" (leading to future success) than males'.

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The theoretical frameworks of this early research were founded in those prevalent at the time, that is, a positivist view that findings were generalizable beyond the contexts in which the research was conducted. Most of the research was quantitative, although sometimes accompanied by qualitative dimensions. The dominant feminist perspective that could be inferred from the stances adopted was that of "liberal" feminism, that is, that females' relative underperformance in mathematics and their under-representation in challenging mathematics offerings at the school level, and in mathematics and science-related courses of study at the tertiary level as well as in related careers, had to be brought up to the levels of those found for males.

In the west, the "Women's Movement" (second wave of feminism) was very active across western societies in the 1970s and 1980s. Within the broader context of women's inferior status in society, girls had been identified as educationally disadvantaged. Women's legal rights and roles in the family and the workplace, as well as sexuality and reproductive rights, were all under scrutiny. Legislation was enacted to address women's demands for a more equitable society. Money was flowing for educational research to address female disadvantage in mathematics (and science), and intervention programs flourished – see Leder et al. (1996) for an overview of a range of these intervention programs, their outcomes, and what was learned from them.

In the 1980s and 1990s, the "founding mother" in the field, Elizabeth Fennema, was joined by a number of eminent scholars. Among them were Gilah Leder and Leone Burton whose books (e.g., Fennema and Leder 1990; Burton 1990) and other scholarly journal articles, handbook contributions (e.g., Leder 1992), and conference papers formed the building blocks for ongoing research in the field. The history of women's place in mathematics (e.g., Henrion 1997), including mathematics education (e.g., Morrow and Perl 1998), and the relationship to mathematics curricula (e.g., Kaiser and Rogers 1995; Perl 1978) were also documented.

The International Commission on Mathematical Instruction [ICMI] had a significant role to play in bringing the field of gender and mathematics education to prominence. The International Organization for Women in Mathematics Education [IOWME] sessions at the ICMI conferences in Budapest (1988), Montreal (1992), and Seville (1996) were watershed events. New scholarship was brought to light, and there were several notable outcomes: Roger and Kaiser's (1995) book introduced various feminist perspectives on gender issues in mathematics learning; Burton's (1990) book included important contributions to the field by authors from a range of international settings; and in Keitel's (1998) book, gender was considered within the broader framework of social justice and equity. ICMI's support for a study on gender and mathematics learning (Höör, Sweden, 1993) was also significant. Ironically, it was at the Höör conference that the all-male leadership of ICMI, the organization representing the field of mathematics education internationally, was openly challenged; this may well have been the catalyst for change. In subsequent years, women in mathematics education have played significant and active roles in the leadership of ICMI.

The "golden era" of research on gender and mathematics education appears to have ended in the mid-1990s. In the West, there was a sense that the "female problem" in society had been solved. For gender and mathematics education, research and intervention funding dried up; governments had other considerations at the top of their agendas. Arguably, too, there was a backlash to the focus on girls' education, and attention switched to boys' educational needs.

One positive and lasting outcome of the era was the mandating of statistical data on educational outcomes emanating from many government sources to be reported by sex. At the international level, there is also easy access to the Trends in Mathematics and Science Studies [TIMSS] data and the Program for International Student Assessment [PISA] data. These data provide researchers with the capacity to examine achievement and participation in mathematics for gender differences, both within and across nations, for the age cohorts taking these tests. It should be noted that affective data are also included in the TIMSS and PISA databases.

The ability to refine investigations for achievement by mathematics content domain and/or by various other equity factors (e.g., socioeconomic background, race, ethnicity, and religious affiliation) is possible from some large-scale international data sources, as well as those available within nations, for example, national testing and competition data. It was the reporting of gender differences from large-scale data sources that provided the initial impetus for research in the field; this must continue. In the contemporary world of the twenty-first century, it is these kinds of data that have sparked concerns and interest in research on gender issues in mathematics learning in the developing world and in Asia.

### Theoretical Considerations

In seeking explanations for observed gender differences in mathematics learning outcomes, a number of explanatory models were postulated in the early period of research in the field. Several focused on explanations for specific aspects of mathematics education including differential elective course enrolments (Eccles et al. 1985), mathematics achievement (Ethington 1992), achievement on cognitively demanding tasks (Fennema and Peterson 1985), and explanations for the relationships between race, socioeconomic background, and gender differences in levels of performance on standardized tests (Reyes and Stanic 1988). Leder's (1990) model was more general. Two groups of factors – student-related and environmental – were identified as interacting contributors to patterns of gender difference in achievement and participation. The postulated models shared several common elements: social environment, significant others, learning context, cultural and personal values, affect, and cognition (Leder 1992).

A major critique of the liberal feminist paradigm framing the early research on gender and mathematics education was that it positioned females as "deficit." In the pursuit of expanding knowledge of gender issues in mathematics education, the explanatory models described above were supplanted by a range of feminist perspectives (e.g., feminism of difference, embracing the ways in which women are different from men; radical feminism, targeting the power/political system that oppresses women; feminist standpoints, founded in the lives and experience of women) and theoretical frameworks from other disciplines to underpin subsequent research endeavors (e.g., postmodernism, rejection of the homogeneity of groups such as girls/boys, instead focusing on the relative truths of individuals; poststructuralism, gender is socially and culturally created through discourse; queer theory, gender is not fixed and does not define the individual; postcolonialism, identifies parallels between women in a patriarchy and recently decolonized countries; racism is implicated).

Fennema (1995) explained that feminist scholars had convincingly argued that male perspectives dominated traditional research approaches and interpretations and that this view was incomplete as female perspectives were omitted. To progress towards gender equity in mathematics education, she urged researchers to embrace "new types of scholarship focused on new questions and carried out with new methodologies" (p. 35) including feminist methodologies, through which the world is viewed and interpreted from a female perspective.

Following the lead of feminist science educators, Burton (1995) challenged the mathematics establishment in questioning the objectivity of the discipline. She argued that mathematics was contextually bound and that from this perspective could be viewed in more human terms; this, she contended, would challenge traditional pedagogical approaches to the teaching of mathematics as well as the content taught. The stages of women's way of knowing (more likely to be "connected") as different from men's (more likely to be "separate") were identified by Belenky, Clinchy, Goldberger, and Tarrule (1986). Becker (1995) adapted Belenky et al.'s model to the learning of mathematics. Kaiser and Rogers (1995) applied McIntosh's evolution of the curriculum model to women and the mathematics curriculum. They identified five phases: *womanless* mathematics, women *in* mathematics, women *as a problem* in mathematics, women *as central to* mathematics, and mathematics *reconstructed*. In line with Burton's challenge of a feminist epistemology of mathematics, Belenky et al.'s gender-related distinction between "connected" and "separate" knowing, and Kaiser and Rogers' curriculum model, research on feminist pedagogies was to follow.

In recent years, there does not appear to be extensive scholarly writing on theoretical developments in the field.

### Methodological Considerations

As noted above, positivism underpinned early research studies on gender and mathematics learning. Although often not acknowledged by the researchers, post-positivism, in which context is recognized as relevant in pursuit of the truth, is identifiable as the epistemological basis of many more recent large- and smaller-scale quantitative studies. Mixed methods research, in which quantitative data are complemented or supplemented by qualitative data, or vice-versa, has been embraced in educational research more broadly and in the field of gender and mathematics learning more specifically. With the advent of cheaper, more reliable, digital technologies in recent times, innovative data-gathering instruments (e.g., mobile devices) and data-gathering sources (e.g., Facebook) have been employed.

### Recent Developments

The advent and pervasive presence, both outside and within mathematics classrooms, of calculators, computers, and ICTs and the mobile devices to access them has introduced a new strand of research into gender issues and mathematics learning. As evidenced by course participation rates and workforce figures, male dominance in the field of computer science and in the world of ICTs is even stronger than in mathematics and the physical sciences. Surrounded by the high expectation that technological advancements will enhance mathematics learning for all and recognizing that another male domain was being introduced into the preexisting male domain of mathematics education, researchers began questioning whether the widespread implementation of these technologies into mathematics classrooms and assessment regimes would challenge or exacerbate gender differences in mathematics learning outcomes (e.g., Forgasz, Vale, and Ursini 2010). Some evidence suggests that females may be disadvantaged by computers and the mandated use of CAS calculators in high-stakes examinations; research is ongoing with respect to the impact of technologies such as the iPad.

Another exciting development is the entry of researchers from Asian, South American, and developing countries including African nations into the field (see Forgasz, Rossi Becker, Lee, and Steinthorsdottir 2010). The common issues highlighted – males' superior mathematics achievement, participation, and attitudes towards mathematics – and the methodological and epistemological approaches adopted resonate with the early research on gender and mathematics learning undertaken in western, English-speaking nations. UNESCO's emphasis on gender mainstreaming (see Vale 2010) has contributed strongly to the efforts being made to the more general goal of achieving equity for women in many societies. Interestingly, the more recent PISA and TIMSS results from several Islamic nations (recent entrants into these international comparative studies) reveal generally low overall achievement levels, with a trend for girls to outperform boys. Clearly factors other than gender per se contribute to these patterns; further research is clearly needed.

Finally, Fennema's (1995) prognostication of the importance of combining neuroscientific research with gender equity considerations has begun but within the framework of broader equity considerations including diversity and special needs (see Forgasz and Rivera 2012). The research has been conducted scientifically and not from feminist perspectives, however. Yet, these intriguing interdisciplinary research findings with respect to sex differences invite further exploration.

## Cross-References

▶ Inclusive Mathematics Classrooms
▶ Poststructuralist and Psychoanalytic Approaches in Mathematics Education
▶ Socioeconomic Class in Mathematics Education

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## Giftedness and High Ability in Mathematics

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### Keywords

Giftedness; High ability; Mathematical expertise; Mathematical discovery/invention; Mathematical creativity

### Definition

Mathematical giftedness is an extremely complex construct exhibited in mathematical invention which is clearly exhibited in work of professional mathematicians. Mathematical giftedness implies high mathematical abilities. Nonetheless, often mathematical giftedness is perceived as inborn personal characteristic, whereas high abilities in mathematics are perceived as a dynamic characteristic that can be developed. The dynamic view of mathematical giftedness assumes that it can be realized only if appropriate opportunities are provided to a person with high mathematical potential. Mathematical talent, which is realized giftedness, is expressed in high-level performance in mathematics that leads to mathematical discoveries and, thus, is closely connected with mathematical creativity.

There is no singular clear definition of mathematical giftedness. Moreover, until recently, the construct of mathematical giftedness was overlooked in mathematics education research for several decades. Thus, this entry synthesizes accounts from two fields of educational psychology: gifted education and mathematics education.

## General Giftedness

In the field of gifted education, gifted students are identified by qualified specialists by virtue of outstanding abilities expressed in exceptional performance. In the adult population, the criteria for giftedness are restrictive, like in the case of Nobel Prize laureates. General giftedness is often measured by means of IQ tests, while a number of theorists have developed broad, multidimensional formulations of giftedness and talent that are widely accepted (Gardner 2003; Sternberg 2000). Gardner's multiple intelligence theory differentiates between not necessarily connected dimensions including verbal-linguistic, logical-mathematical, and visuospatial intelligences. Sternberg claims that giftedness is a function of analytical, practical, and creative abilities accompanied with personal wisdom. Several models postulated giftedness as being the result of the complex interactions of cognitive, personal-social, and sociocultural traits and environmental conditions (e.g., Renzulli 2000; Milgram 1989).

In the field of gifted education, mathematical giftedness is usually regarded as a special type of specific giftedness which is opposed to general giftedness. However, most models of general giftedness can be applied to mathematical giftedness associated with mathematical abilities and skills.

## Mathematical Abilities and Skills

The precise acquisition of mathematical abilities involves a broad range of different general cognitive skills, including spatial perception, visuospatial ability, visual perception, visuomotor perception, attention, and memory. Together these skills enable the acquisition, understanding, and performance of various mathematical activities. Mathematical activities deal with five main types of mathematical objects: number and quantity, shape and space, pattern and function, chance and data, and arrangement, while successful mathematical performance involves modeling and formulating, manipulating and transforming, inferring and drawing conclusions, argumentation, and communication.

During the past half century, only a small number of systematic studies devoted to mathematical giftedness were performed. Krutetskii's (1976) study on high mathematical abilities in schoolchildren is seminal to the field and remained unique for several decades. It introduced components of high mathematical ability in schoolchildren which included the abilities to grasp formal structures; think logically in spatial, numeric, and symbolic relationships; think critically; generalize rapidly and broadly; and be flexible with mental processes. According to Krutetskii students with high mathematical abilities are able to switch from direct to reverse trains of thought and to memorize mathematical objects, schemes, principles, and relationships. These students appreciate clarity, simplicity, and rationality and can be characterized by the general synthetic component called mathematical cast of mind.

## Mathematical Giftedness and Creativity

At times mathematical abilities are measured using the nonverbal portions of psychometric tests like SAT-M. The main criticism of these tools is that they do not test creativity since creativity is fundamental to the work of a professional mathematician, while the ability to discover mathematical objectives and find inherent relationships among them requires mathematical creativity.

The connection between mathematical giftedness and creativity leads to an eight-tiered (from 0 to 7) hierarchy of mathematical gift (Usiskin 2000) assuming that the abilities of professional mathematicians are at levels 5, 6, and 7. The most creative mathematicians that discover new mathematical theorems and invent new mathematical concepts are at the highest (7th) by virtue of their creative ability. This hierarchy implies that while mathematical creativity implies mathematical giftedness, the reverse is not necessarily true (Sriraman 2005).

The notion of mathematical giftedness and its relationship to mathematical creativity is quite clear with respect to research mathematicians; however, it is rather vague with respect to high school students. This duality reflects the distinction between *absolute* and *relative* creativity (Leikin 2009, cf. Big C and Little c defined by Csikszentmihalyi 1996). *Absolute creativity* is associated with mathematical discoveries at a global level. *Relative creativity* refers to discoveries of a specific person within a specific reference group.

When connecting between high mathematical abilities and mathematical creativity, researchers express a diversity of views. Some researchers argue that creativity is a specific type of giftedness; others feel that creativity is an essential component of giftedness, while others suggest that these are two independent human characteristics. Thus, analysis of the relationships between mathematical creativity and giftedness is an important question for future research.

## Mathematical Giftedness, Problem Solving, and Insight

High-level problem-solving expertise (e.g., success in solving Olympiad problems) often serves as an indicator of mathematical giftedness in schoolchildren. High achievements in school mathematics usually reflect students' problem-solving proficiency on the topics that they have studied in school; however, they are not an indicator of mathematical giftedness, since they do not reflect students' independent mathematical reasoning.

Mathematical invention, which is an integral part of the activities of research mathematicians, consists of four stages: initiation, incubation, illumination, and verification (Hadamard 1945). Special attention is given to illumination which involves a large measure of intuitive thinking that leads to mathematical insight. Insight exists when a person acts adequately in a new situation, and as such, insight is closely related to creative ability. Thus, success in insight-based problem solving can serve as an indication of mathematical giftedness among school students.

Insight is viewed as a trait central to the construct of general giftedness, in which gifted children outperform their average-achieving peers in problem solving because of their increased tendency towards insight. Accordingly, students with high ability in mathematics have been found to understand an insight-based problem immediately and to solve it quickly.

## Development of Mathematical Ability

Better understanding of the nature of mathematical giftedness at the relative (e.g., school) level can inform mathematics educators of the ways in which school mathematics should be taught to students who can become research mathematicians. This understanding can lead to a special instructional design and mathematical curricula that can be suitable for these students including the choice of mathematical problems for MG students.

The construct of mathematical potential accepts the dynamic perspective on mathematical giftedness (Leikin 2009; Sheffield 1999). The mathematical potential of a student includes abilities (analytical and creative), affective factors (including motivation), and personal characteristics (including commitment). These factors can be advanced and developed if a student is provided with challenging learning opportunities that take into consideration his/her ability, personality, and affect. These leaning opportunities help a mathematically gifted student to realize his/her mathematical potential and become a talented mathematician. Some insight on learning opportunities provided to mathematically gifted students can be learned from nearly five decades of experience of Kolomgorov's mathematical schools in Russia (Vogeli 1997).

## Research on Mathematical Giftedness

In the past, giftedness, creativity and high abilities in mathematics did not receive sufficient research attention in the fields of gifted education and mathematics education. Fortunately, during the last decade, attention to the nature and nurture of mathematical giftedness and creativity has increased (Leikin et al. 2009); as a result, the International Group for Mathematical Creativity and Giftedness was established (http://igmcg.org/). Several directions can be traced in the research conducted on mathematical giftedness.

The relationship between mathematical creativity and mathematical giftedness is at the focus of attention of several research groups nowadays (Leikin and Pitta, accepted). Some studies demonstrate that mathematical creativity is a subcomponent of mathematical ability, whereas others show that differences in creativity in gifted students and in those who are not identified as gifted are task dependent: the higher the mathematical insight required for the problem's solution, the stronger between-group differences in the students' mathematical creativity. Study of professional mathematicians is a rich source for understanding of the relationship between mathematical creativity and mathematical giftedness. Starting from Hadamard (1945) these studies demonstrate special qualities of their reasoning in terms of the inventiveness of their mathematical mind, as expressed in illuminations, mathematical imagery, and inner need for rigorous proof.

While differences between the research findings can be related to the differences in study populations and study methodologies, the question of the relationship between creativity and giftedness in mathematics remains open for future systematic investigation. One of the more challenging questions for research in mathematics education is the relationship between creativity and expertise, as expressed in solving Olympiad problems.

Some studies have explored problem-solving strategies used by mathematically advanced students as compared to strategies employed by those who are not identified as being advanced in mathematics. These studies demonstrate that students with higher abilities are more successful in solving complex mathematical problems and that their heuristics in solving mathematical problems lead to this success. Still, the underlying mechanisms for their success can be the focus of mathematics educational researchers. Special qualities of mathematical understanding of mathematical concepts in mathematically gifted students can be seen as an additional promising and fascinating direction for future research.

Brain research is another direction in educational research that has been gaining the attention of mathematics education researchers. In the field of general giftedness, several studies have demonstrated neuro-efficiency effect (lower brain electrophysiological activity associated with solving problems) in the gifted population. Following advances in brain research, a research group at the University of Haifa has revealed/demonstrated that the nature of general giftedness differs from that of excellence in mathematics (Waisman et al. 2012). The group hypothesizes that excellence in school mathematics is a necessary but not sufficient condition for mathematical giftedness and that generally gifted students who excel in school mathematics have high potential to become talented research mathematicians.

To conclude, research on mathematical giftedness is a relatively new field in mathematics education. This fascinating field calls on mathematicians and mathematics educators to gain a better understanding of the nature and structure of high mathematical abilities, of the ways in which future talented mathematicians can be identified in school and in which ways they can and should be educated in order to fulfill their mathematical potential and to further develop mathematics as a scientific field.

## Cross-References

- ▶ Creativity in Mathematics Education
- ▶ Critical Thinking in Mathematics Education
- ▶ Logic in Mathematics Education
- ▶ Mathematical Ability
- ▶ Problem Solving in Mathematics Education
- ▶ Visualization and Learning in Mathematics Education

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## Heuristics in Mathematics Education

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### Keywords

Discovery-based learning; Heuristics; Polya; Problem solving

### Definition

In this entry we examine Polya's contribution to the role of heuristics in problem solving, in attempting to propose a model for enhancing students' problem-solving skills in mathematics and its implications in the mathematics education.

### Characteristics

Research studies in the area of problem solving, a central issue in mathematics education during the past four decades, have placed a major focus on the role of heuristics and its impact on students' abilities in problem solving. The groundwork for explorations in heuristics was established by the Hungarian Jewish mathematician George Polya in his famous book "*How to Solve It*" (1945) and was given a much more extended treatment in his Mathematical Discovery books (1962, 1965). In "*How to Solve It*," Polya (1945) initiated the discussion on heuristics by tracing their study back to Pappus, one of the commentators of Euclid, and other great mathematicians and philosophers like Descartes and Leibniz, who attempted to build a system of heuristics. His book also included advice for teaching students of mathematics and a mini-encyclopedia of heuristic terms. The role of heuristics and his 4-step model for problem solving impacted enormously on the teaching of problem solving in schools.

The term "Heuristic" comes from the Greek word "Evriskein," which means "Discover." According to the definition originally coined by Polya in 1945, heuristics is the "study of means and methods of problem solving" (Polya 1962, p. x) and refers to experience-based techniques for problem solving, learning, and discovery that would enhance one's ability to solve problems. A heuristic is a generic rule that often helps in solving a range of non-routine problems. Heuristics, such as Think of a Similar Problem, Draw a Diagram or a Picture, Working Backward, and Guess and Check, can serve different purposes such as helping the student to understand and represent the problem, simplify the problem, identify similarities with other problems, and to identify possible solutions. These heuristics, often used in combinations, can be used to solve different types of problems, though there is no guarantee that applying these heuristics will be successful.

Heuristics are an important aspect of mathematical problem solving, especially if we refer to

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them as the capabilities for mathematical reasoning that enable insightful problem solving. Beyond those proposed by Polya, the appropriate inclusion of more general heuristics like spatial visualization, diagrammatic and symbolic representations in complex novel problems, and the recognition of mathematical structures in the teaching and learning of problem solving might result in enhanced student problem-solving behavior (Goldin 2010).

Based on Polya's contribution, extended and more refined lists of heuristics have often been proposed by researchers, and quite often they have been included in official documents and mathematics curricula around the world. Among others, students should be exposed to and know when to use the following heuristics: (a) Try it out; take the role of other people and try to do what they would do. Make use of objects and other (electronic) media to represent the situation or problem. (b) Use a diagram and/or a model of the problem to create a diagrammatic description of the problem and to visualize the problem data. (c) Organize data in systematic lists and look for patterns might help the solver to identify how data is related to the problem question and to perceive patterns in the data. (d) Work backwards; looking at the required end result and working backwards can be especially useful in problems involving a series of steps. (e) Use before after concept; compare the situation before and after the problem is solved. This comparison can shed light on the cause and lead to a possible solution. (f) Use guess and check; make an educated guess of the answer and check its correctness. Use the outcome to improve the next guess and look for patterns in the guesses. (g) Make suppositions; studying the problem data and make suppositions (assumptions without proof) to form the basis for further and better thinking will reduce the number of possible solutions. (h) Restate the problem to better understanding the problem and identifying important factors of the problem. (i) Simplify the problem; try to make a difficult problem simpler, by changing complex numbers to simple or by reducing the number of factors in the problem. The solution to the simplified problem may help in solving the original problem.

While theories of mathematics problem solving have placed a focus on the role of teaching heuristics for an enhanced problem-solving performance, research from Begle (1979) to Schoenfeld (1992) has a consistent outcome that classroom teaching of problem-solving heuristics does little to improve students' problem-solving abilities. There is, of course, a number of constraints related to the teaching and learning of heuristics. First, in a number of problem-solving approaches, problem solving is taught through textbook sections in which students are presented with a strategy (e.g., finding a pattern), then are given practice exercises using the strategy, and finally they are tested on the strategy. When the strategies are taught in this way, they are no longer heuristics, in the sense described by Polya.

A second constraint is related to the nature of heuristics. Despite their long history and although heuristics have descriptive power in describing experts' problem-solving behaviors, there is little evidence that these heuristics could also serve well as prescriptions to guide novices' next steps during ongoing problem solving. This problem lies, according to Begle (1979, p. 145–146), in the fact that heuristics are "both problem- and student-specific often enough to suggest that finding one (or few) strategies which should be taught to all (or most) students are far too simplistic." In line with Begle (1979), Schoenfeld (1992) concluded that a better "understanding" of heuristics is needed, since most heuristics are really just names for large categories of processes rather than being well-defined processes in themselves. To overcome this constrain, Sriraman and English (2010) contended that "understanding" heuristics means to knowing when, where, why, and how to use heuristics and other tools, including metacognitive, emotional (e.g., beliefs), and social (e.g., group-mediated) tools.

A third constraint related to the appropriate teaching of heuristics for enhanced problem-solving skills is related to teachers' skills. As Burkhardt (1988) identified, the task of teaching heuristics is harder for teachers, because (a) mathematically, teachers should provide constructive and formative feedback to students' different approaches in solving problems; (b) pedagogically, teachers should carefully plan their interventions and feedback and assist students using the least possible help; and (c) personally, teachers should be equipped with experience, confidence, and self-awareness, in order to work well with problems without knowing all the answers requires.

How to overcome the above constrains? In his review on heuristics, Schoenfeld (1992) concluded that better results could be obtained by (a) teaching specific (rather than general) problem-solving heuristics that better link to structurally similar problems, (b) teaching metacognitive strategies that could help students in effectively deploying their problem-solving heuristics, and (c) improving students' views of the nature of problem solving in mathematics, by enhancing their productive beliefs, while eliminating their counterproductive beliefs. Further, as English and Sriraman (2010) noted, next research steps in the area of heuristics in problem solving need to develop operational definitions that enable the mathematics education community to answer more prescriptive, than descriptive, questions like the following: "What does it mean to "understand" problem-solving heuristics and other tools?" "How, and in what ways, do these understandings develop and how can we foster this development?" "How can we reliably observe, document, and measure such development?"

The legacy of Polya's contribution to heuristics in problem solving is not restricted to a list of strategies used by experts or novices when solving problems, but rather implies for the significance of problem solving in mathematics and the necessity to find appropriate teaching and self-regulated methods to enhance students' problem-solving skills.

## Cross-References

▶ Inquiry-Based Mathematics Education
▶ Problem Solving in Mathematics Education

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# History of Mathematics and Education

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## Keywords

Epistemology; History and Pedagogy of Mathematics; International Commission on Mathematical Instruction; Interdisciplinary teaching; New Math reform; Original sources; Problem solving; Activity-based teaching; Mathematical proof; Construction of concepts and theories; Modelization; Ethnomathematics

## History and Epistemology for Mathematics Education: An Ever-Increasing Interest

In the last 30 years or so, integrating history of mathematics (HM) in mathematics education (ME) has emerged as a worldwide intensively studied area of new pedagogical practices and specific research activities. However, the interest on the HM in the context of ME dates back to the second half of the nineteenth century. Mathematicians like F. Klein and A. De Morgan and historians like P. Tannery and G. Loria showed an active interest on the role of the HM in education. Already at that time, history enters into textbooks, e.g., in France in Rouché and Comberousse (Barbin et al. 2008, ch. 2.2.). At the beginning of the twentieth century, this interest was revived as a consequence of the discourse and the related debates on the foundations of mathematics. Poincaré criticized Hilbert's axiomatic approach and declared that the history of science should be the "principal guide for the educator." Later, history became a resource for the various epistemological approaches, like Bachelard's *historical* epistemology (Bachelard 1938), Piaget's genetic epistemology (Piaget and Garcia 1989), and Freudenthal's *phenomenological* epistemology (Freudenthal 1983), at the same time stimulating the formulation of specific ideas and conclusions on the learning process (Lakatos 1976; Brousseau 1997; Ernest 1994) (see ► Learning Study in Mathematics Education).

The interest on the history and epistemology of mathematics became stronger and more competitive in the 1960s and 1970s in response to the "New Math" reform. Those supporting the reform were strongly against "a historical conception of education" ("à bas Euclide" declares Dieudonné), whereas, for its critics, history appeared like a "therapy against dogmatism," conceiving mathematics not only as a language but also as a human activity.

Since 1968 ME has constituted a standard subject in regularly organized international meetings. In 1972 a working group on the "History and Pedagogy of Mathematics" was organized by Ph.S. Jones during the 2nd *International Congress on Mathematics Education* (ICME 2), and in 1976 the *International Study Group on the relations between the History and Pedagogy of Mathematics* (known afterwards as the *HPM Group*) was created as an international study group affiliated to the *International Commission on Mathematical Instruction* (ICMI). For the history and the activities of this group, which has been playing a leading role in this area, see Fasanelli and Fauvel (2006).

In fact, the eight points which constituted the original focus and aim of the HPM Group and to some extent achieved so far, are still pertinent today (see Fasanelli and Fauvel 2006) to promote international contacts and exchange information in this area, to promote and stimulate interdisciplinary investigation, to further a deeper understanding of mathematics' evolution, to assist in improving instruction and curricula by relating mathematics teaching and its history to the development of mathematics, to produce relevant material for the teachers' benefit, to facilitate access to this material and to historical sources, and to promote awareness of the relevance of the HM for mathematics teaching and its significance for the development of cultures.

In the mid 1980s, the French network of the IREMs (*Instituts de Recherche sur l'Enseignement des Mathématiques*) began to organize every 2 years a Summer University on the History and Epistemology in Mathematics Education. Since 1993, this was extended on a European scale constituting the *European Summer University on the History and Epistemology in Mathematics Education* (ESU), which gradually has become a major international activity in the spirit of the HPM Group (Barbin et al. 2010). This spirit goes beyond the use of history in teaching mathematics and conceives mathematics as a living science with a long history, a vivid present and an unforeseen future, together with the conviction that this conception of mathematics should be not only the core of its teaching but also its image spread out to the outside world.

The gradually increasing interest of mathematicians, historians and mathematics teachers and educators in this area, has led to various research activities and didactical experiments, which were analyzed, and their results were disseminated in the context of regularly organized local and international meetings and were presented in numerous publications in international journals, collective volumes, and conference proceedings. Some standard works in chronological order (with detailed extensive bibliography therein) are NCTM 1969/1989, Commission Inter-IREM (1997), Swetz et al. (1995), Calinger (1996), Fauvel and van Maanen (2000), Katz (2000), Bekken and Mosvold (2003), Katz and Michalowicz (2004), Barbin and Benard (2007), Knoebel et al. (2007), Barbin et al. (2010), and Katz and Tzanakis (2011).

There are three different – though interrelated – types of contributions of didactical/educational research and the associated experimental work on the role of HM in ME in the last 30 years, epistemological, cultural, and didactical.

## Epistemological Contributions

Bachelard and Lakatos' influence clearly appears in the research conducted on the role of problems and on constructing and rectifying concepts and theories. This research should be placed both in the context of pedagogical constructivism of the 1980s and 1990s, as well as, in the area of "problem solving" (see ▶ Problem Solving in Mathematics Education). Here, history plays a crucial role, since it provides specific pertinent examples of problems on the basis of which concepts were invented and/or transformed (see, e.g., Commission Inter-IREM 1997; Fauvel and van Maanen 2000, section 7.4.7; Katz and Michalowicz 2004).

More generally, history allows for a deeper analysis of mathematical activities, thus motivating and stimulating research in relation to "activity-based teaching," promoted in the 1990s. A lot of research work in this context consists of determining the issues at stake and the practices adopted concerning mathematical reasoning. They show that rigor and the evaluation of mathematical proof have been subjected to debate and controversy among mathematicians. Actually, fundamental notions like rigor, evidence, and proof have been different in different historical periods; that is, (meta) ideas and (meta) concepts that today are taken for granted in their present form are actually the product of a historical development; there is a *historicity* inherent to them and maybe it is more appropriate to use plural number when referring to them (Barbin and Benard 2007). This fact gave rise to ideas about learning processes of school mathematics. From this point of view, history has what has been called a "*replacement role*" ("*rôle vicariant*" in French) by offering to teachers the possibility to approach and explore pieces of mathematics, which are not included in the official school curricula, and in this way to often replace what is usual with something different and/or unusual.

Since the 1990s, a lot of research has been conducted on number systems, equations, geometrical constructions, the role of technical instruments in mathematics, the history of proof, etc. (Calinger 1996). In addition, the historical study of the range of applicability of concepts has led to a critical analysis of school programs (e.g., on the history of probability and statistics; see Barbin 2010; Katz and Tzanakis 2011, ch. 16).

More recently, many works propose to connect history and semiotics in order to analyze the role of script and figures in the evolution of mathematics, concerning both invention of concepts and the mode of reasoning (see ▶ Mathematical Proof, Argumentation, and Reasoning). There are several international meetings and publication in this context (see, e.g., Hanna et al. 2010).

## Cultural Contributions

In many works it is claimed that the main cultural aspect of history is to provide a different image of mathematics both to teachers and – more importantly – to students, on which their more positive relation with mathematical knowledge can be founded. In fact, history allows placing mathematics in the philosophical, artistic, literary, and social context of a certain period.

Thus, teachers could link mathematics to philosophy, or history; e.g., the history of the concept of perspective, which is also interesting for the teachers of plastic arts, stimulated many works (e.g., Commission inter-IREM 1997). Similarly, the relation between mathematics and literature leads to cultural insights if seen in a historical perspective. This could consist of the intrusion of mathematics into a roman through this roman's human characters, but mathematics could also inspire the subject or the structure of a roman.

Other research activities have shown the way HM leads to the history of science. In particular, reading a text often requires placing it in relation to the author's scientific preoccupations, prejudices, and concerns. Sometimes, the solution of a problem requires establishing passages or developing analogies among different disciplines. It is interesting for ME to study the circulation of problems, concepts, methods, or modes of writing (scripts) between mathematics and other sciences (e.g., see the work on vectors Barbin 2010).

Research on the history of complex numbers constitutes a privileged domain to unfold the different aspects of cultural or interdisciplinary development (see ▶ Interdisciplinary approaches in Mathematics Education). This case articulates and connects mathematics to physics and philosophy; additionally, it questions mathematical invention and the link between reality and the status of mathematical truth (Fauvel 1990; Barbin 2010).

In the last few years, the relation between mathematics and other disciplines has been subsumed in education by the concept of modelization. On this issue, proposals have been put forward that seem to be incompatible with each other if one ignores the different conventions adopted for this concept in its short history. The conception of mathematics as an "experimental science" – also used in education – has given rise to historical reflections, e.g., on the comparison between mathematical and physical experiments.

This multifaceted aspect of the character of mathematical notions and concepts revealed through the work done on the HM supports the idea for an interdisciplinary teaching that has been promoted since 2000. The movement for interdisciplinarity has been officially incorporated in curricula through pedagogical innovations in secondary education, and in this context the interest in the history of science has been stimulated (Barbin et al. 2010).

There are two areas which have been recently developed on the intersection among mathematics, culture, and societies: on the one hand, the history of ME, which forms part of the HM in general, has contributed to research in education proper and has led to several international meetings (see ▶ History of Mathematics Teaching and Learning); on the other hand, the research on ethnomathematics initiated by U. d'Ambrosio makes appeal on history, given that the investigated methods and practices can be traced back to old ones that were transmitted to the present era (see ▶ Ethnomathematics).

## History of Mathematics in the Classroom

There is a gradually increasing number of works introducing a historical aspect in the mathematics classroom (Fauvel 1990); as a consequence, the activities and publications in the context of the HPM Group have been enriched. A comprehensive presentation is given by Fauvel and van Maanen (2000); for subsequent developments, see Katz (2000), Katz and Michalowicz (2004), Shell-Gellasch and Jardine (2005), Knoebel et al. (2007), and Katz and Tzanakis (2011). This does not mean that the research conducted concerns a line of approach of teaching history to students as an independent subject, but rather, to orient the teacher towards enriching his/her teaching by taking into account ideas based on epistemology and history or directly introducing historical elements. The aim of "introducing a historical perspective in mathematics teaching" is not to approach a subject in the classroom or at home in a way completely detached from conventional teaching. Rather, it should be meant as the stimulation of historical or epistemological reflections of the teacher in connection with his/her teaching (Barbin 2010) to give important dates for a concept, to explain its historical significance, to refer and/or read original texts, to solve "historical problems," etc.

Reading original texts allows for a "cultural shock" by directly immersing mathematics into history. Therefore, the majority of researches insist on the necessity to read original texts, not in relation to our present knowledge and understanding, but in the context where they were written. It is this line of approach which becomes a source of "epistemological astonishment" by questioning knowledge and procedures that "have been taken for granted" so far. Thus, reading original texts has a strong virtue of "*reorientation*" ("*vertu dépaysante*" in French). A lot of works in the last 20 years present numerous resources for reading original texts and the variety of activities related to this reading. It gives the opportunity to introduce methods that may not be taught today and/or to compare different methods of solution (e.g., Fauvel and van Maanen 2000; Knoebel et al. 2007; several chapters especially in Swetz et al. 1995; Barbin et al. 2010; Katz and Tzanakis 2011).

The introduction of a historical dimension in ME requires appropriate teacher training, however (Fauvel and van Maanen 2000, ch. 4). Since the creation of the HPM Group, a large number of studies have been devoted on the conditions of such preservice and in-service training. To this end, a large number of monographs and anthologies addressed to students and teachers have appeared in the last 30 years. A direct approach in this context – though not the only one – is to give undergraduate courses based on historical material (e.g., Katz and Tzanakis 2011; Knoebel et al. 2007).

The analysis of numerous teaching experiments has led to specific pedagogical and didactical questions. On one hand, such experiments – like any other pedagogical innovation – aim to have a value by themselves. But, they are not easily reproducible since they depend on the teacher's culture and the resources at his/her disposal. On the other hand, these experiments should be evaluated/assessed in relation to their own objectives, which do not often correspond to the conventional conceptions of evaluation and assessment. These two issues constitute the starting point of new ideas and trends on developing further a historically inspired and epistemologically driven approach to teach specific pieces of mathematics and/or to design mathematics curricula.

## Cross-References


- ▶ Ethnomathematics
- ▶ History of Mathematics Teaching and Learning
- ▶ Interdisciplinary Approaches in Mathematics Education
- ▶ Learning Study in Mathematics Education
- ▶ Mathematical Proof, Argumentation, and Reasoning
- ▶ Problem Solving in Mathematics Education


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# History of Mathematics Teaching and Learning

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## Keywords

Curriculum; General education; Mathematics in the global world; Professional education; Teacher education; Textbooks

## General Education and Professional Training in Mathematics

The first known systematic teaching of mathematics started in the Third Millennium in states of Mesopotamia, where scribal schools – *edubba*, the houses of tablets – prepared the scribes who had to work for the state administration and were required to master writing and accounting techniques. Similar processes are observed in Ancient Egypt. Thus, for a long period, the goal of teaching was professional training.

Mathematics became a subject of general education for the first time in the city states of Greece, when a new class of free citizens governing their state emerged. This form of general education practiced two distinct patterns: (1) rhetoric and dialectic as qualifications for political activity and (2) mathematics as a certain complement. This two-sided general education became later conceptualized as the *trivium* and the *quadrivium*, together constituting the *septem artes liberales*, which became a characteristic of general education in Europe. Professional training, as related to manual work, turned to be practiced by the lower social strata. In countries of Islamic civilization, institutionalized education was limited to basic teaching of reading. Acquiring practical knowledge or studying for a learned profession depended on an individual's decisions.

In European states, two parallel systems were institutionalized yet in premodern times – general education and vocational training provided in private or corporate forms (guilds). Gradually these were transformed into parallel forms of classical secondary schools and socially lower-ranking schools that provided training for commercial and technical professions (their curriculum assigned an important role to applied mathematics). Largely by the end of the nineteenth century, these schools rose in social status and quality and began to rival the classical schools. Internationally, the situation was addressed in a variety of ways. One way was to run parallel types of schools, differing in the degree of teaching languages and sciences. Another way was to integrate both parallel types into one "middle" school but organizing the tracking of students according to their supposedly better abilities or professional expectations.

With the enormous expansion of the educational system from the 1960s in industrialized countries because of social changes and technological advances in the professions, "Mathematics for All" became the goal for the entire pre-college education. Likewise, from the 1980s onward, "Mathematics for All" became a popular conception in developing countries, calling for equal access to quality teaching of mathematics for everybody (Schubring 2012).

## Mathematics in Primary Schooling

Primary schools were often the last to be institutionalized within educational systems, and when they began to be established in the seventeenth–eighteenth centuries, mathematics was not their major focus. Eventually, arithmetic became one of the "three Rs" (along with reading and writing), providing basic education for daily use, which included rudimentary techniques of calculating. The rule of three, with its various applications in converting measures, indicated the highest level of teaching for a long time. Typically, primary school teachers for this subject were poorly prepared.

The situation began to improve in the second half of the eighteenth century, because of the Enlightenment. Significantly, teacher education first became a concern for state initiatives. The term "normal school," predominantly used in many countries from the nineteenth century on, first referred to such state-run teacher education institutions in Austria, in Naples, and from 1795 in France. From the 1780s, teacher seminaries were analogous institutions in various German states. The ideas of the Swiss pedagogue J. H. Pestalozzi (1746–1827) had an enormous influence in Europe, from the early nineteenth century onward; he called to transform dull drill and rote learning into approaches for active methods and to convert the practice of reckoning into a deeper understanding of elementary mathematics.

In the same vein, arithmetic had to be complemented by basic notions of geometry. The German pedagogue F. W. A. Fröbel (1782–1852) developed didactic materials for such geometry teaching. Yet, including geometry into primary school teaching remained highly controversial throughout the nineteenth century; governments feared that pupils – and their teachers – would be too highly educated. Therefore, the initiatives of F. A. W. Diesterweg (1790–1866) for including geometry into teacher training at Prussian seminaries were interrupted. This strict confinement was due to the social status of primary schools: nearly everywhere, they constituted a separate school system for the lower social classes, with schools, curriculum, and teacher education all a world apart from secondary schools. Yet, it was in institutions for teacher training that pedagogical and methodological approaches for teaching (elementary) mathematics first began to be developed.

Only during the twentieth century did primary schools become the first step in a consecutive system, which all students had to pass to continue on in secondary schooling. In this process, the syllabus was reformed, and basic arithmetic was replaced by fundamental concepts of school mathematics. In large measure owing to the New Math and Modern Mathematics Movements in the 1960s, the primary school syllabus became an integral part of the entire school mathematics coursework.

## Mathematics in Secondary Schooling

Secondary schools differentiated from the universities by the first half of the sixteenth century and thus shared with them the same social and professional orientations: to prepare upper social strata for university studies and hence for learned professions. As a consequence, classical languages dominated the secondary schools – of both the Catholic and Protestant educational systems in Western Europe – that were rivalling each other. In the Jesuit colleges, mathematics was reduced – according to their general curriculum, the *Ratio Studiorum* of 1599 – to brief teaching in the last grade (the philosophy grade); the Protestant gymnasia at first taught mathematics as arithmetic in the lower grades and slowly introduced geometry in the upper grades. Since school attendance was not compulsory, the parents were left to decide when their sons would enter the Gymnasium (or college) and which preparation they would get before entering. Sources for France, for example, show that most students left the colleges before the philosophy grade, thus without having experienced any teaching of mathematics.

During the eighteenth century, various developments led to establishing more teachings of mathematics, generally in somewhat rivalling and parallel types of schools, like the *Realschulen* for the middle classes and the *Ritterakademien* for the nobility in various German states, or in colleges as in several Catholic states (by attaching engineer training to existing colleges). As a consequence, mathematics achieved a stronger status in the Gymnasia. Some non-Jesuit orders such as the *Oratoire* in France also taught more mathematics. The next critical step came mid-eighteenth century with the foundation in France of military schools to prepare engineers; there – based on concepts of the Enlightenment – mathematics became the principal teaching subject.

One of the impacts of the French Revolution was the establishment of the first system of public education. Latin and mathematics became the two pillars of general education in French secondary schools. Other countries followed this pattern. In particular, Prussia offered three components of neo-humanist general education: classical languages, history and geography, and mathematics and the sciences. Yet, this strong role of mathematics was not permanently assured: during the nineteenth century, France almost returned to the Jesuit model, while in Germany, only Prussia continued with mathematics as a major teaching subject, and the classical languages dominated other German states (Schubring 1991). Italy, after its unification in 1861, basically assigned mathematics a secondary role.

Characteristic of the various functions that mathematics can assume in a school curriculum was the threefold type of secondary schooling in Germany: the *humanistisches Gymnasium*, with Greek and Latin; the *Realgymnasium*, with only Latin; and the *Oberrealschule*, with no classical language but modern languages. Mathematics was a major subject in all three types but had their different profiles.

In the second half of the twentieth century, the lower and middle grades of the secondary schools typically provided a common curriculum in mathematics for all students. The upper grades, however, often differentiated according to curricular profiles (there, mathematics could be optional or a certain course of mathematics was obligatory).

## Curriculum

It is often believed that the mathematics curriculum has essentially been the same in all countries over the centuries. This belief is based on the similarity of some superficially descriptive terms, like algebra and geometry. In reality, history shows enormous differences in the curriculum among countries, particularly because of diverse epistemological conceptions of school mathematics and methodological approaches to the subject.

From the beginning of a somewhat broadly organized teaching in premodern times, there was already a clear difference between a Euclidean approach to geometry and an anti-Euclidean one, first propagated by Petrus Ramus (1515–1572); later, influenced by him, algebraizing approaches appeared and, even later, during the French Revolution, the analytic ones. The opposition between geometric and algebraic-analytic approaches characterizes the spectrum of school mathematics curricula at the secondary level.

Since secondary schools used to be dominated by classical languages, at least until the end of the nineteenth century, mathematics followed this pattern and likewise emphasized classical geometry – in some countries (England, Italy) even by directly using Euclid's *Elements*. The analytic approach was in general short lived, appearing only at the beginning of the nineteenth century.

Overcoming a static curriculum, which utilized the synthetic methodology of geometry and was unconnected to scientific progress, became the motto of the reform movement, first in Germany and France, and then, directed by Felix Klein and the IMUK (ICMI), of the first international reform movement in the early twentieth century. The reformers wanted functional thinking to permeate the entire curriculum. The introduction of the function concept and the elements of calculus became the characteristics of this reform movement.

From then on, school mathematics tried to keep up a better pace with the progress of mathematics. The main goal of the second international reform movement, from 1959, which was known as the New Math or the Modern Mathematics Movement, was to align school and modern mathematics even more tightly, constructing the curriculum on the basic structures of mathematics. Although later many ideas of this movement were rejected, school mathematics finally became structured, from the primary grades, according to fundamental concepts of mathematics in arithmetic, algebra, geometry, calculus, and, as a recent innovation, probability theory and statistics.

## Textbooks for Mathematics

Throughout the millennia, textbooks constituted the main resource for the teaching of mathematics. In the epochs before the invention of the printing press, the uniqueness of the manuscript, not being reproducible, led to teaching practice consisting of its oral reading to the students. In fact, a genuine qualification for teaching was not even desired: knowledge was regarded as "classic" and canonical; its static character was enhanced by the few existing educational institutions. Striving for new knowledge was even considered suspect, and original productivity appeared primarily in the form of commentaries on canonical textbooks. Moreover, the overall culture of orality enforced the leading role of the textbook and compelled teachers to function as the "organ" of the textbook.

In fact, for extended periods, only two textbooks were broadly used for teaching: Euclid's *Elements of Geometry* (about −300) in Europe and parts of the Islamic civilization and the *Jiu Zhang Suan Shu*, the Nine Chapters of Arithmetic Technique (about −200) in China and East Asia. Although both were likely not been composed as textbooks for teaching, they were used as such. Euclid's text or an uncountable number of diverse adaptations of it constituted the standard material for secondary schools in many European countries, particularly in Catholic colleges where at least its Book I was required.

The printing press stimulated the publication of an enormous number of arithmetic textbooks for practitioners in the vernacular as well as new textbooks for the university and secondary school level. Noteworthy were textbooks algebraizing mathematics, such as Antoine Arnauld's *Nouveaux Élémens de géométrie* (1667) and subsequent works by members of the *Oratoire* in France (Prestet, Reynaud, Lamy). Another trend was textbooks for a mundane public (Clairaut 1741 and 1746).

The establishment of systems of public instruction created new dimensions (Schubring 2003). Following its centralistic policy, France first assigned only one and then later a very limited number of textbooks for the entire country. Few authors, like S. -F. Lacroix, became entrepreneurs, dominating the schoolbook market. Other countries, like neo-humanist Prussia, emphasized the autonomy of the teacher with regard to method and let him choose his textbook. Textbook writing was provided according to the respective values of education either mostly by university mathematicians (France, Italy, and in some periods Russia) or mostly by school teachers (Germany). In a few cases, some textbooks, like Legendre's book on geometry, continued to be used internationally. Predominantly, however, textbooks were now published exclusively for use in their respective countries. The former type of single book for a teacher and his students gave way to more differentiated sets including schoolbooks for students, methodical guides for teachers, collection of problems and exercises, and booklet with solutions.

## Mathematics Learning for Girls

Even when primary education was available for girls, they were for a long time excluded from attending public secondary schools. The feminist movement in the second half of the nineteenth century was instrumental in establishing eventually separate schools for girls. The history of such schools in various countries has been poorly studied. These first schools offered fewer career opportunities for girls than for boys; in particular, mathematics only played a minor role, given the persistently strong prejudices negating women's ability to understand mathematics. At best, they were attributed intuition instead of abstract thinking. The curriculum for these schools thus focused on intuitively accessible geometric concepts. Secondary schools created for girls in Italy in 1923 featured "drawing" as the only subject with some remote mathematical kinship. In Nazi Germany, the curriculum for girls became reduced even further, focusing only on those geometric forms which might be of use in the household.

By the social reforms of the 1960s and the expansion of the secondary schools, the girls' schools merged with the boys' schools almost everywhere, and both girls and boys were taught the same curriculum. No longer did the curriculum maintain a female inferiority in mathematical thinking.

## Teachers of Mathematics

The professionalization and special training of mathematics teachers are recent developments. For a long time, teachers used to be self-taught persons, practitioners, or generalists. The first time teacher training became institutionalized in primary schools (see Section "Mathematics in Primary Schooling"). For Catholic secondary schools, the various religious orders practiced rudimentary forms of training for their novices; for Protestant schools, largely the graduates of the Theology Faculty came teaching to the schools when they could not find a parish (then they taught mainly classical languages). For teaching arithmetic, Gymnasia used to hire a practitioner. The first specialized teachers of mathematics at the *Gymnasia* are known in the early eighteenth century only (in the kingdom of Saxony).

France did not establish teacher education even after the Revolution and left it to the individual's preparation for a *concours*. Later on, the *École normale supérieure* prepared candidates for this *concours*, the *agrégation*. It was Prussia that reformed its Philosophy Faculty from 1810 by charging it with the scientific formation of teachers, particularly in mathematics. From the 1820s, this education was complemented by a subsequent probationary year for training in the teaching practice (Schubring 1991). While various profiles of scientific formation emerged for future mathematics teachers in different countries, the basic problem remained: How would qualification in mathematics be complemented by qualification in teaching practice? A good overview of the situation during the first half of the twentieth century is provided by the international reports of the IMUK/CIEM at the 1932 Congress of Mathematicians (see *L'Enseignement Mathématique* vol. 32, 1933, 5–22).

Only during the 1970s did a broader concept of professional qualifications become established in numerous countries, now including pedagogical qualifications and studies in mathematics education in the university, followed by probationary training in schools. In some countries, the education of teachers for the primary grades was elevated to university level; yet, it remained largely unspecialized for mathematics and included preparation for teaching various subjects.

## Mathematics in the Global World

Mathematics has been created and developed around the world, and each culture made its own distinctive contribution to its development (D'Ambrosio 2006). The modern system of mathematics education, however, for all the variety that it exhibits across different countries, owes a great deal to structures and conceptions that emerged during the first half of the sixteenth century in Western Europe due to specific economic, social, and cultural developments and were decisively shaped later, during the time of the Enlightenment. The manner in which Western Europe's influence spread and took root around the world varied from country to country.

In Japan, the Meiji Restoration (officially announced in 1868) led to Westernization and ushered in the broad use of foreign textbooks and the recruitment of foreign teachers (Ueno 2012). In China, as has been noted, the development of mathematics and of the teaching of mathematics has a history that is many centuries long, and at certain stages China was far ahead of the countries of Europe. By the nineteenth century, however, China was clearly and appreciably falling behind in science and technology, which led to its defeat in a number of wars. The response to these defeats was modernization, which may be to a certain extent equated with Westernization: new educational institutions began to appear and new programs and methods of teaching, borrowed from the West, began to be used (Chan and Siu 2012).

The Ottoman Empire's system of mathematics education developed in a largely similar way. While in the countries that made up this empire interest in astronomy and mathematics, and consequently in an education in these subjects based on Arab sources, was noted by travelers as early as the eighteenth century, a crucial step was taken with the establishment of national military schools, in which the teaching of mathematics was conducted in accordance with European models (Abdeljaouad 2012).

Another pattern is exemplified, for example, by Tunisia, which at one time belonged to the Ottoman Empire – European-type schools were later set up here by French colonial authorities (Abdeljaouad 2014). Such a pattern was also characteristic of many other countries in Africa, Asia, and Latin America: European colonial authorities established schools for European settlers, as well as for a narrow segment of local elites, thereby nonetheless introducing into these countries more modern mathematics education practices – making use of European textbooks, exams, methods of teaching, and either European teachers or at least teachers who had been trained in Europe.

Note that the process of borrowing from other countries was not always unproblematic. Researchers have pointed out that, for example, "without the brutal intrusion of Western powers, development of the Chinese culture in the political, social and scientific arenas may have achieved a totally different but harmonious existence" (Chan and Siu 2012, p. 471). Even in Russia, where active employment of Western European teaching materials and teachers began as early as the first half of the eighteenth century, foreign influences in education were not infrequently later perceived as hostile (Karp 2006). Mathematics education was often part of political discussions.

The complicated process in which national systems of mathematics education were formed in developing countries is part of more recent history. Only very gradually did a national work force of teachers and centers for their preparations began to appear in these countries, along with textbooks and teaching materials. The colonial powers left these countries largely illiterate and mathematically illiterate. The development and often even the establishment of an education system based on practices available in the world and aimed not at an elite, but at all students, was and in many instances remains a crucial problem. Such international organizations as UNESCO, as well as separate countries, including countries belonging to hostile political blocs, have provided assistance with the development of education, including mathematics education. In the process, distinctive local features were quite frequently ignored (Karp 2013). Meanwhile, the preservation of indigenous and culturally specific features is particularly important in the context of increasing tendencies toward globalization.

## Research into History of Mathematics Education as a Field

The history of mathematics education as a scholarly field is still in the process of formation. To be sure, many significant studies were conducted as early as the nineteenth or early twentieth centuries (see International Bibliography); for example, the first doctoral dissertations in mathematics education in the United States completed in 1906 under the supervision of David Eugene Smith were devoted precisely to the history of mathematics education.

In recent decades, the development of mathematics education as a scientific discipline (Kilpatrick 1992) has led to a growing interest in its history. This is attested to by the creation of special ICME Topic Study Groups, the appearance of a special journal devoted to the history of mathematics education, the organization of special scientific conferences, etc. (Furringhetti 2009). All such activity facilitates the formation of shared standards of research and methodology.

The history of mathematics education, like any historical discipline, is based first and foremost on the analysis of primary sources. It is important, however, to conceive of these sources in a sufficiently broad manner, not limiting research to "administrative history" (Schubring 1988) – that is, the history of decrees concerning education or even standards and curricula. Objects and sources of study include textbooks, students' notebooks, exam questions and answers, complaints and their analysis, biographical documents, diaries, letters, memoirs, journalistic, and even imaginative writing.

Perhaps even more important is not stopping at a purely descriptive approach: that is, to seek not only to establish the events that have taken place but also to understand their position in the context of other events and social historical processes. The very choice of what to teach or offer on exams is evidently determined not only by strictly mathematical but also by social considerations, whose meaning and content must be elucidated (Karp 2011). The role and place of the mathematics teacher and of the subject of mathematics itself; the interaction between higher and secondary education; the mutual influences among various cultures in teaching; the causes of, attempts at, and outcomes of reforms – these and other areas of research are today the most worthy of study.

Special mention should be made of the importance of research in the history of mathematics education in developing countries. Usually little is known about education in these countries during the pre-colonial period, yet mathematics was in one way or another a part of culture everywhere. Nor have interactions between local cultures and various European cultures been sufficiently studied, even though education in the colonies of different European countries was by no means identical. Nor was the formation of mathematics education during the postcolonial period in these countries everywhere alike. Research in these directions must continue.

## Cross-References


- ▶ Curriculum Resources and Textbooks in Mathematics Education
- ▶ Gender in Mathematics Education
- ▶ History of Research in Mathematics Education
- ▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes
- ▶ Professional Learning Communities in Mathematics Education


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# History of Research in Mathematics Education

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## Keywords

Research; Mathematics education; History; Academic field; Psychology; Mathematics

## Definition

An account of activities and events concerned with the development of disciplined inquiry in mathematics education as a flourishing academic enterprise.

## Main Text

Although mathematics has been taught and learned for millennia, not until the past century or so have the nature and quality of teaching and learning mathematics been studied in any a serious manner. Clay tablets from ancient Babylonia (c 1900 BC to c 1600 BC), for example, show that students in the scribal school were expected to solve problems involving quadratic polynomials (Høyrup 1994, pp. 4–9), but no available evidence indicates how much drill and practice either they received or their instructors thought they needed. As of 1115 BC, applicants to the Chinese civil service had to pass an examination in arithmetic (Kilpatrick 1993, p. 22), but as far as anyone knows, no one ever investigated how well their examination performance predicted their job performance. In Plato’s *Meno*, he relates how, in the fifth century BC, Socrates helped a slave boy discover that doubling the side of a square apparently squares its area. Plato does not, however, say how well the boy fared with similar geometry problems once his teacher was no longer around. Mathematics education is a long-established field of practice; research in mathematics education, a relatively recent enterprise.

Over the centuries, teachers of mathematics in various countries have offered reflective accounts of their work, often writing textbooks constructed around teaching techniques they developed out of their own experience. Only during the nineteenth century, however, as national educational systems were established and the training of teachers moved into colleges and universities, did people begin to identify themselves as mathematics educators and begin to conduct research as part of their scholarly identity (Kilpatrick 1992, 2008). Not until 1906 were the first doctorates in mathematics education granted – to Lambert L. Jackson and Alva W. Stamper, students of David Eugene Smith at Teachers College, Columbia University (Donoghue 2001). Within the next few decades, research in mathematics education gradually began to be conducted in several countries as lectures in mathematics education were offered and graduate programs in mathematics education became established in universities.

## Mathematics Education as an Academic Field

The education of teachers, which had often been a hit-or-miss affair, did not become a field of professional studies until the nineteenth century. Although teacher-training schools had begun in France and Prussia late in the seventeenth century, only in the eighteenth century were normal schools – very much influenced by the ideas of the Swiss pedagogue and reformer Johann H. Pestalozzi – established in European countries (Cubberley 1919). In 1829, the American geographer William C. Woodbridge, who in the previous 4 years in Europe had observed schools in Prussia and Switzerland and had visited Pestalozzi, tried unsuccessfully to establish in Hartford, Connecticut, a teachers seminary modeled after the Prussian version. In 1831, he observed: "In those of the countries of Europe where education has taken its rank as a science, it is almost as singular to question the importance of a preparatory seminary for teachers, as of a medical school for physicians" (quoted by Cubberley 1919, p. 374). Education in general had slowly been entering the university since the eighteenth century, beginning with a chair of education established at the University of Halle in 1779, but not until the late nineteenth and early twentieth centuries were such chairs established elsewhere, and only then did school mathematics start to become an object of scholarly study (Kilpatrick 2008).

Many of the early researchers in mathematics education were mathematicians who had become interested in how mathematics is done. For example, the editors of *L'Enseignement Mathé matique*, Henri Fehr and Charles-Ange Laisant, sent a questionnaire to over 100 mathematicians to learn how they did mathematics. The report of their survey, which was published in 11 installments in the journal from 1905 to 1908, was essentially a list of verbatim responses to their questions. In contrast, the French mathematician Jacques Hadamard later undertook a similar but less formal inquiry into the working habits of mathematicians in America that went somewhat deeper into the methods and images they used (Kilpatrick 1992). Other early researchers were psychologists who were developing an interest in how children think about and learn mathematical ideas. Beginning in 1875, with Wilhelm Wundt's establishment of a laboratory in Leipzig and William James's establishment of one at Harvard, dozens of psychological laboratories were established in Europe, Asia, and North America (Kilpatrick 1992). Psychologists such as Alfred Binet, his colleague Jean Piaget, Max Wertheimer, Otto Selz, and Lev Vygotsky investigated mental ability and productive thinking using mathematical tasks. Psychology was becoming the so-called master science of the school: "Psychology . . . became the guiding science of the school, and imparting to would-be teachers the methodology of instruction, in the different school subjects, the great work of the normal school" (Cubberley 1919, p. 400). Together, mathematicians and psychologists began the efforts that would lead to research in mathematics education.

## Comparative Studies of School Mathematics

In 1908, the International Commission on the Teaching of Mathematics (ICTM) was formed at the Fourth International Congress of Mathematicians in Rome. Its purpose was "to report on the state of mathematics teaching at all levels of schooling around the world" (Kilpatrick 1992, p. 6). In 1912, at the Fifth International Congress in Cambridge, England, some 17 countries presented reports, and by 1920, the countries active in the ICTM had produced almost 300 reports (Schubring 1988; Furinghetti 2008). The international comparisons based on these reports, however, were essentially restricted to descriptions by a handful of mathematicians or educators in each country of activities that they were aware of. They did not engage in large-scale, systematic surveys of the school mathematics curriculum, nor did they visit classrooms to record instructional practices. Nonetheless, they had begun the process of looking across countries to get a better perspective on mathematics education around the world.

In the last half century, researchers have undertaken a variety of international comparative assessments of students' mathematical knowledge and of teachers' knowledge of pedagogy and mathematics. They have also compared mathematics teaching across countries using video records of lessons. (For an analysis of the levels at which these comparisons have been made, see Artigue and Winsløw 2010). Considerable progress has been made in both the thoroughness with which such comparative studies have been done and the sophistication of the data collection and analyses. Although these studies can be criticized for being too oriented toward Western practice and inadequately sensitive to Asia-Pacific cultures (Clements and Ellerton 1996), they have had, in many countries, considerable influence on curriculum, teaching, and educational policy. For an account of the development of international collaboration in mathematics education during the past century, see Karp (2013).

## Becoming Scientific

In trying to make their field scientific, educational psychologists looked to the natural sciences for models, and in much the same way, some mathematics educators seeking to establish their field as a science took those sciences as models. They studied mathematics learning under controlled laboratory conditions, testing hypotheses about the effects of various "treatments," and making careful measurements of the learning achieved. Influential examples were studies by the psychologist Edward L. Thorndike in the early years of the twentieth century. Using a control group whose performance was compared with that of an experimental group (with students assigned randomly to one of the two groups), Thorndike demonstrated that practice by the experimental group in performing certain tasks such as judging the size of rectangles did not improve their performance in – that is, did not transfer to – judging the size of triangles (Kilpatrick 1992). Thorndike's research studies dealt a major blow to arguments that mathematics ought to be taught and learned because the logical thinking it promoted transferred to other realms. He argued that his research showed that transfer was much more limited than mathematics teachers appeared to assume.

Thorndike not only published important books on the psychology of arithmetic and the psychology of algebra in which he promoted the psychology he termed connectionism; he also published a series of arithmetic textbooks that was widely used in schools. Connectionism became the forerunner of the behaviorism that came to dominate much of research in mathematics education in the United States from the 1930s through the 1950s (Clements and Ellerton 1996). Although other psychologists, such as Charles H. Judd, Guy T. Buswell, and William A. Brownell, performed research studies that called Thorndike's work into question, thereby developing a psychology of the school subjects that mathematics educators found more congenial (Kilpatrick 1992), connectionism and its successor behaviorism exerted a much stronger influence on research methodology in mathematics education for many years and not just in the United States.

Elsewhere in the first decade of the twentieth century, some psychologists were looking at errors and difficulties that children were having in arithmetic. Paul Ranschburg in Budapest, in particular, began the study of differences in calculation performance between normal children and low achievers in arithmetic. In 1916, he coined the term *Rechenschwäche* (dyscalculia) for severe inability to perform simple arithmetic calculations (Schubring 2012). Like Thorndike, Ranschburg attributed children's successful performance to their possession of *Vorstellungsketten* (chains of association), but his research method relied more on observation of differences between existing groups (normal and low achieving) than on experimentation.

Psychologists gradually stopped being so concerned about emulating the natural sciences and began to develop their own techniques for studying learning, and researchers in mathematics education followed. For example, in the movement known as "child study" (Kilpatrick 1992), which had appeared in Germany and the United States at the end of the nineteenth century, researchers looked at the development of concepts in young children using techniques of observation and interview. Although mathematics was not often the focus of child study research, it did give rise to a number of descriptive, naturalistic studies. Less than a century later, research on the learning of mathematics had burgeoned. A survey in the 1970s, for example, located some 3,000 published studies of mathematics learning (Bauersfeld 1979).

## Studying the Teaching of Mathematics

As mathematics educators began to study children's mathematics learning and thinking, they increasingly recognized that laboratory studies present a restricted view of those processes; however, they are conceived. Children do most of their learning of mathematics in school classrooms along with other children, and their thinking about mathematical concepts and problems is much influenced by others, including their teacher. The psychologist Ernst Meumann, who had studied with Wundt in Leipzig, was one of the first to address what he called "experimental pedagogy" and in 1914 published a volume in which he looked at the didactics of teaching specific school subjects (Schubring 2012). Meumann was the forerunner of researchers who were later in the century to establish a critically important field of research, especially in Germany and France: the *didactics of mathematics* (Artigue and Perrin-Glorian 1991; Biehler et al. 1993). Although the didactics of mathematics began with a psychological orientation, it came under the influence of other fields – anthropology and philosophy, in particular – as it was increasingly located in university departments of mathematics and began to become established as one of the mathematical sciences.

Didactics of mathematics, however, was not the only research effort to address mathematics teaching. In a number of studies conducted in the first half of the twentieth century, components of teaching or characteristics of teachers were linked to learners' performance in efforts to understand what might constitute effective teaching. Researchers eventually moved from such simple "process-product" models to more sophisticated efforts that attempted to capture more of the complexity of the teaching-learning process, including the knowledge and beliefs of the participants as well as their activities during instruction. For an account of the gradual elaboration of research models for studying mathematics teaching, see Koehler and Grouws (1992).

In later developments, researchers attempted to go deeper into questions of what constitutes classroom practice in mathematics and how that is experienced by teachers and learners. In particular, they studied how discourse is structured in mathematics classes, how norms are established in classrooms for learning and doing mathematics, and how teachers and learners build relationships based on getting to know each other (Franke et al. 2007). Research on teaching and teachers has become a major strand of current research in mathematics education, and those studies now extend from preschool to tertiary instruction.

An especially fertile development of recent decades has been the growth of research on technology and digital environments for mathematics teaching and learning. Physical tools have been used for centuries to assist the teaching and learning of mathematics, and an examination of how those tools have been used can help put into perspective the use of computing technology today (Roberts et al. 2013). In an early review of how electronic technologies had been studied in mathematics education research, Kaput and Thompson (1994) lamented the paucity of technology-related research publications. That situation has changed dramatically since that review, as numerous recent books (e.g., Guin et al. 2005; Hoyles and Lagrange 2010) and journals (e.g., *International Journal for*

*Technology in Mathematics Education*; *Journal of Computers in Mathematics and* Science Teaching) attest.

## A Flourishing Academic Enterprise

The last half century has witnessed a growing flood of research activity in mathematics education that has been an integral part of its growth and development:

> Today an astonishing profusion of books, handbooks, proceedings, articles, research reports, newsletters, journals, meetings, and organizations is devoted to mathematics education in all its aspects. A search of the scholarly literature on the Web for the phrase *mathematics education* yields 125,000 hits; a search of the entire Web yields almost 9 times that number. (Kilpatrick 2008, p. 38).

One measure of the maturation of the field of mathematics education is that researchers have begun to study its history. A major milestone was the founding in 2006 of the *International Journal for the History of Mathematics Education*. The history of the field had been discussed at various international conferences beginning in 2004, and a series of biennial conferences devoted to the topic began in Iceland in 2009.

As the field of mathematics education has grown, research in the field has grown even faster. The subject matter of research studies has broadened to include such topics as the school mathematics curriculum, assessment in mathematics, the education of mathematics teachers and their professional development, the sociopolitical context of learning and teaching mathematics, teaching mathematics to students in special education programs, and the politics of mathematics education. The methods used to conduct research now go well beyond experimentation to include case studies of teachers and students, surveys of attitudes and beliefs, and ethnographies of cultural practices.

Organizations of researchers have been formed that range from those of international scope, such as the International Group for the Psychology of Mathematics Education (IGPME, or PME), to organizations within one or several countries, such as the Canadian Mathematics Education Study Group (CMESG), the French *Association pour la Recherche en Didactique des Mathématiques* (ARDM), and the Mathematics Education Research Group of Australasia (MERGA). For a comprehensive survey of international or multinational organizations in mathematics education, see Hodgson et al. (2013). Many of these organizations hold regular conferences on research and publish research journals. Mainstream journals that have been publishing research for more than four decades, such as *Educational Studies in Mathematics* and the *Journal for Research in Mathematics Education*, have lately been joined by more specialized research journals such as the *Journal of Mathematics and Culture*, started in 2006, and the *Journal of Urban Mathematics Education*, started in 2008. For an account of the growth of journals and research conferences in mathematics education, see Furinghetti et al. (2013). The sheer volume of research activity being reported in these journals and at these conferences is staggering. A comprehensive portrayal of research activity in mathematics education today is no longer possible; the terrain is simply too extensive and diverse to be captured in toto.

## Cross-References

- ▶ Anthropological Approaches in Mathematics Education, French Perspectives
- ▶ Constructivist Teaching Experiment
- ▶ Design Research in Mathematics Education
- ▶ History of Mathematics Teaching and Learning
- ▶ International Comparative Studies in Mathematics: An Overview
- ▶ Teacher as Researcher in Mathematics Education
- ▶ Theories of Learning Mathematics

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## Hypothetical Learning Trajectories in Mathematics Education

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### Keywords

Learning; Teaching; Constructivism; Teacher thinking; Learning progressions

### Definition

Hypothetical learning trajectory is a theoretical model for the design of mathematics instruction.

It consists of three components, a learning goal, a set of learning tasks, and a hypothesized learning process. The construct can be applied to instructional units of various lengths (e.g., one lesson, a series of lessons, the learning of a concept over an extended period of time).

### Explanation of the Construct

Simon (1995) postulated the construct *hypothetical learning trajectory*. Simon's goal in this heavily cited article was to provide an empirically based model of pedagogical thinking based on constructivist ideas. (*Pedagogical* refers to all contributions to an instructional intervention including those made by the curriculum developers, the materials developers, and the teacher.) The construct has provided a theoretical frame for researchers, teachers, and curriculum developers as they plan instruction for conceptual learning.

Simon (1995. P. 136) explained the components of the hypothetical learning trajectory:

> The hypothetical learning trajectory is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process – a prediction of how the students' thinking and understanding will evolve in the context of the learning activities.

There are a number of implications of this definition including the following:

- Good pedagogy begins with a clearly articulated conceptual goal.
- Although students learn in idiosyncratic ways, there is commonality in their ways of learning that can be the basis for instruction. Therefore, useful predictions about student learning can be made.
- Instructional planning involves informed prediction as to possible student learning processes.
- Based on prediction of students' learning processes, instruction is designed to foster learning.
- The trajectory of students' learning is *not* independent of the instructional intervention used. Students' learning is significantly affected by the opportunities and constraints that are provided by the structure and content of the mathematics lessons.

To elaborate the last point, the second and third components of the hypothetical learning trajectory, the learning activities and the hypothetical learning process, are interdependent and co-emergent. The learning activities are based on anticipated learning processes; however, the learning processes are dependent on the nature of the planned learning activities. Clement and Sarama (2004a, p. 83) reaffirmed this point.

> Although studying either psychological developmental progressions or instructional sequences separately can be valid research goals, and studies of each can and should inform mathematics education, the power and uniqueness of the learning trajectories construct stems from the inextricable interconnections between these two aspects.

They went on to define learning trajectories as follows.

> We conceptualize learning trajectories as descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain (c.f. Clements 2002; Gravemeijer 1999; Simon 1995) (p. 83).

According to Simon (1995), a hypothetical learning trajectory was part of a *mathematics teaching cycle* that connects the assessment of student knowledge, the teacher's knowledge, and the hypothetical learning trajectory. The cycle is meant to capture a progression in which an instructional intervention is made based on the hypothetical learning trajectory. Student knowledge/thinking is monitored throughout. This monitoring leads to new understandings of student thinking and learning, which, in turn, leads to modifications in the hypothetical learning trajectory. The mathematics teaching cycle also stresses that, in the context of teaching, teachers develop additional knowledge of mathematics and mathematical representations and tasks. All modifications in teacher knowledge contribute to changes in the revised hypothetical learning trajectory. Thus, an implication of the mathematics teaching cycle is that a big part of good teaching is the ability to analyze student learning in order to revise the instructional approach.

The mathematics education research community picked up the hypothetical learning trajectory construct, and 9 years after the original article, Clements and Sarama (2004b) edited a special issue of *Mathematics Thinking and Learning* on hypothetical learning trajectories. Although the hypothetical learning trajectory construct grew out of constructivist ideas, it has been adapted for use with social learning theories (e.g., McGatha et al. 2002).

Two lines of research grew out of the original work on hypothetical learning trajectories. The first, conducted by Simon and his colleagues, is an attempt to explicate the mechanisms of conceptual learning, that is, to provide a framework for generating hypothetical learning processes in conjunction with learning activities. (See Tzur and Lambert 2011; Simon et al. 2010; Tzur 2007; Simon et al. 2004; Simon and Tzur 2004; Tzur and Simon 2004). Whereas research grounded in constructivist ideas has a tradition of modeling students' thinking at various points in their conceptual learning, postulation of the hypothetical learning trajectory construct called for modeling the learning process itself, the means by which the students' thinking changes as they interact with the instructional tasks and setting.

The second line of research, which grew out of the original hypothetical learning trajectory work, is research on learning trajectories in mathematics (also referred to as "learning progressions"; see discussion of learning progressions in this volume). Learning progressions research is an attempt to provide an empirical basis for instructional planning.

> Trajectories involve hypotheses both about the order and nature of the steps in the growth of students' mathematical understandings and about the nature of the instructional experience that might support them in moving step-by-step toward their goals of school mathematics (Daro et al. 2011, p 12).

Not only have a significant number of researchers gotten involved in this line of research, but the *Common Core Standards* (CCSSO/NGA 2010) in the United States has leaned heavily on the learning progressions work to date. A key issue as research on learning progressions develops is whether a central idea in Simon's hypothetical learning trajectory will be maintained. That is, will the learning process continue to be seen as interrelated with the instructional approach or will various stakeholders in mathematics education seize on particular learning progressions as *the* way that students learn. The quote above from Daro et al. seems to imply that there is a set of learning steps, and then instruction is built to foster that sequence of steps. This stands in contrast to a view that any particular sequence of steps is in part a product of the instructional experiences provided to the students. Clements and Sarama pointed to an important implication of the perspective based on Simon's original definition:

> Thus, a complete hypothetical learning trajectory includes all three aspects. . . . Less obvious is that their integration can produce novel results. . . . The enactment of an effective, complete learning trajectory can actually alter developmental progressions or expectations previously established by psychological studies, because it opens up new paths for learning and development.

## Cross-References

▶ Constructivism in Mathematics Education

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## Immigrant Students in Mathematics Education

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### Keywords

Diversity; Multicultural; Multilingual; Immigration; Ethnomathematics

### Definition

"Immigrant students" refers to the case when students (or their parents) were born in a country other than the one they are currently living in and attending school.

### Characteristics

The topic of the mathematics education of immigrant students has become quite prominent in different parts of the world. An indication of this is that one of the survey teams at the 11th International Congress on Mathematical Education (ICME) in 2008 focused on mathematics education in multicultural and multilingual environments. One of the four themes under this survey team was "the mathematics teaching and learning of immigrant students." This article is largely based on the work undertaken to address this theme for that survey team (Civil 2012). The pressing need to address the mathematics education of immigrant students is reflected in the following quote by Gates (2006):

> In many parts of the world, teachers – mathematics teachers – are facing the challenges of teaching in multiethnic and multilingual classrooms containing immigrant, indigenous, migrant, and refugee children, and if research is to be useful it has to address and help us understand such challenges. (p. 391)

This quote mentions four diverse groups – immigrant, indigenous, migrant, and refugee children. The research reviewed for this article will focus on immigrant students. However, the research with indigenous communities contains much relevant information to the teaching and learning of immigrant students. One example is the work from an ethnomathematics perspective that emphasizes engaging indigenous communities in the development of the teaching and learning of mathematics, hence bringing in the communities' knowledge, experiences, and approaches as valuable resources (Meaney 2004; Lipka et al. 2005).

It is important to acknowledge that there is large diversity among immigrant students. This article focuses on some general characteristics that are likely to impact the mathematics teaching and learning of low-income, immigrant students, whose first language is different from the language of schooling in

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the receiving country. It is organized around five themes: educational policy and immigration, different forms of mathematics, teacher education in an immigration context, multilingualism and mathematics teaching and learning, and immigrant parents' perceptions of mathematics education.

## Educational Policy and Immigration

It is important to understand the educational policies in place with regard to the education of immigrant students. Whether those are grounded on seeing immigrants as a resource or as a problem is likely to affect the schooling experiences of immigrant children. In their account of the many faces of migration in the world, King, Black, Collyer, Fielding, and Skeldon (2010) discuss two prominent models of integration, multiculturalism and assimilation. They note that while multiculturalism may have been the model in some European countries, more recently "a swing back to assimilation has occurred, with greater demands on immigrants to learn the host-country language and subscribe to core national values" (p. 92). The research addressing the mathematics teaching and learning of immigrant students underscores the potential negative impact of some educational policies and of a general public discourse that frames immigration as a problem (Civil 2012). Such a framing is likely to affect teachers who may view the diversity of approaches to doing mathematics that immigrant students often bring (e.g., different algorithms) as problematic rather than as an opportunity to learn. More research is needed to examine the possible connections among educational policies, public views on immigration, and the mathematics education of immigrant children. The complexity of the situation calls for interdisciplinary teams, where in addition to the expected expertise in mathematics and mathematics education, there is expertise on the political and policy (social, educational, language, in particular with respect to immigrant students) scene in the context (country, region) of work.

## Different Forms of Mathematics

The relationship between mathematics and culture/context has been widely described (Bishop 1991; Nunes et al. 1993; DiME 2007; Presmeg 2007; Abreu 2008). This body of research stresses that mathematics is not culture-free and illustrates the complexity of the relationship between different forms of mathematics, in particular between in- and out-of-school mathematics. Immigrant students are quite likely to bring with them different ways of doing mathematics. These differences may be obvious, such as using different algorithms for arithmetic operations, or subtler, such as emphasis of topics studied. Immigrant students may have also experienced different pedagogical approaches from those in the receiving country (e.g., teacher lecturing vs. group work). Depending on their context of immigration, they may bring approaches that are more related to out-of-school mathematical practices. Issues related to the gap between in-school mathematics and out-of-school mathematics and transitions across contexts are well documented (de Abreu et al. 2002; Nasir et al. 2008; Meaney and Lange, 2013).

The research surveyed in Civil (2012) from different countries points to some general findings concerning these different forms of mathematics. One such finding is that schools and teachers are often not familiar with the mathematical knowledge that immigrant children may bring with them. A belief that mathematics is universal and culture-free may lead teachers to not see these different forms and focus only on the different languages at play (home and school) as the main issue that affects immigrant students' learning of mathematics. Another finding is related to the concept of valorization of knowledge (Abreu and Cline 2007). That is, different forms of mathematics may be given different valorization, and it is often the case that immigrant children's mathematical knowledge may not be valued as much as the "expected" mathematical knowledge in the given school context. Even in the cases in which teachers are aware of these different forms of mathematics, they may not have the appropriate background knowledge, preparation, or support to develop learning experiences that reflect and build on these different forms. Thus, this finding underscores the need to make sure that teacher education programs prepare teachers to not only acknowledge different approaches to doing mathematics but also to learn how to build on those in ways that are inclusive for immigrant students.

## Teacher Education in an Immigration Context

The research surveyed on this topic addresses teachers' attitudes, beliefs, and knowledge with respect to the teaching and learning of immigrant students. Overall, teachers feel unprepared to address the mathematical learning needs of immigrant students (Favilli and Tintori 2002). As mentioned earlier, language seems to be the main factor of concern for teachers. However, in the different studies surveyed, researchers point to other areas that should be addressed when working with teachers of immigrant students. One such area is the need to pay more attention to the cultural nature of learning (Abreu 2008). Another area is the need to confront deficit views towards immigrant students. These views are often grounded on public discourse about immigration rather than on a direct knowledge of the students and their families and can lead to teachers not valuing the mathematical knowledge that immigrant students bring with them (Alrø et al. 2005; Gorgorió and de Abreu 2009; Planas and Civil 2009).

One approach to engaging teachers in learning about their immigrant students and their families is based on the concept of funds of knowledge (González et al. 2005). Through ethnographic home visits, teachers learn about their students' and families' experiences, knowledge, and backgrounds. They can then build on this knowledge in their classroom teaching. In Civil and Andrade (2002), this approach is applied to the teaching and learning of mathematics.

Although there is considerable research in teacher education and diversity in general terms (not necessarily specific to mathematics), still we know little about how effective teachers for diverse students developed their knowledge and dispositions (Hollins and Torres Guzman 2005). In mathematics teacher education, although there is a large body of research addressing teachers' mathematical knowledge and beliefs about teaching and learning mathematics, there seems to be little research about teachers' beliefs about equity, in particular in the areas that are likely to apply to immigrant students (race, culture, ethnicity, language, and socioeconomic background) (Forgasz and Leder 2008). Efforts in mathematics teacher education need to emphasize that mathematics is not culture-free and may have to be more upfront in engaging teachers to discuss topics that are likely to create discomfort and may lead to resistance to diversity (Rodriguez and Kitchen 2005; Sowder 2007).



## Multilingualism and Mathematics Teaching and Learning

As mentioned before, for education policymakers and many teachers and school personnel, limited knowledge of the language of instruction seems to be the main (if not the only) obstacle that immigrant students need to overcome. Thus, different educational systems across a variety of countries attempt to address "the language problem" through systems that segregate immigrant students for all or part of the day to focus on learning the language of instruction. Researchers in the teaching and learning of mathematics with immigrant students raise questions about the implications of these language policies on the learning of mathematics (Alrø et al. 2005; Civil 2011; Barwell 2012; Setati and Planas 2012). Barwell (2012) provides an overview of some of the key themes in multilingual mathematics classrooms through a discussion of four tensions. One such tension is around school language and home languages. Researchers in mathematics education in multilingual classrooms call for a focus on the strengths that multilingualism provides rather than on the fact that immigrant students may lack proficiency in the language of schooling (Moschkovich 2002; Barwell 2009; Clarkson 2009). Research shows the complexity behind code-switching and language choice in mathematics classrooms (Adler 2001; Moschkovich 2007; Planas and Setati 2009) and how code-switching is actually a resource towards students' learning of mathematics rather than a deficit. This body of research also points to the need to develop models of teaching in multilingual mathematics classrooms that are not based on a monolingual view of teaching and learning mathematics (Clarkson 2009).

A focus on language as an obstacle may prevent teachers from seeing the mathematical knowledge that immigrant students bring with them. An important question to consider is to which extent placement decisions in mathematics classes are based on students' knowledge of this subject or on their level of proficiency in the language of instruction (Civil 2011; Civil et al. 2012).

## Immigrant Parents' Perceptions of Mathematics Education

Immigrant parents also concur in naming language as the main obstacle towards their children's learning of mathematics. A question to raise is whether a focus on language as the main obstacle to overcome may prevent parents from assessing their children's overall mathematical experience in school (Civil 2011; Civil and Menéndez 2011).

The research reviewed in Civil (2012) with immigrant parents in some European countries and in the USA points to some common themes despite the diversity in countries of origin. Three related perceptions stand out (see also Abreu 2008 for some similar themes): (1) a lack of emphasis on the "basics" (e.g., learning of the multiplication facts) in the receiving country, (2) a higher level of mathematics teaching in their country of origin, and (3) schools as less strict in the receiving country (i.e., discipline, homework). Underlying these perceptions is the concept of valorization of knowledge, which affects teachers as well as parents.

These perceptions underscore the need for schools and teachers to establish meaningful communication with immigrant parents. Parents tend to bring with them different ways to do mathematics that are often not acknowledged by the schools, and conversely, parents do not always see the point in some of the school approaches to teaching mathematics.

## Some Implications

Based on the literature reviewed, here are some key points to keep in mind when addressing the mathematics teaching and learning of immigrant students. Efforts should be made to focus on the knowledge and experiences that immigrant students and their families bring rather than on what they lack (e.g., limited knowledge of the language of instruction). Seeing diversity as a resource rather than as a problem could enhance the learning opportunities in mathematics for all students in the classroom. Through a deeper understanding of their students' communities and families (e.g., their funds of knowledge), teachers can work towards using different forms of doing mathematics as resources for learning.

The diversity of languages plays a prominent role in the mathematics education of immigrant students. Barwell's (2012) four tensions can serve as a document for discussion with teachers to see multilingual classrooms as complex and rich environments for the learning and teaching of mathematics. This may call for the need for mathematics teachers to seek the expertise of language teachers and/or linguists to further understand the strengths of multilingualism in communicating about mathematics.

Finally, the research reviewed on the mathematics education of immigrant students makes clear the need for a holistic approach to their education. Such an approach should include multiple voices and participants (parents, teachers, school administrators, community representatives, and the students themselves).

## Cross-References

▶ Ethnomathematics
▶ Indigenous Students in Mathematics Education
▶ Urban Mathematics Education

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## Immigrant Teachers in Mathematics Education

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The level of cross-border human movements – temporary or permanent – in the current world order is unprecedented. The transnational mobility of teachers – whether these are young and newly qualified teachers looking for a different lifestyle for a few years or teachers who migrate permanently to a new nation – is thus a relatively new phenomenon which challenges in many countries traditional or authentic images of what school teachers look like. At the same time, these immigrant and foreign teachers alleviate to some extent the problem of teacher shortage facing many nations. In particular, there has been a lack of teachers who are qualified to teach mathematics in countries such as Australia, the UK, and the USA.

Optimizing the professional socialization of immigrant and foreign teachers in their respective host cultures has direct implications for the pedagogical qualities of their practice. Yet, research into this aspect of teachers' lives has been lacking in the mathematics education research arena. This may be due to the fact that the proportion of immigrant and foreign teachers in any education system is still relatively small, further masked by the illusion of similar skin colors (e.g., white American teachers practicing in Australia). Also, the acculturation experiences of many of these teachers normally remain silent, even when these teachers may leave the education system and subsequently seek employment in unrelated professions such as taxi driving. Or, perhaps, researchers have underestimated the potential for mathematics teachers to encounter dissonance specific to content and its pedagogy during their respective acculturation processes.

After all, compared to their peers teaching other subjects, school teachers of mathematics themselves may be less prepared for these cultural differences in their professional settings (Seah 2005b). There seems to be a widespread perspective in the society of a culture-neutral mathematics discipline, one which believes that mathematical knowledge constitutes absolute truth and that there are standard ways of 'doing mathematics.'

Yet, immigrant and foreign teachers of mathematics do find it "different" teaching mathematics in schools in a different culture (Seah 2005b). After all, mathematics is socially constructed knowledge (Bishop 1988). Even if the same 'Western' mathematics is being taught at school in the home and host cultures, there are very likely different ways of finding the answers to the same questions (e.g., using a computer algebra system, or not) and/or different ways of organizing the student learning activities (e.g., group discussions vs. individual seatwork).

How then do immigrant and foreign teachers of mathematics respond to perceived cultural dissonance in their professional work? In some instances, these teachers may be helpless, leading or adding to the acculturation stress they may already be experiencing. At other times, a range of responses have been observed, ranging from "status quo" on the one extreme (i.e., ignoring the host culture's norms and continuing to enact the home culture's) to accommodation on the other extreme (i.e., embracing the host culture's ways). Relatively more empowering for the immigrant teachers, however, was the adoption of responsive strategies which strike a balance somewhat between these two extremes, namely, assimilation, amalgamation, and appropriation (Seah 2005b). In particular, the appropriation response involves the interaction of the home and host cultures in productive and empowering ways, such that the pedagogical discourse of the individual immigrant/foreign teacher develops and extends beyond the current form associated with the respective cultures. The crucial role of cultural values is emphasized (see, e.g., Seah 2005a).

From a social ecological perspective, these responses to cultural dissonance are influenced by the immigrant/foreign teachers' own life experiences and personal characteristics as well as by the increased ease in the maintenance of relations with family and friends in the home countries (facilitated by global connectedness and transnational connectivity) (Bhattacharya 2011). The range of the teacher responses can also be understood in the context of great within-group diversity, understandably so when so many ethnicities and races are involved in the collective group of immigrant and foreign teachers.

From the critical pedagogy perspective, the acculturation experiences of immigrant/foreign teachers are seen in terms of "the codification of what counts as authentic culture to be studied as well as practiced in school [which] negatively impacts students and teachers who negotiate non-mainstream identities" (Subedi 2008, p. 57).

The range of responsive strategies which immigrant teachers of mathematics use flexibly to negotiate the differences in cultural values that they perceived are also aligned with the postcolonial theorists' view (e.g., Bhabha 1997). That is, in the face of minority practices, teachers possess the capacity to resist, subvert, or negotiate. Their situative cognition (Whitfield et al. 2007) also serves to problematize teaching across cultures in this regard. Furthermore, given the similar SES status in the home and host cultures, the subsequent portability of the teachers' respective social capital (Bhattacharya 2011) probably also facilitates teacher agency.

Research into how the pedagogical activities of immigrant and foreign teachers of mathematics needs to be ongoing, not just because we have limited knowledge and understanding in this aspect of mathematics education, but also because such findings will have direct implications to the professional well-being of the immigrant/foreign teachers (of mathematics) and to the quality of mathematics learning amongst their students.

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## Inclusive Mathematics Classrooms

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### Keywords

Classroom practice; Language; Culture; Ethhnomathematics; Inclusion

### Definition

The term "inclusive" has been widely adopted within the special needs discourse and is frequently associated with this field. However a much expanded view of "inclusive" is used in this entry. The term "inclusive" is used here to refer to those students who traditionally have been excluded from success in school mathematics. This may be on the basis of gender, social background, culture, race, and language. The focus of the inclusivity can be directed at any one or more of these target areas. The focus of the work may include the practices within the classroom through to policy at school or government levels that shape the practices in the classroom. Inclusivity may include innovations in curriculum, pedagogy, and/or assessment. It can be shaped by psychological discourses aimed at developing characteristics within the students such as motivation, self-esteem, confidence, and resilience as well being shaped by sociological discourses that consider the wider social and political contexts of mathematics classrooms.

### Characteristics

Three key areas are evident in the research associated with inclusive classrooms. These include the practices within the classroom, the ways in which language use is implicated in gaining access (or not) to learning, and success in school mathematics and the mathematics itself. These are considered in the following sections.

#### Classroom Practice

The focus of inclusive mathematics classrooms is varied. Boaler (1997a) explored how practices adopted by UK teachers shaped the learning of students. She found that the use of group work in heterogeneous classrooms produced significant mathematical learning for those students. In subsequent work in the USA, she (Boaler and Staples 2008) found that schools adopting complex instruction (Cohen and Lotan 1997) improved their learning outcomes for some of the most disadvantaged students in California. The approach drew on a wide range of research to develop inclusive practices (such as group work, use of home language) to enable all students' access to deep mathematical learning. Both of these studies drew out the importance of heterogeneous groupings in classrooms in enhancing mathematics learning for students who typically are at risk of failing in schools.

Ability grouping is widely adopted in mathematics classrooms with a wide range of international studies (Boaler 1997b; Zevenbergen 2005) indicating that it is far from inclusive. Studies have shown that while top grouped students are exposed to high levels of mathematics, the pacing of the lessons and the pressure imposed by the teaching may be detrimental to learning. Worse still are the experiences of those students in the lower groups who frequently reported poor teaching but also the internalization of failure and a poor concept of self as learner of mathematics. Mathematically and psychologically, ability grouping can have detrimental impact on learning, but the sociology of ability grouping also indicates that there is a strong correlation between social background and the levels in which students are placed in ability groups.

The use of pedagogical aids in classrooms also relates to inclusion, or not, of students. In his work with textbooks, Dowling (1998) illustrated the relationship between the types of textbooks used in the UK and social background. In this work, Dowling showed how students from working-class backgrounds were more likely to experience restricted mathematical texts than their middle-class peers. Similarly, in their analysis of wide-scale testing, Cooper and Dunne (1999) showed how students interpreted and solved mathematical questions and how their responses were shaped by the background of the students. Students from working-class families were equally as likely as middle-class students to solve esoteric problems, but the working-class students were more likely to misinterpret contextual problems and locate them in a nonmathematical discourse and provide an incorrect response.

### Language Use in Classrooms

Many students may be excluded from mathematics classrooms due to factors related to language. Zevenbergen (2000) argued that success in mathematics classrooms was about "cracking the code" of the linguistic practices within the classroom. Migrant students (Planas and Setati 2009) may come to class where their language is different from that of the dominant culture. Classrooms may have many languages, like in some contexts such as South Africa (Setati and Adler 2000) or in some parts of the USA (Moschkovich 1999), where there are home languages but these are not the language of instruction. In some contexts, such as remote Australia (Watson 1988), New Zealand (Meaney et al. 2012), or Canada (Borden 2013), where there are Indigenous people attending mainstream schools, the language of instruction may not be that of the home, and for some of these students, the language of instruction is a foreign language as it is only spoken in the school context. Collectively this diversity in languages and their relationship to the mathematics classroom creates challenges for inclusive classrooms.

### Mathematics

Being able to engage with mathematics is central to inclusive mathematics classrooms. Providing an impoverished mathematics further excludes students from the study of mathematics, so it is necessary for inclusive classrooms to offer mathematics that enables deep learning. Scaffolding learning is central to developing strong mathematics. Some authors (Powell et al. 2009) have focused on developing deep mathematics for all students, but most notably those from diverse backgrounds. Others (Gutstein 2003) have argued strongly for a mathematics that is located in a social and political context to enable students to see the power of mathematics to enable them to better understand their social circumstances. In contexts, such as Canada (Lipka 2009), where the First Nation people have world views and ways of interacting that may not be represented in and through the curriculum, appropriate scaffolding has been developed while embracing aspects of the culture and building mathematics around the cultural mathematics. Cultural approaches may also favor the validation of mathematics embedded in the culture (see entry on ethnomathematics) where the students "unfreeze" the mathematics in cultural activities such as basket weaving (Gerdes 1988) or everyday mathematics of workplaces (Noss 1998; Zevenbergen and Zevenbergen 2009).

Countries and smaller jurisdictions will also create policies to shape the ways in which schools develop their practices. These policies vary considerably from country to country. Results of policies and international practices are widely discussed by researchers working in studies such as TIMMS and PISA and should be referred to in these sections of this encyclopedia.

## Cross-References

▶ Ethnomathematics
▶ Indigenous Students in Mathematics Education

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# Indigenous Students in Mathematics Education

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## Keywords

Culture; Language; Underperformance of students; Indigenous knowledge; Ethnomathematics

## Definition

Definitions of Indigenous people differ, with some countries acknowledging Indigenous people, while other countries labelling similar groups as tribal or minority nationalities (Sanders 1999). In mathematics education, a definition that has been used in Australasia considers Indigenous students to be those who belong to communities who originally controlled the land and developed distinctive cultures before the arrival of Europeans but who are presently attending educational institutions which closely resemble those of industrial countries (Meaney et al. 2012). It is important to recognize that Indigenous cultures are heterogeneous and many sets of behaviors, understandings, or cultures typify different groups of Indigenous people, even when different Indigenous groups live in the same country.

In this entry, we describe some Indigenous people's mathematical activities prior to colonization, how such activities have been used in school mathematics, and the use of an Indigenous language as the language of instruction in mathematics classrooms. The final section considers issues which surround the underperformance of some Indigenous groups in mathematics education and the approaches adopted by government organizations and schools to improve the situation.

## Traditional Mathematical Activities

Long before Indigenous people were recognized by the United Nations (Sanders 1999), anthropologists had recorded mathematical practices of Indigenous people. However, not all anthropologists were willing to accept the information from Indigenous people because of preconceptions about the type of knowledge that mathematics was and the level of intellectual sophistication that Indigenous people could reach (Harris 1990).

An interest in cognitive development contributed in the 1970s to a number of cross-cultural studies being undertaken which documented the range of mathematical practices that Indigenous people participated in. However comparisons with children's development in Western cultures indicated that the outcomes of this mathematical development could be quite different (Lancy 1983). This led to suggestions that Indigenous communities should reproduce home lives similar to those of Western communities, an idea that was resisted by Indigenous communities as it left little opportunity for Indigenous children to maintain their own culture (Cantoni 1991).

## Use of Indigenous Mathematical Activities in School

From their work with the Kpelle in Liberia, Gay and Cole (1967) proposed the need for school mathematics to recognize *Indigenous mathematics*. Since the 1980s, the role of Indigenous mathematics in supporting Indigenous cultures and contemporary schooling was recognized by the emerging research discipline of ethnomathematics (Denny 1986; Gerdes 1988; D'Ambrosio 1992). With the inclusion of ethnomathematical perspectives, Indigenous students are expected to achieve better results because they would feel that their backgrounds and experiences are valued in the classroom, mathematics can be developed by others outside of Western culture, and mathematics has relevance to their lives outside the classroom. However, there is little research which has documented such outcomes (Meaney and Lange 2013).

In Papua New Guinea, a major reform was undertaken to recognize the mathematics that children came to school with. However, research by Esmonde and Saxe (2004) in one remote community conducted in the first few years after the reforms showed that Hindu-Arabic system was the more dominant counting system known to students. The local counting system was used in a very restricted way. Students indicated that they believed that the local system could not be used for numbers greater than 27, even though many adults could count to very large numbers in this language. On the other hand, Matang (2005), also in Papua New Guinea, found that students using the counting systems in his home language, Kâte, were better able to transfer their understandings to the English counting system.

On the other side of the world, a program based on ethnomathematics has reported good results for Indigenous students. Lipka and colleagues found that the use of culturally based mathematics teaching with Yup'ik students in Alaska resulted in significant improvement in standardized test results (Kisker et al. 2012). The materials incorporated not just culturally-relevant contexts but also participation structures. The development of materials has been done over several decades in collaboration with Yup'ik communities which retain many of their traditional customs.

## Indigenous Languages and the Teaching of Mathematics

For many Indigenous groups, decisions about what language should be used for teaching mathematics are often political and not just about what is cognitively appropriate. In the nineteenth century, some Indigenous students were taught mathematics in their native languages as a consequence of missionaries and governments' assimilationist policies (Meaney et al. 2011). However, over time a number of overt and covert policies were introduced that shifted schooling for Indigenous students to the colonizers' languages. In countries, such as Fiji (Bakalevu 1999), policies about the language of instruction are still in place even when independence had been granted many years previously.

There has also been much discussion about the most appropriate language in which to learn a Western knowledge area such as mathematics because of the intimate relationship between culture, language, and mathematics. Berry (1985) in his discussion of the teaching of mathematics in Botswana emphasized the "distance" between the language of the learner and the language of the curriculum developer. In looking at the problems Botswana children were having in learning school mathematics, Berry suggested that even where a mathematical register was developed in the Indigenous language, there could still be a clash between the different underlying cognitive structures of the mathematics register and the Indigenous language. This could result in children failing to learn mathematics. Research from the 1980s has focused on issues to do with developing the mathematics register in an Indigenous language (Meaney et al. 2011).

Similar issues were identified by Denny (1980) in the translation of mathematics curriculum materials from English into Inukitut, the Inuit language. In contrast, Denny (1980) proposed using Inuktitut, the Eastern Canada Inuit language, in a "learning-from-language" approach where the development of mathematical concepts could grow out of the concepts children learnt through being Inuktitut speakers.

Mendes (2005) reported on work in Brazil where Indigenous teachers produced written mathematics problems. Their languages had been written down only recently and this allowed for some experimentation. The format of the problems often incorporated aspects of oral culture and pictures so that the problems could be considered as being different to those found in Western mathematics classrooms.

In New Zealand, Māori communities' push to revitalize their language resulted in mathematics classes in Māori-immersion schools being taught in the Māori language (Meaney et al. 2011). This has lead to many different challenges being overcome in order to use Māori for the teaching of mathematics. However, research into how to overcome these challenges has shown that aspects of the language, such as the large number of logical connectives, are very useful for students to discuss mathematics.

## Current Issues Concerning Mathematics Education for Indigenous Students

The rise of minority and Indigenous peoples' movements usually has incorporated a strong educational focus in order to produce political and economic emancipation. As a consequence, there has been a surge in interest in how to facilitate the teaching of mathematics to Indigenous students.

Differences between Indigenous communities and Western schools' ways of valuing knowledge often contribute to the non-Indigenous society and its teachers labelling Indigenous communities as being deficient and the contribution that these communities may have to offer to the teaching of mathematics ignored (Meaney et al. 2012). National or international testing highlights the underperformance of Indigenous students in school mathematics. Although in places such as Australia and New Zealand, these results have led to education initiatives, the focus has been on the Western mathematics needed for these students to become economically sustainable adults. Unfortunately the constant reiteration of how poorly Indigenous students perform in these tests "is likely to produce in teachers, policy makers, the general public and Indigenous students themselves a belief that Indigenous students cannot learn or utilise mathematics in their everyday lives" (Meaney et al. 2012, p. 68). An alternative approach would be to investigate what contributes to Indigenous students being successful. For example, Lipka and his colleagues (Kisker et al. 2012) continue to show that working with Indigenous communities can result in students gaining more from school mathematics experiences as well as strengthening ties to their Indigenous culture.

## Cross-References


- ▶ Bilingual/Multilingual Issues in Learning Mathematics
- ▶ Cultural Diversity in Mathematics Education
- ▶ Cultural Influences in Mathematics Education
- ▶ Mathematical Language

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# Informal Learning in Mathematics Education

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## Keywords

Formal learning; Informal learning; Mathematical learning outside institutions; Street mathematics; Ethnomathematics; Knowledge transfer; Consciousness of concepts; Psychological theory of the conceptual fields; Knowledge transfer; Levels of conceptualization

Even though the characterization of "informal learning in mathematics education," as well as its goals, is still problematic, the thematic has a clear place within contemporary approaches to education. Many of the questions examined by professional practitioners and researchers in education lead to considerations about mathematical learning outside of institutions. In any case, this area of studies cannot be addressed by a single theory, a single disciplinary field, or even a single research topic. It must be studied from the perspective of multiple theories, research methods, and data analysis. We hold that psychology, anthropology, and educational sciences have all played a most important role in the history of ideas about informal learning in mathematics education and in the area's research initiatives.

In keeping with Greenfield and Lave's (1982) conception of "Informal learning styles," we consider "informal learning" to be knowledge and capabilities acquired and developed outside of an established system, hence the opposite of "formal learning" which means knowledge gained from within a school framework.

Using a terminological approach as a starting point does not simply lead to a discussion about the proper use of terms. Instead, the very choice of signifiers, such as "formal, informal," to characterize education and learning raises fundamental theoretical questions. By questioning the term "informal learning in mathematics education" as determined by its pragmatic use, it appears that key concepts such as consciousness, status of knowledge, transfer, and context are fundamental in understanding the nature of "informal learning," placing them in various relevant theoretical frameworks. Are we talking about the context or about the learning process?

Thus, if the term "informal" is problematic, how does one conceptualize learning? It must remain clear that here we consider the learning process, not its product. Even though human potential abilities are universal, their realization and the form they take across multiple learning opportunities depend on culture.

One of the goals of comparative cultural psychology is the analysis of this variability of human behavior (Bril and Lehalle 1988). The diversity of behavior is not inconsistent with the universality of the process. Learning is a field of study that allows understanding both sides of the issue – diversity and universality – through the analysis of the construction process of these behaviors (Bril 2004).

But can we speak of "informal learning" as part of a formal discipline like mathematics? This leads to different theoretical positions with different educational implications. The negative answer to that question considers mathematics as a formal discipline, universal and decontextualized; the positive response considers mathematics as a cultural product.

We believe that even though learning processes are universal, they can express themselves in different ways depending on the context in which they are manifested, taking different "forms" or being "formalized" in different ways. However, the formal/informal dichotomy to account for the different forms of the actual learning process seems inadequate. There is every reason to believe that a model incorporating a dialectical relationship between formal and informal and a gradient to situate learning between the two poles would be better able to account for the phenomenon we study.

The terminological ambiguity inevitably compels us to reflect upon the characteristics of learning. Research studies identified by Acioly-Régnier (2004), as shown by the following issues, have addressed:

- What concepts and ideas do researchers use to address issues of learning?
- To what extent and by what criteria does the research in this area hierarchically order (or not) these kinds of learning?
- How can we investigate researchers' conceptions in the study and analysis of learning processes and their activation in specific contexts?

These questions raise a number of dichotomies that researchers are forced to confront in order to clarify the theme: formal versus informal context; context versus no context; explicit versus implicit; conscious versus unconscious; concrete versus abstract, etc. Consider, for example, the socio-historical theory of Vygotsky which offers different perspectives to address these binary opposites when he examines, for example, the question of scientific concepts versus everyday concepts, which are central to the informal learning of mathematics.

## From Terminology to Conceptualization: Consciousness, Status, and Knowledge Transfer

Vergnaud (1999) discusses the polarization of these two types of concepts by Vygotsky. He comments on this idea by considering a more nuanced view found in other writings in which

Vygotsky argues that the development of spontaneous concepts and scientific concepts are closely linked processes that exert on one another a constant influence (in Yves Clot 1999, p. 55).

### Consciousness of Concepts Versus Non-consciousness Concepts

For Vygotsky, the spontaneous or daily concept is unconscious since it is always directed to the object it represents, rather than to the very act of thinking that grasps the object. Only when a daily concept is integrated into a system can it become conscious and voluntary. Thus, in the literature, the features of a daily concept are designated as "non-conscious," "unsystematic," or "spontaneous," while those of a scientific concept are described as "conscious" or "systematic."

Therefore, the different research paradigms, different theories, mobilized theories, and the various scientific disciplines which are interested in this subject of study seem to agree at least on one point: that learning can be non-intentional or unconscious in informal situations. Even when one consciously engages in a learning process, for example, a craft, a game, or a task of everyday life, he or she may not be aware that in his or her subconscious, several concepts needed to conduct the task may be hiding. Of course, this is also partly true in academic learning when students are expected to know they are there to learn specific contents which are well determined and verbalized. Therefore, we argue that informal learning can take place as much in formal educational settings as in non-formal settings. We aim to enrich the original definition of informal learning by relying on a distinction between the concept of learning and that of "learning context." In other words, we do not subordinate the qualification of learning to the context where it takes place.

We are more concerned with the cognitive processes implemented than in contexts, although contexts clearly play a role in triggering these processes. We recall here the Vygotskian perspective where cognition and consciousness are not the causes but the products of human activity.

The core concepts in the psychological perspective I adopted (Acioly-Regnier 2004) in analyzing research on informal learning are those of consciousness and the *focus* of consciousness. We adopt the psychological theory of conceptual fields of Gerard Vergnaud to illuminate the notion of both in-school and out-of-school concepts. This theory of conceptualization of reality incorporates aspects of the situation, the concept itself, and the subject. This theoretical framework allows us to identify and study knowledge in terms of its conceptual content, to analyze the relationship between concepts as explicit knowledge and as operational invariants that are implicit in one's behavior in a situation, and to deepen the analysis of relationships between signifiers and signified.

The theory defines the concept as a tripolar system constituted by signifiers, situations, and operative invariants. The set of signifiers allows the representation, communication, and treatment of the concept. The second set refers to situations where the concept operates and the idea of reference. The set of operational invariants refers to the signifiers.

This model allows one to distinguish between school and non-school situations from the perspective of the focus of consciousness. In schools, the focus of consciousness seems to be mainly directed to the bipolar relationship meaning ↔ invariant procedure leaving aside the set of reference situations. The weakness of the learner appears in difficulties to recognize situations, out of school or in school, in which the concepts developed are relevant. For example, the learners know their lessons but do not know how to apply the definitions they have learned. In contrast, in non-school education settings, the focus of consciousness is directed to the bipolar relationship situation ↔ invariant procedure, neglecting the resource provided by the signifiers. In this case, the weakness of the learner lies in the lack of symbolic resources that enable him or her to further develop knowledge learned in a specific situation (Frade et al. 2012).

### Transfer of Knowledge and Abilities

Like the notion of consciousness, the notion of knowledge transfer has been a key concept in the theoretical framework of research on the relationships between "formal learning" and "informal learning." The usual formulation of the problem often addresses concerns regarding the academic failure of disadvantaged social groups or of cultural minority groups.

Interest in informal mathematics increased with research findings showing that children who failed in schools displayed at work mathematical abilities that required conceptual understanding similar to those implicit in school mathematics (Carraher et al. 1982, 1993). This led to the development of what came to be known as "Street mathematics" which aimed at identifying the conceptual invariants underlying mathematical procedures developed at work.

Carraher, Carraher, and Schliemann (1982), in their study of street vendors in Recife, Brazil, observed that they were able to perform arithmetic operations in daily work activities without being able to formalize the written arithmetic taught in school. Similar conclusions about the ability to generalize were drawn by Lave's study (1979, cf. Greenfield and Lave 1982) among tailors in Liberia and by Greenfield and Childs (1977) on weavers.

Greenfield and Lave (1982) argue that when the experimental task is similar to the task where learning took place, tailors and weavers, just like school subjects, are able to solve new problems. However, neither school experience nor everyday experience led to transfer when problems deviate significantly from the circumstances in which learning initially took place. Acioly (1985) and Schliemann and Acioly (1989) demonstrated in a study with lottery vendors in Recife, Brazil, that school experience alone does not play a major role in mathematics performance in work situations. They observed that the lottery vendors' performances were composed and hybrid, taken from learning at work as well as years of schooling.

Among the educational contribution of these results, we have witnessed the birth of work aimed at implementing the learning approaches found in out-of-school settings onto school settings.

### The Question of Context

Another central question common to all work on informal versus formal learning refers to the notion of context. One prevailing view connects decontextualization to formal learning, and informal learning to contextualization. We believe that this association is insufficient to account for properties of our object of study. As Schliemann and Carraher (2004) propose, school mathematics has a context that is neither concrete nor tangible, but is as real as the sales context in the markets. It is therefore necessary to consider the characteristics of contexts.

Looking at the acquisition of certain forms of knowledge, Jean Lave and Etienne Wenger (1991) have tried to place it in social relationships with situations of co-participation. This participation refers not just to local events that trigger certain activities with certain people, but to a larger process that progressively integrates the active practices of social communities and leads them to construct their identities in relation to these communities. Learning, thus, is not seen as mere acquisition of knowledge by individuals, but as a process of social participation. The nature of the situation plays a significant role in determining the acquisition process. From this viewpoint, differentiation formalized by the notions of contextualization and decontextualization has no relevance, because cognition can be seen only as part of a process of social participation in context.

### Status of Knowledge

From the discussion of the three basic concepts common to most studies, that is, consciousness, transfer, and context, emerges the issue of the status of knowledge and learning. We can already distinguish two main research approaches. The first approach prioritizes formal learning and analyzes informal learning by taking formal concepts as paradigms. The other considers that informal learning has a similar status to that of formal learning and suggests what is called "informal mathematics" to be considered as part of the curriculum.

## Informal Learning in Mathematics Education

Research on mathematical knowledge in informal work situations shows the limits of school learning and even proposes assigning a greater role to life experience and outside of school practices in the development of further knowledge (Lave 1977; Greenfield and Lave 1982; Reed and Lave 1979).

To consider the relationship between the nature of the task and one's more familiar type of learning requires a prior discussion of some important points. Problem solving, including mathematical problem solving, is a formal educational activity, through written calculations and the search for true and correct solutions. But it is also an informal education activity, characterized by frequent use of calculations performed often mentally, using approximation and estimation to reach results.

## About the Paradigms of Research on Informal Learning in Mathematics Education

### Ways of Approaching the Issue and Methods

The literature on informal learning in mathematics education mainly deals with the description of the importance of informal learning as a valid mode of knowledge acquisition, with methods used by learners in informal situations, and with ways to support and assess informal learning, highlighting the analysis of local procedures to resolve problems which, although far removed from those validated by formal education, are recognized and recommended by a specific social environment.

Current work is guided by the theory of action, by a focus on "Culture and Cognition" and by the study of ethno mathematics. The methods of data collection, used in isolation or in a procedure of cross-fertilization, include ethnographic observation, clinical interview of the Piagetian type, and quasi-experimental methods. The differences in theoretical frameworks and methodology of this research replicate methodological biases repeatedly recognized in the history of research.

Most research in this area either engaged in so-called conceptual aspects and neglected the social factors, or focused on social aspects and neglecti an in-depth analysis of the concepts themselves.

Saxe and Posner (1983) consider the strengths and weaknesses of cross-cultural research on the development of number concepts, associated with the Vygotskian or the Piagetian approach and conclude that each of these theories contributes to the analysis of cultural universals and of specific cultural aspects of number concept formation in children. On the one hand, the Piagetian approach provides a formulation of the manner in which numerical operations grow, but does not analyze the mechanisms by which social factors contribute to the formation of numerical thinking. On the other hand, the Vygotskian approach, as taken by American psychologists (Cole et al. 1971; Cole and Scribner 1974, Wertsch 1979, cf. Saxe and Posner 1983), treats the cultural experience as a differentiated theoretical construct and, even though they do not deny the importance of concepts, they do not provide a deep analysis of numerical concepts.

### Models of Formal Knowledge and Informal Learning in Mathematics

Among research studies taking a formal knowledge model for analysis of informal learning, we find those based on levels of conceptualization. Conceptualization is built in stages. The important thing is to identify the level rather than identifying the absence or presence of a given concept. These considerations provide a theoretical basis for the idea that, to solve a mathematical problem, individuals implement representations, and that these representations tell us about their level of conceptualization. Note however that, in the psychological literature, problem solving is often distinguished from concept formation. Problem solving is viewed as a new combination of behaviors and procedures dependent on prior knowledge, while the formation of concepts is taken as the emergence of new categories, new ways of conceptualizing the world, with new objects and new properties of these objects.

However, for Vergnaud (1987), this distinction is invalid, because it underestimates two things: the role of problem solving in concept formation and the role of representation and concepts in problem solving. We know that these representations are based on a conceptual core, as well as on contextual characteristics of specific situations. From this point of view, in what is regarded as the conceptual core, the actions and procedures individuals implement while performing a task always refer to concepts. This is true even if this knowledge is expressed in terms of practical activities, and in the particular context of their culture. Note that if these concepts are not necessarily conscious for the individual, the researcher must postulate their existence to understand the actions and procedures, and especially the systematic variations that can be found. It is from a perspective similar to that proposed by Vergnaud that the large majority of studies by psychologists on the subject of informal learning, including the studies on "street mathematics" by the Brazilian group from Recife, have been developed.

### Model of Daily Learning and Informal Learning in Mathematics

Among those considering informal learning as knowledge that has the same status as formal knowledge, we find the ethnomathematics approach. Ubiratan D'Ambrosio, one of its founders, proposed an "ethnomathematics curriculum". The idea of "program" is understood in the sense given by Lakatos. The direction of thought ethnomathematics seeks is the consideration of past and present stories of different social groups. This program of research on the history, philosophy, and epistemology of mathematics has pedagogical implications which provide in no way a substitute for formal mathematics by academic "math people." The ethnomathematics program is also presented as a theory of knowledge. Research methods must assume an attitude of respect for the mathematical abilities of the learner. This involves a respect for the cultural historical perspective to understand the development of concepts in the field of informal mathematics as well as in academic cultures.

The issue of cultural diversity is a central discussion issue. Indeed, this is in part to avoid the trap of praising the exotic and also to find an appropriate articulation of two movements from very different sources: one that seeks integration of subcultures marginalized in school curricula, and the other seeking to give access to the students of dominated subcultures to the knowledge of the dominant culture. This group also takes into consideration the games of power relations implicit in this task, and their implications in the development and implementation of curricula. It also aims to problematize these issues in the very formation of teachers.

## An Alternative Approach

Most of the research on informal learning and the concepts studied are actually located within a particular culture and also present in formal education. This dual membership creates an ambivalence that impedes their recognition and identification by researchers. It does call for the construction of analytical invariants, taking into account the concept studied while controlling for specific situations and contexts.

Informal learning is not reduced to the simple acquisition of practical skills. It also relies on a process of conceptualization. Levels of this conceptualization are built, based on internal processes actualized in a given social and cultural context that imposes limitations as well as favorable conditions. As such, curricular and extracurricular activities have the power to inhibit the development of certain dimensions of the concepts. This inhibition casts a shadow on the concepts. In a significant proportion of research on informal learning, anthropological variables constructed from social and cultural factors guide the interpretations of the cognitive functioning of individuals. The use of these variables to explain and understand what cognitive functioning is influenced by the theoretical framework and by the method of data collection.

It is therefore important to pay attention to levels of conceptualization of different complexity that can be triggered by specific contextual situations. Problem-solving procedures should be considered as being related to the culture in which it takes place. This requires considering the fact that different cultures solve the "same" problem by different routes, although the results may appear similar. It does not mean that there are no problems common to many cultures, but that the specific characteristics of each culture, not always easy to discern, determine specific practices. The theme of "informal learning in mathematics education" requires, obviously, research centred on issues that include all the above features.

## Cross-References


- ▶ Cultural Influences in Mathematics Education
- ▶ Informal Learning in Mathematics Education
- ▶ Learning Difficulties, Special Needs, and Mathematics Learning
- ▶ Mathematical Knowledge for Teaching


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# Information and Communication Technology (ICT) Affordances in Mathematics Education


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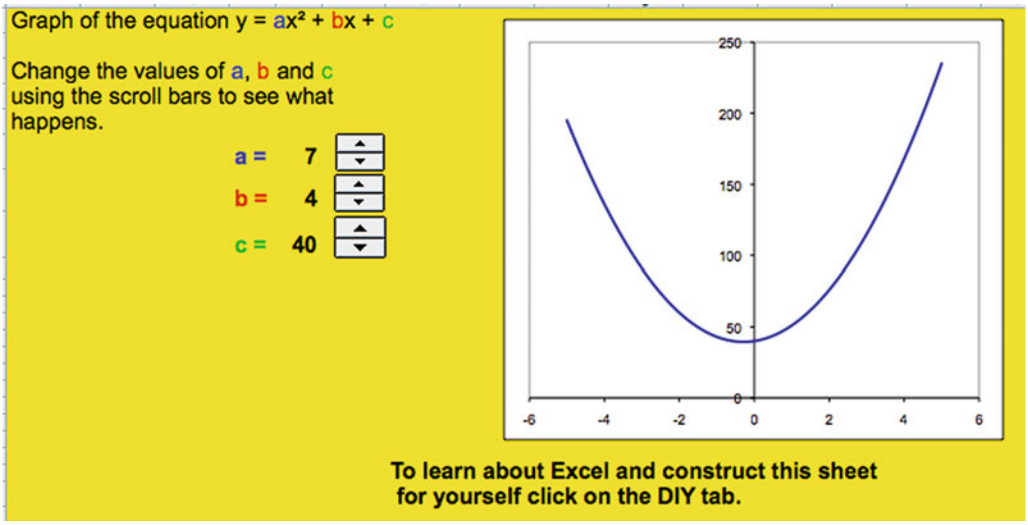
## Keywords

Static mathematics; Dynamic symbol systems; Dragging; Invariance; Mediation; Social technology


## Definition

New forms of technology that enhance access to core mathematical concepts through dynamic representations and classroom connectivity.



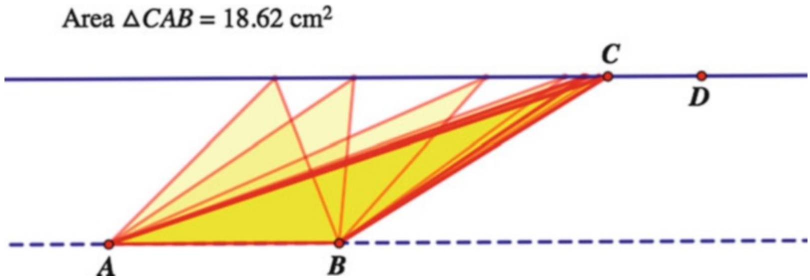

**Information and Communication Technology (ICT) Affordances in Mathematics Education, Fig. 1** Spinners in spreadsheets

## Characteristics

*Information* has changed over the past 10 years. Information can be thought of as a knowledge base, and with advances in technology, access to this knowledge is increasing on a daily basis. Knowledge is growing and the impact of such growth on education is wide and varied. In addition to thinking of information as the accumulation of knowledge, it can also be thought of as how knowledge can be represented, and in mathematics education, this has certainly evolved rapidly over the past decade in terms of the representational affordances of new technologies both software and hardware. Information is now embedded in representational media. In mathematics education, this has enabled a transformation of the mathematics from static to dynamic symbolic systems through which teachers and learners can access knowledge and think. Representational media can be both static and dynamic. Mathematical figures can be inert pictures or images as well as dynamic, constructible, and deformable objects.

Within these broad categories of interaction, we can further describe such systems as both discrete and continuous in terms of how users can interact and navigate the concepts represented. For example, a spreadsheet can offer a discrete input system through the manipulation of tables of data either statically formed or dynamically formed through sliders or spinners (see Fig. 1).

Similarly, certain software allows for continuous input through dragging parts of a figure or some controller to manipulate and change constructions in some mathematically meaningful way. Consider the following example. The area of a triangle is measured by its altitude and base by a standard formula Area = ½ * base * height. The preservation of this relationship over a wide range of similar triangles can be illustrated in dynamic geometry environments (see Fig. 2). Here vertex A is dragged across a line that is defined as parallel to the line upon which base BC is constructed. Since parallel lines preserve perpendicular distance of separation and the base BC is fixed, the trace of all triangles ABC has equal area, but students often think that the triangles are changing area as their perception of the lateral shape, being stretched, implies for them a change in area. The area measurement tool (as a different notation) is used to illustrate the resulting variation or invariance under the action of dragging a particular vertex. Dragging vertex A yields no change in area.




**Information and Communication Technology (ICT) Affordances in Mathematics Education, Fig. 2** Dynamic areas of triangles

While the measurement tool reports this invariance, it does not prove why such an action yields this result. It does allow an environment for the learner to explore what *changing* properties of a triangle are relevant in determining its area, i.e., height and base.

In both examples, mathematical information is mediated through interacting within the environment. Changing values of a parameter, or dragging a vertex, allows for information to pass back and forth between the user and the environment. The environment can guide the user just as the user guides what changes within the environment. We refer to this as coaction (Moreno-Armella et al. 2008). The representations are linked so that information is tightly bound across the representations. For example, changing the parameter "a" simultaneously changes the concavity of the quadratic in its graphical form.

Technology has offered and afforded representations and interactions between representations for a long time. These have been in terms of symbolic manipulators, where computational duties are offloaded to the microprocessor and new actions are linked to traditional notation systems. But in addition, there is now support for new interactive notation systems. Specific examples of such software environments in mathematics education span various subject areas including data analysis (Fathom, TinkerPlots®), geometry and number sense (Geometer's Sketchpad®, Cabri-Geometre), and algebra (SimCalc MathWorlds®). These modern affordances have been translated into mathematics classrooms as a mode to enhance access: offering students the ability to see through abstract constructs or symbolic figures.

Essentially, information in mathematics education is evolving within the representational media by which people wish to use for the purposes of learning and teaching. This is more broadly referred to as a representational infrastructure (Kaput et al. 2001; Kaput and Roschelle 1998; Kaput and Schorr 2008) whose elements can be used in huge varieties of combinations tuned to specific curricular objectives, student needs, and pedagogical approaches (Hegedus and Roschelle 2012).

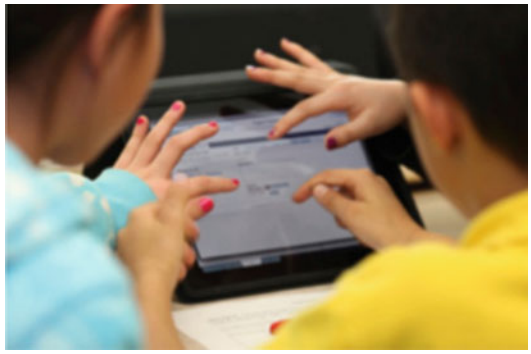
*Communication* has been a critical aspect in the evolution of mankind and in recent decades the advancement of knowledge. As symbolic species (Deacon 1997), language and the brain have coevolved, and since the evolution of external supports of memory some 35,000 years ago (Donald 2001), language has been expressed through ever-changing forms of media. We refer to communication as human actions in terms of speech or physical movement (e.g., gesture) or digital inscriptions through modern-day interfaces.

Communication can be one or a combination of several modalities of human expressiveness through writing, talking, and physical action. It can also be a technical infrastructure by which, and through which, students and teachers can project their personal work into a public workspace. Hence, communication in a technological workspace can also be thought of as an infrastructure with various interacting elements (human and digital) to produce affordances for mathematics education. Mathematical work can be shared for the purposes of comparison, extension, or accumulation of ideas. Networks have been essential in allowing various researchers to exploit such affordances by connecting various small and robust technologies together wirelessly for various educational purposes.

As more handheld devices become ever present in the lives of children as well as adults, it is important to address how such a technological boom advances or transforms communication in mathematics education. Communication as a transfer protocol is not sufficient to describe the (mathematical) educational affordances of such advances. Even in social networking, people do not only share information in a traditional sense presented above but can also be part of a community where ideas are developed, thinking evolves, and identities are formed. In mathematics education, the student experience of "being mathematical" (Nemirovsky et al. 1998) has become a joint experience, shared in the social space of the classroom in new ways as the mathematical constructions of each student can be aggregated in common representations (Roschelle et al. 2010; Brady et al. 2013) and form participatory simulations (Stroup 2003; Stroup et al. 2005; Wilensky and Stroup 1999, 2000). Cognitive activity can now be distributed in the socio-material space (Hutchins 1996). Similarly changed are how students interact mathematically with each other and their teacher and, critically, how their personal identity manifests in their shared mathematical experience in the classroom.

Advances in mathematics education have arisen where both information and communication have been treated as an integrated system. Hegedus and Moreno (2009) have described how the integration of representational and communicational infrastructure yield forms representational expressivity – charged by the dynamic affordances of the technologies and the opportunities for social mediation of ideas – in terms of physical (e.g., gestures) and verbal forms of communication.

As technology becomes more "social," we should be aware of the enhanced forms of mediation that emerge. These can exist through the representational media as a result of the technology in terms of graphical and computational affordances. They can also exist through social mediation in how we share and transfer ideas and use technology locally as well as globally. For example, the portability of handheld devices – such as iPads – allows students to pass ideas around a table via the tablet or push ideas up to a server for public display. New technologies



**Information and Communication Technology (ICT) Affordances in Mathematics Education, Fig. 3** 10-year-olds using iPads

also offer multimodal affordances, which will evolve over the next decade. Allowing students and teachers to use various sensory modalities (e.g., sight, touch, sound) in mathematics education will transform the landscape of mathematical discovery. And within modalities, there are new affordances. Allowing users multi-touch offers mathematical affordances. For example, each touch can be an input. Such inputs can be processed into one or more outputs thus establishing a mapping of a set of inputs to outputs with some well-defined rule or function – a critical concept in mathematics still to be fully utilized in mathematics classrooms today. Figures 3 and 4 illustrate how such systems can be integrated into elementary school classrooms infusing social engagement from small groups to whole class discussion via classroom networks.

Such forms of mediation have been broadly described in mathematics education as semiotic mediation which include embodied actions of pointing, clicking, changing, grabbing, and dragging parts of mathematical constructions (Falcade et al. 2007). Such actions mediate between the object and the user who is trying to make sense of, or induce some particular attribute of, the diagram or prove some theorem. In addition, such mediation can be established within the social setup of the classroom. Excellent summaries of how theories of semiotic mediation have impacted the design and implementation of certain technologies (e.g., computer algebra systems and dynamic

**Information and Communication Technology (ICT) Affordances in Mathematics Education, Fig. 4** Dynamic multiplication on the iPad

geometry) into mainstream classrooms can be found in Drijvers et al. (2009) and more broadly in Hoyles and Lagrange (2009).

In summary, information and communication technology needs to be reconceptualized and redefined in this digital era. Information and communication need to be tightly integrated. The affordances of such systems have been described here in principle but need further investigation in terms of transforming the activity domain and social landscape of the mathematics classroom.

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▶ Technology and Curricula in Mathematics Education

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# Inquiry-Based Mathematics Education

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## Keywords

Constructivism; Student-centered pedagogy; Problem solving; Adidactic; Scientific debate; Modeling; Experimental practice; Teachers' practice; Pre-service and in-service training

## Related Terms (and Acronyms)

Inquiry-based education (IBE), Inquiry-based learning (IBL), Inquiry-based science education (IBSE), and Inquiry-based teaching (IBT)

## Definition

*Inquiry-based mathematics education* (IBME) refers to a student-centered paradigm of teaching mathematics and science, in which students are invited to work in ways similar to how mathematicians and scientists work. This means they have to observe phenomena, ask questions, look for mathematical and scientific ways of how to answer these questions (like carrying out experiments, systematically controlling variables, drawing diagrams, calculating, looking for patterns and relationships, and making conjectures and generalizations), interpret and evaluate their solutions, and communicate and discuss their solutions effectively.

The role of the teacher in such a setting is different to traditional teaching approaches: pedagogies make a shift away from a "transmission" orientation, in which teacher explanations, illustrative examples, and exercises dominate, towards a more collaborative orientation, in which students work together on "interconnected," "challenging" tasks. Here, the teacher's role includes making constructive use of students' prior knowledge, challenging students through effective, probing questions, managing small group and whole class discussions, encouraging the discussion of alternative viewpoints, and helping students to make connections between their ideas.

### Sociopolitical Background

In recent years, IBME and generally IBE has met a real success especially in educational policy and curriculum documents but also in developmental in-service and pre-service professional development courses and projects. The reasons for this wide popularity of IBE may be found in the alarming decline in young people's interest for sciences and mathematics studies, attested in most countries in the world, especially in Europe and North America, as well as the poor results of many countries in mathematics and science in international evaluations like PISA. In Europe, for instance, this led to political reactions at various levels. A famous report known as Rocard's report (Rocard et al. 2007) incriminated (among other causes) the "deductive approach," in which "the teacher present the concepts, their logical – deductive – implications and gives example of applications" resulting in students lacking interest, considering science and mathematics to be extremely difficult, and being not able to apply their knowledge in bigger and maybe unfamiliar contexts. Instead of this traditional education, the experts advocate the promotion of IBE and refer to Linn et al. (2004) to promote IBE: "By definition, inquiry is the intentional process of diagnosing problems, critiquing experiments, and distinguishing alternatives, planning investigations, researching conjectures, searching for information, constructing models, debating with peers, and forming coherent arguments." This led the EU to invest a lot of money to support research projects to promote widespread dissemination of these pedagogies in order to improve Europe's capacity for innovation. For more details on the different European projects on the implementation of IBE, see www.proconet-education.eu.

### Historical Background

Historically the importance of inquiry in education is generally attributed to the American philosopher and educator John Dewey (1859–1952). In his book published in 1910, he acknowledged the importance of inquiry in child's attitude towards science: "This scientific attitude of mind might, conceivably, be quite irrelevant to teaching children and youth. But this book also represents the conviction that such is not the case; that the native and unspoiled attitude of childhood, marked by ardent curiosity, fertile imagination, and love of experimental inquiry, is near, very near, to the attitude of the scientific mind" (Dewey 1910, p. iii).

Moreover Dewey insists on the process through which inquiry develops: "There is continuity in inquiry. The conclusions reached in one inquiry become means, material and procedural, of carrying on further inquiries" (Dewey 1938, p. 140). He also puts forward the importance of action on objects, rather than language in scientific thinking: "The authors of the classic logic did not recognize that tools constitute a kind of language which is in more compelling connection with things of nature than are words [. . .] Genuine scientific knowledge revived when inquiry adopted as part of its own procedure and for its own purpose the previously disregarded instrumentalities and procedures of productive workers" (Dewey 1938, p. 94).

Dewey's perspective on education implies a practice of teaching based on projects closely linked to students' life and interests and to the development of inquiry habits of mind considered as generic. However, the details of Dewey's work are usually diluted in more general approaches, despite the relevance of his work for contemporary reflection in education (Hickman and Spadafora 2009). Historically, IBE at first concerned sciences rather than mathematics. In this sense, one major event was the publication of the *National Science Education Standards* in the USA in 1996. From there a wide spectrum of IBSE approaches and practices emerged and developed (Barrow 2006), with various definitions that the 2000's revised NSRS tried to summarize in 5 points:

- Students create their own scientifically oriented questions.
- Students give priority to evidence in responding to questions.
- Students formulate explanations from evidence.
- Students connect explanations to scientific knowledge.
- Students communicate and justify explanations.

In the PRIMAS project (www.primas.eu 2011) these are embedded in broader picture capturing what could be meant by an inquiry-based teaching practice in science and mathematics; see Fig. 1.

### IBME and Mathematics Education Research

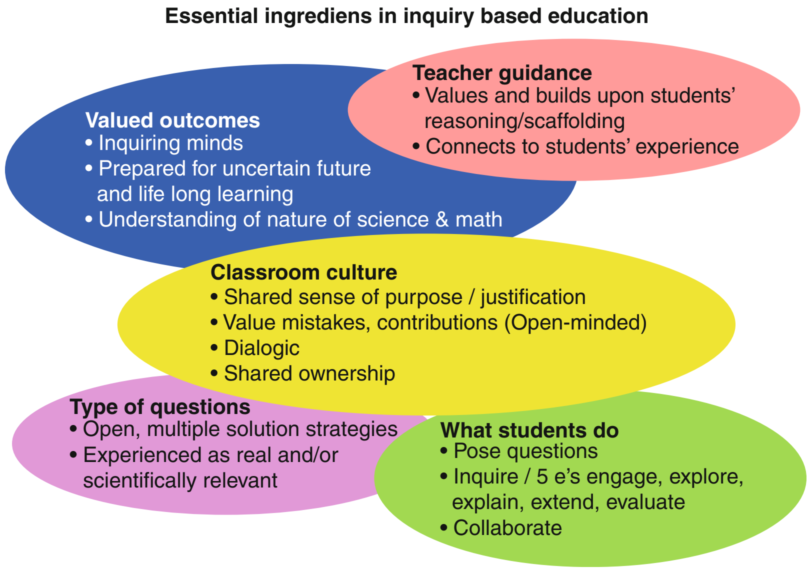
The focus on inquiry in mathematics education is more recent than in science. It is based on the increasingly shared view that mathematics and sciences education are closely connected, that mathematics is not purely deductive, and that mathematical concepts may be grasped through some experimental practice. However, the migration in mathematics led to some specificities, especially a strong connection with problem solving, a long tradition in mathematics education (see, e.g., Rocard et al. pp. 9–10).

Although the term IBME has not been traditionally used, several research works and theories in mathematics education can be linked to it. Artigue and Blomhoej (Maass et al. in press) have made an overview of these links offering a well-documented and illustrated analysis. Even if they do not claim of course to be exhaustive, they reviewed several different trends and theories, namely, problem-solving tradition, theory of didactic situations, realistic mathematic education, modeling perspectives, anthropological theory of didactics, and *dialogical and critical* approaches.

These authors conclude this review by a reflection on the possible conceptualization of IBE in mathematics education. Such a theoretical concern has been missing so far, due to the fairly recent migration from science.

### Evidences from Research of IBME Benefits

Considering the sociopolitical background, as depicted above, the success of IBME as a remedy to all problems is barely questioned.




**Inquiry-Based Mathematics Education, Fig. 1** The working definition of IBE in the PRIMAS project

However, this issue is more complex when approached from a research perspective. One of the most extensive surveys was published recently in the context of science (Minner et al. 2010), but it is also relevant for mathematics. It took into account 138 studies (mostly in the USA) published between 1984 and 2002 and tried to evaluate the impact of IBL on students' competencies in sciences. One of their first duties was to develop a framework in order to measure the level of IBL in the instructional intervention at stake in each study. "In this framework, inquiry science instruction can be characterized as having three aspects: (1) the presence of science content, (2) student engagement with science content, (3) student responsibility for learning, student active thinking or student motivation within at least one component of instruction – question, design data, conclusion or communication" (p. 478). Based on this framework, their overall conclusion is that "the evidence of effects of inquiry-based instruction from this synthesis is not overwhelmingly positive, but there is a clear and consistent trend indicating that [. . .] having students actively think about and participate in the investigation process increases their science conceptual learning" (p. 493).

Concerning mathematics, there are also several studies that point some various positive effects of IBME on students' achievements, motivation, autonomy, flexibility, etc. There has also been a concern on the type of students for whom IBME could be more beneficial, but these studies lead to a mosaic of evidences from which it is not always easy to draw some general conclusions. Yet, an overview of these results with references to several studies can be found in the article by Bruder and Prescott in (Maass et al. in press). Furthermore, the political pressure due to the supposedly radically positive effects of IBME on students' achievements in and motivation for mathematics is an opportunity for the implementation of IBL in day-to-day teaching but may also elude some research necessity. Still large-scale studies on the implementation of IBL and its effects in mathematics education are missing.

## Research on Teachers' Practices Regarding IBME

In spite of research evidences and political pressure, IBME remains quite marginal in day-to-day mathematics teaching and often limited to softer versions compared to more ambitious experiments. This raises the issue of the role to be given to IBME in teachers' training and professional development courses, based on research works on teachers' practices (see, for instance, the ICMI study (Even and Ball 2009) or Grangeat (2011) and the results of the European project S-team http://www.s-teamproject.eu/). In particular, it seems essential (yet not sufficient) that teachers have a chance to experience this type of teaching personally in their own mathematical or professional training; in other words, the paradigm of inquiry could serve as a model for designing activities with trainees. This issue is specifically stressed in research works on communities of inquiry (see, e.g., Jaworski et al. 2007) or the model of lesson studies in Japan (see, e.g., Inoue 2010).

Another concern is that professional development courses need to start off from teachers' needs, to be relevant to day-to-day teaching, and should engage teachers in reflecting on their teaching practice and on their beliefs on what they consider as good mathematics education. This is also important in relation to the teachers' need for legitimacy in relation to students, parents, and colleagues.

In order to be effective, professional development courses need to develop on a long-term perspective, allowing teachers to learn about inquiry-based education, to try out inquiry-based pedagogies in their teaching, and to reflect on it in the next meeting. However, including IBME in pre- and in-service teacher education is not sufficient to establish a sustainable teaching practice in mathematics in which IBL plays a substantial role. Systemic support from school policy is of course crucial. In particular, curricula and external assessment need to include some inquiry dimension; more information is to be found in the article by Maass and Doorman (Maass et al. in press).

## Cross-References

▶ Anthropological Approaches in Mathematics Education, French Perspectives
▶ Argumentation in Mathematics
▶ Argumentation in Mathematics Education
▶ Collaborative Learning in Mathematics Education
▶ Communities of Inquiry in Mathematics Teacher Education
▶ Constructivism in Mathematics Education
▶ Critical Thinking in Mathematics Education
▶ Dialogic Teaching and Learning in Mathematics Education
▶ Didactic Situations in Mathematics Education
▶ Interdisciplinary Approaches in Mathematics Education
▶ Learner-Centered Teaching in Mathematics Education
▶ Lesson Study in Mathematics Education
▶ Mathematical Modelling and Applications in Education
▶ Mathematics Teacher Identity
▶ Models of In-service Mathematics Teacher Education Professional Development
▶ Models of Preservice Mathematics Teacher Education
▶ Motivation in Mathematics Learning
▶ Realistic Mathematics Education
▶ Teacher Beliefs, Attitudes, and Self-Efficacy in Mathematics Education
▶ Theories of Learning Mathematics



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## Instrumental and Relational Understanding in Mathematics Education

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### Keywords

Instrumental; Relational; Skemp; Understanding; Algorithms; Knowledge; Conceptual; Procedural

### Characteristics

Richard Skemp (1919–1995) was a British mathematics educator and educational psychologist who was very prominent in the field of mathematics education in the 1970s. Skemp's writings [particularly his two books – *The Psychology of* Learning Mathematics (1971) and *Intelligence*, Learning, and Action (1979)] articulated a theory of intelligent learning, in which relational and instrumental understanding played a prominent role. Skemp first popularized the terms relational and instrumental understanding in an article published in 1976 in *Mathematics Teaching*. This article was subsequently published in the United States in 1977 in the *Arithmetic Teacher* (a professional journal published by the American organization National Council of Teachers of Mathematics) and also included as a chapter in an expanded American edition of *The Psychology of Learning Mathematics*, published in 1987.

According to Skemp, credit for the origination of the terms relational and instrumental understanding should properly go to Stieg Mellin-Olsen (1939–1995). Mellin-Olsen was a Norwegian mathematics educator and theorist who (like Skemp) was very prominent in mathematics education internationally for many years. The instrumental/relational distinction appears to have been originally proposed by Mellin-Olsen and then was explored in more depth in a comparative study of English and Norwegian mathematics curricula on which Skemp and Mellin-Olsen collaborated (Skemp and Mellin-Olsen 1973, as cited in Mellin-Olsen 1981).

In introducing the terms relational and instrumental understanding, Skemp notes that while *understanding* may be a commonly stated goal for both teachers and students in mathematics education, this term can actually hold multiple meanings. Skemp writes that many math educators likely conceptualize understanding as he does, as *knowing what to do and why*, which he refers to as relational understanding. In contrast, he points out that some students and teachers may have a different way of thinking about understanding – more akin to *rules without reasons* or what he calls instrumental understanding. Skemp notes that instrumental understanding was not something that he had previously considered to be understanding at all.

Other than providing the memorable phrases *knowing what to do and why* (for relational) and *rules without reasons* (for instrumental), Skemp does not provide an explicit or elaborated definition of relational and instrumental understanding. However, it is possible to extrapolate what he appears to mean with these terms through a close reading of this seminal work. Instrumental understanding involves "memorising which problems a method works for and which not, and also learning a different method for each new class of problems" (Skemp 1987, p. 159), is a desire to know "some kind of rule for getting the answer" (p. 155) so that a student can "latch on it and ignore the rest" (p. 155), involves knowing "a multiplicity of rules rather than fewer principles of a more general application" (p. 155), is about developing "proficiency in a number of mathematical techniques" (p. 156), may be potentially useful in the short term but in the longer term is quite detrimental, and generally involves conceiving of mathematics as a set of isolated, unrelated set of techniques ("fixed plans" (p. 162)) which should be memorized. Relational understanding is described in even less detail – but with the clear assumption that relational is defined by what all that it is not – as the opposite of instrumental. A person with relational understanding has developed a "mental map" (p. 162) or "conceptual structure" (p. 163) of the mathematics that he/she is learning.

Note that in his writings, Skemp uses the adjectives relational and instrumental to modify a host of different nouns. Most prominently, Skemp writes about relational and instrumental *understanding*, to describe kinds of knowledge that learners may develop. Similarly, Skemp also writes about relational and instrumental *knowledge*, as well as relational *schemas*. In addition, Skemp writes about instrumental and relational *mathematics*, to suggest that (for example) the mathematics that is taught when a teacher holds instrumental goals for student learning is quite different from the mathematics that is taught when the teacher holds relational goals for student learning. Skemp also uses the phrases relational and instrumental *thinking*, which seem to be used synonymously with understanding. In one instance Skemp refers to relational *mathematicians*, which appears to refer to mathematicians who use relational thinking. Finally, by implication Skemp also writes about relational and instrumental *teaching*, where relational teaching seeks the development of relational understanding and instrumental teaching seeks instrumental understanding.

Although Skemp was clearly a proponent of relational understanding, given the prevalence of teaching geared toward instrumental understanding, he attempts to articulate what might be some benefits of thinking instrumentally. First, he notes that it is usually easier to develop instrumental understanding; "if what is wanted is a page of right answers, instrumental mathematics can provide this more quickly and easily" (Skemp 1987, p. 158). Second, instrumental understanding can provide a more immediate and apparent set of rewards, provided one applies rules correctly to generate correct answers. Third, instrumental thinking often leads to the correct answer more quickly and reliably than relational thinking. As a result, Skemp notes that, "even relational mathematicians often use instrumental thinking" (p. 158). (He notes that, "This is a point of much theoretical interest, which I hope to discuss more fully on a future occasion" (p. 158), although there is no evidence that he returned to this particular topic in his later writings.)

In terms of the advantages of relational understanding, Skemp notes four. First, Skemp claims that relational understanding is more adaptable – meaning that relational knowledge can allow students to be able to modify a known problem-solving strategy so that it is helpful for solving unfamiliar problems. Second, Skemp notes that while relational mathematics is harder to learn, it is easier to remember. While instrumental thinking necessitates remembering a large number of rules, relational thinking involves also knowing how all of the rules are interrelated, and Skemp claims that knowing these interrelationships between rules ("as parts of a connected whole" (Skemp 1987, p. 159)) results in longer-lasting learning. Third, Skemp claims (based on evidence from uncited "controlled experiments using non-mathematical material" (p. 159)) that relational learning requires fewer extrinsic rewards and punishments to learn. And fourth, Skemp claims that the development of relational knowledge leads learners to seek out new knowledge and continue to learn relationally.

Although he articulates advantages of both instrumental and relational understanding and also notes the presence of many contextual and situational factors in schools that may push teachers toward advocating instrumental understanding, Skemp clearly advocates for relational understanding. He describes a personal anecdote where the benefits of relational understanding, and how it differs from instrumental learning, became very clear to him. While in a strange town to meet with a colleague, Skemp notes that he learned a small number of routes for getting around, such as between his hotel and his friend's office and between his hotel and the university dining hall. Knowledge of these set of fixed routes or plans was certainly quite useful. But when he had free time, he began to explore – not explicitly to learn new routes between points of interest but rather to "learn my way around" (Skemp 1987, p. 162) and see what might be of interest. His goal for exploring the town was to "construct in my mind a cognitive map of the town" (p. 162). Although an observer viewing Skemp walking around town might not be able to distinguish the differences between these two types of activities, for Skemp these activities had very different goals. In the first case, the goal was merely to get from point A (e.g., his hotel) to point B (his friend's office). But in the second case, his goal was to further develop his knowledge of the town. Skemp connects the first kind of activity with instrumental understanding, where one develops a set of fixed plans that enable one to reach a certain set of goals. These plans provide a prescription for what to do next – e.g., take the second right and cross the street by the cafe. Each step of the plan is guided solely by the local situation – the instruction "take the second right" is only useful and comprehensible when one has correctly completed all immediately preceding steps. As a result, one is very limited in what can be accomplished in terms of navigating through the town, given such a small and fixed set of plans. In contrast, the second kind of activity is similar to relational understanding, in that the development of a mental map of the town could enable Skemp to travel from any starting point to any ending point in the town.

In addition to advocating a focus on relational understanding, Skemp also notes that he considers it potentially problematic when students and teachers hold mismatched views on what understanding means – such as when teachers desire that students develop relational understanding, while students only seek instrumental understanding (and vice versa). Similarly, teachers might hold a different view of understanding than the text that they are using. Skemp proposes that such mismatches are endemic and often unrecognized by mathematics educators.

It is worth noting that, since the mid-1980s, mathematics educators have come to rely upon a different terminological framework for describing mathematical understanding. Instead of Skemp's relational and instrumental understanding, Hiebert's *conceptual and* procedural knowledge (Hiebert and Lefevre 1986) has become dominant in both the research and policy arenas. These two terminological distinctions are not isomorphic (Haapasalo and Kadijevich 2000; Star 2000). In addition, some scholars have raised concerns about the terminological distinction between conceptual and procedural knowledge, including whether this framework has resulted in misunderstandings and misplaced priorities (Star 2005, 2007) as well as communication and collaboration difficulties between different groups of scholars who study mathematical understanding (Star and Stylianides 2013).

The field currently lacks consensus on which framework(s) are optimal, but clearly Skemp's notion of instrumental and relational understanding will and perhaps should continue to be widely used by mathematics educators throughout the world for advancing important conversations about mathematical understanding.

## Cross-References

▶ Algorithms
▶ Theories of Learning Mathematics

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# Instrumentation in Mathematics Education

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## Keywords

Appropriation; Artifact; Document; Instrumental genesis; Instrumentalization; Instrumentation; Orchestration; Resources; System of Instruments

## Definition

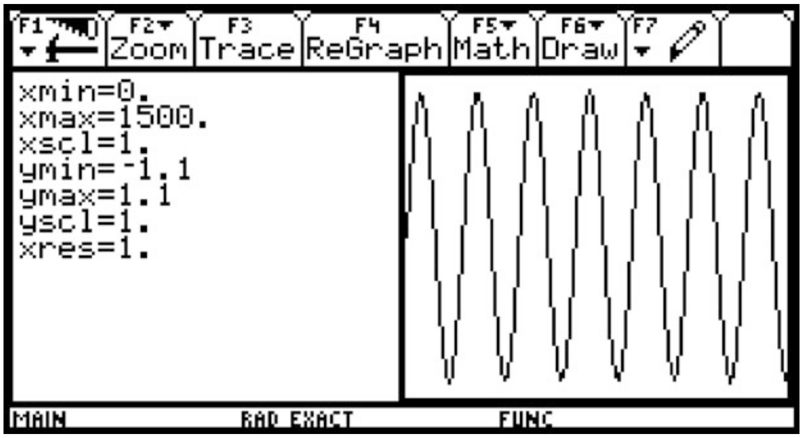
In order to define *instrumentation* in the context of mathematics education, it is necessary to define *instruments*: at this stage of this article, we do not differentiate between *instruments* and *artifacts*, i.e., we regard them as things that are *created* and *used* by humans to help, assist, support, enlarge, and empower their activity. Instrumentation is the action to give someone an *instrument*, or the process by which someone acquires an instrument, in order to perform a given activity. The notion of instrumentation is part of a network of concepts; we will focus here on the main dialectical relationships between them.

## Instrumentation and Instruction

Contrary to the common perception that mathematics is a pure mental activity, the importance of instruments in mathematical activity has been largely acknowledged: "the development of mathematics has always been dependent upon the material and symbolic tools available for mathematics computations" (Artigue 2002, p. 245). What is true in general is all the more true for these essential parts of mathematical activity that are "teaching and learning mathematics." Proust (2012), for example, noticed the richness of *school material* already available for teaching mathematics in Mesopotamia, 4,000 years ago: "the resources of masters result therefore from a complex and two-way process between learning and scholarship, involving memory, oral communication, writing, and probably material artifacts" (p. 178).

In a survey conducted for the centennial of ICMI, Maschietto and Trouche (2010) provided evidence that the *interest* in and *influence* of instruments for mathematics teaching and learning had been questioned for a long time. For example, in the case of ICT, they noticed that "the ease and speed of computations disrupt the organization of mathematical work: when a computation is long and difficult, it is necessary to be sure of its relevance before tackling it; whereas, when a computation can be made by simply pushing a key, it is possible to store sets of results, and only afterwards embark on the process of sorting them, according to the objectives associated with the task in hand" (p. 34).

This leads us to a comprehensive view on instrumentation, seen not only as an *action* (by which someone acquires an instrument) but also as *the influence of this action* on a subject's activity and knowledge. This view is coherent with the origin of the word: *instrument* and *instruction* have the same Latin root "instruo," meaning to build and to




**Instrumentation in Mathematics Education, Fig. 1** A confusing (for some students) representation of the function ln x + sin x

assemble. This view is in line with Vygotsky's work, situating human activity in a world of history and culture, where the instruments, psychological as well as material, are essential. Vygotsky (1962) quoted Francis Bacon (1600) saying: "Nec manus, nisi intellectus, sibi permissus, multam valent: instrumentis et auxilibus res perficitur" (human hand and intelligence, alone, are powerless: what gives them power are *instruments* and assistants provided by culture – our translation). Vygotsky (1981) wrote also that "by being included in the process of behavior, the psychological tool alters the entire flow and structure of mental functions. It does this by determining the structure of a new instrumental act, just as a technical tool alters the process of a natural adaptation by determining the form of labor operations" (p. 137).

With more and more complex instruments (e.g., calculators) being used in the mathematics classroom, mainly used by students, it has become clear that this process of "alteration" needs to be further investigated. Tools contribute to the shaping of students activities (Noss and Hoyles 1996) – and to associated knowledge. Guin and Trouche (1999) found that students' answers to the question "Does the f function defined by f (x) = ln x + sin x have a limit + ∞ as *x* approaches + ∞?" depended to a large extent on the environment.

If students had a graphic calculator, due to the oscillation of the observed graphical representation (Fig. 1), 25 % of them answered that this function had no limit. Within a group of students of the same level without a graphic calculator, only 5 % of wrong answers were collected. The students' work was thus altered by a "confusing" graphical representation of the function, a representation which was *understood* as the true mathematical object, encapsulating, for the students, all its properties.

This type of phenomena in new technological environments was studied at the end of the last century (e.g., Lagrange et al. 2003), and based on this research, a theoretical approach focusing on the link between mathematics instrumentation and instruction emerged.

## The Dialectical Relationships Between *Artifact* and *Instrument*

The need for a theoretical approach of instrumentation led researchers in mathematics education to turn towards scientific domains researching instruments and cognition, in particular the field of cognitive ergonomic. Verillon and Rabardel's studies (1995) followed the work of Vygotsky's theorization, focusing on learning processes involving instruments. They stressed the essential difference between an *artifact* (given to a subject or acquired by him) and an *instrument* as a psychological construct: "The instrument does not exist in itself, it becomes an instrument when the subject has been able to appropriate it for himself and has integrated it" (p. 84). In this frame, an instrument can be considered as a mixed entity made up of an artifact component (an artifact or the part of an artifact mobilized in the activity) and a cognitive component (what a subject learned from/for using the artifact in this context). The development of an instrument is a complex process, which Verillon and Rabardel coined *instrumental genesis*. They claim that this process needs time and influenced by the artifact's characteristics (its *potentialities* and its *constraints*), to the subject's history (his/her knowledge and former method of working) and to his/her activity, when working with a problem to be solved. Following this approach, one can generally speak of "*an* artifact" (e.g., a hammer or a calculator), but one has to be more specific when talking about an instrument: *the* instrument of somebody, for performing a given type of task, at a given step of its development.

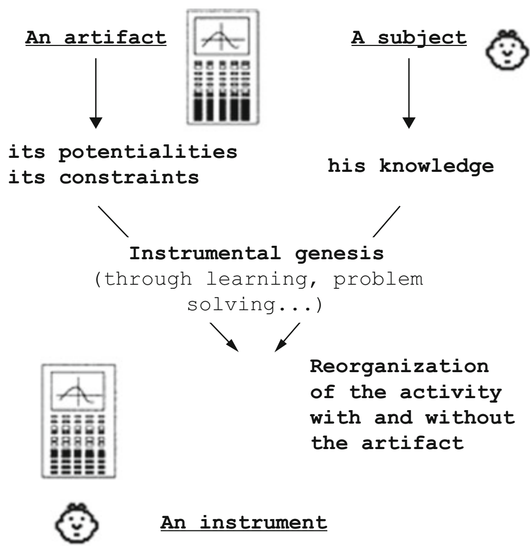
This frame leads us to clarify our initial definition: instrumentation is the action by which a subject acquires an *artifact*, and the *effect of this action* on the subject, who develops, from this artifact, an instrument for performing a task. An instrument is thus made up of an artifact component and a cognitive component (knowledge necessary for/from using the artifact for performing this type of task).

In the field of mathematics education, several French researchers (e.g., Guin and Trouche 1999; Artigue 2002) appropriated this theoretical framework for analyzing the effect of the integration of ICT (e.g., Computer Algebra System) in mathematics learning (Fig. 2).

They developed what became internationally known as the *instrumental approach of didactics of mathematics* (Guin et al. 2005), in interaction with other theoretical approaches (Drijvers et al. 2012). This approach has the following advantages:

– It situates the effects of artifacts not as "parasites," but as essential components of learning processes (Fig. 2) to be integrated by the teacher.
– It leads, through the notion of genesis, to the analysis of instrumentation and learning as long-term processes.
– The notion of genesis leads to consider an instrument as something living: it was born "to do something" and goes on living across a field of mathematical problems.

Finally, this approach leads to consider instrumentation at the heart of the "dialectics between

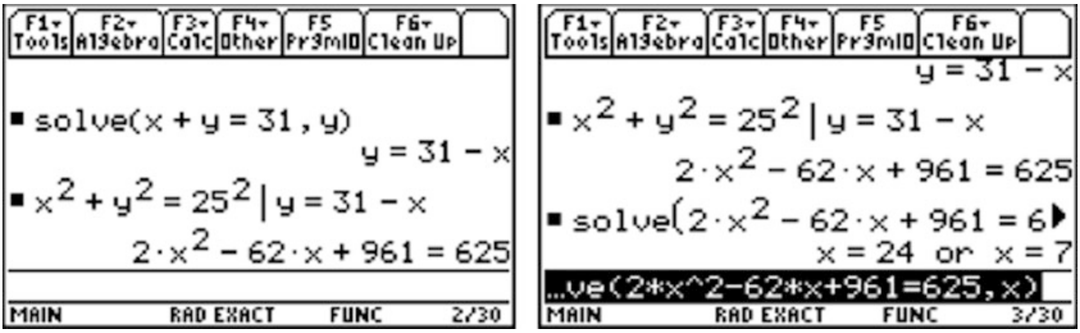



**Instrumentation in Mathematics Education, Fig. 2** A schematic representation of an instrumental genesis (Guin and Trouche 1999, p. 202)

technical and conceptual work" (Artigue 2002). Drijvers (in Guin et al. 2005) illustrates this dialectics by showing how a student uses a calculator for solving a system of two equations with two unknown. He extracts y from the first equation and then replaces the expression of y (function of x) in the second equation and finally solves this equation containing only one unknown, x. The action developed by the student (Fig. 3) can appear as a sequence of gestures (isolate-substitute-solve) on the keypad of the calculator, but it requires considerable knowledge.

For example, "the fact that the same solve command is used on the TI-89 for numerical solutions and for the isolation of a variable requires an extended conception of solve: it also stands for taking apart a variable and for expressing one of the variables in terms of one or more others in order to process it further" (Drijvers et al. 2010, p. 227). Each instrumental genesis thus appears both as a process of appropriating an artifact for doing something and a process of learning something on mathematics.

Learning new things in mathematics could engage new ways of using the artifact: beyond the instrumentation process, there is actually a dialectic relationship between an artifact and




**Instrumentation in Mathematics Education, Fig. 3** The result of a sequence of gestures (isolate-substitute-solve) on the keypad of a calculator, as it appears on its screen

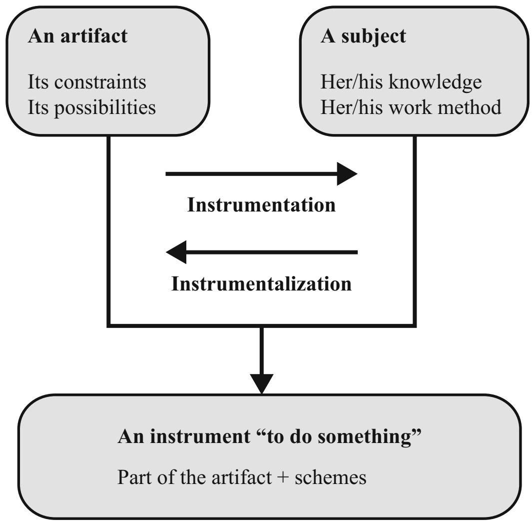
the instrument developed by its integration through the subject's activity.

## An Essential Dialectic Relationship Between Instrumentation and Instrumentalization

Looking at Fig. 2, it clearly appears that, at the root of each instrumental genesis, there are two "protagonists": an artifact and a subject. Up to now, we have just considered the effect of the first one on the second (more exactly: the effect on the subject acquiring the artifact in terms of his/her activity and knowledge), that is to say, the instrumentation process. Yet Verillon and Rabardel (1995) consider that instrumental geneses are made up of two interrelated processes:

– An *instrumentation process* (directed towards the subject)
– An *instrumentalization process* (directed towards the artifact)

This second process appears on the schema representing an instrumental genesis (Fig. 4). It has been described by Guin et al. (2005, p. 156), in the case of calculators: "This process is the component of instrumental genesis directed towards the artifact. Instrumentalization can go through different stages: a stage of discovery and selection of the relevant functions, a stage of personalization (one fits the artifact to one's hand) and a stage of transformation of the artifact, sometimes in directions unplanned by the designer: modification of the task bar, creation of keyboard shortcuts, storage of game programs, automatic execution of some tasks (calculator manufacturers' websites and personal web sites




**Instrumentation in Mathematics Education, Fig. 4** Instrumentation and instrumentalization, seen as two essential components of instrumental geneses (Guin et al. 2005). An instrument is here defined as a mixed entity composed of a part of the artifact and a scheme, a scheme being, according to Vergnaud (1996), the invariant organization of activity to perform a type of task, including rules of action and specific knowledge, product and spring of the activity

of particularly active users often offer programs for certain functions, methods and ways of solving particular classes of equations etc.). Instrumentalization is a differentiation process directed towards the artifacts themselves."

However, this process remained quite hidden in the first studies analyzing the integration of ICT in mathematics education in light of instrumental approach of didactics. For example, in

Artigue's seminal work (2002), the word instrumentation is quoted 20 times (and appears in the title), while the word instrumentalization is quoted only one time (and is not practically used in the didactical analysis). This can be explained by the following: firstly, the instrumental approach of didactics has been developed for analyzing the unexpected effects of artifacts *on* students' mathematics learning, giving to *instrumentation* processes a major importance; secondly, the artifacts at stake (as Computer Algebra System or Dynamic Geometry Software) were complex and quite closed – in these conditions, the effects of student's action on the given artifacts did not appear at a glance; they were hidden (e.g., using a calculator to store games).

There is perhaps a third, deeper reason linked to a first "classical" reading of Vygotsky, as Engeström et al. (1999, p. 26) pointed out: "Both in the East and in the West, it has been almost a truism that internalization is the key psychological mechanism discovered by the cultural-historical school [. . .] Symptomatically, Vygotsky's writings that deal with creation and externalization, especially the Psychology of art, have received very little attention. And it seems to be all but forgotten that the early studies led by Vygotsky, Leont'ev and Luria not only examined the role of given artifacts as mediators of cognition but were also interested in how children created artifacts of their own in order to facilitate their performance."

In subsequent studies (e.g., Trouche and Drijvers 2010), instrumentation and instrumentalization appear to be mentioned in a more balanced way, as two inseparable ingredients of every instrumental genesis. This evolution is linked to several factors: a better mastering of the instrumental approach, perceiving the relationship between artifact and subject as essentially dialectic; a deeper and more comprehensive view of "appropriation processes" (to appropriate something means to make something *proper*, to *customize* it); and a wider view of what an artifact is (§ 5).

This leads us to reformulate our initial definition: instrumentation and instrumentalization are two intrinsically intertwined processes constituting each instrumental genesis, leading a subject to develop, from a given artifact, an instrument for performing a particular task; the instrumentation process is *the tracer* of the artifact on the subject's activity, while the instrumentalization process is the tracer of the subjects' activity on the artifact.

### From a *Set of Artifacts* to a *System of Instruments*: The Crucial Notion of "Orchestration"

We have, up to now, explained the dynamics of making *one* instrument from *one* artifact. Actually, the situation is a more complex, for at least two reasons:

– Firstly, a student has a set of artifacts (. . .) at his/her disposal, for performing a particular task (paper/pencil, rule, compass, calculator). A single computer can be considered as a *toolbox*, including a set of artifacts (e.g., CAS, spreadsheet, word processing). The trend of digitalization is at the same time a trend of miniaturization, a trend of gathering very different artifacts in the same envelope (e.g., MP4 or digital tablet), and a trend of facilitating the switch from one representation to another, from one application to another. Under these conditions, for each type of task, a student will develop *an* instrument, by using and appropriating *several* artifacts. Beyond the treatment of one type of task (solving a type of equation, studying a type of function, etc.), each mathematical problem usually requires the simultaneous activation of several instruments, related to several types of tasks. A student needs to develop, from a set of artifacts, a coherent system *of instruments*. The combination and articulation of several instruments demand a *command of the process* (Trouche 2004) and requiring assistance from the teacher.
– Secondly, the development of an instrument by a given subject is never an isolated process. The instrumental geneses always combine individual and social aspects. Particularly in a teaching context, students usually have to face the same type of task at a given moment, and they simultaneously develop their instruments in the same context. That requires another level of combination of different instruments by the teacher.

The necessity of combining, on a coherent manner, different instruments in action leads to the notion of *instrumental* orchestration. Trouche (2004) has introduced this concept to model the work of a teacher taking into account, when designing her teaching, the set of artifacts available for each student and for the classroom, and the stage of development of the different students' instruments. As in the case of an orchestra, an instrumental orchestration stands to make the different student instruments playing together with the same objective (execute a work or solve a problem). Designing an orchestration needs to carefully choose a mathematical problem, according to the didactical goals, to anticipate the possible contribution of the artifacts to the problem solving, and to anticipate, in this context, the possible instrumentation of students by these artifacts. An orchestration appears thus as a musical score, pinpointing different phases for the problem solving and, in each phase, the monitoring of the various artifacts (how the artifacts could be mobilized by the students and by the teacher).

Drijvers et al. (2010) deepened this notion, showing the necessity, for the teacher, to adjust, on the spot, her monitoring of the artifacts: they named "didactical performance" the way a teacher adjust her orchestration due to her understanding of the stage of development of each student instrument. Actually, orchestrations appear thus as resources assisting teacher activity, developing into teachers instruments through the two processes of instrumentation and instrumentalization (the didactical performance being, in this point of view, an expression of instrumentalization).

## Conclusion: From Student Instrumentation to Teacher Instrumentation

Starting from a learner's instrumentation point of view, we would like to conclude this article by a teacher's professional development point of view, asking the question: what elements are instrumenting a mathematics teacher activity? Certainly these are textbooks, different software (dedicated, or not, to mathematics), various repertoires of mathematical problems and orchestrations (see above), but also students' reactions, colleagues' comments, and, in the thread of digitalization, much more: Gueudet and Trouche (2009) pinpointed this dramatic change in teachers' interactions, using emails, websites, forum, blogs, etc.

Gueudet and Trouche (ibidem) took into account this metamorphosis and enlarged the instrumental approach: they named "resources" (instead of artifacts) all the "things" that are supporting teacher activity and "documents" what is developped by a teacher to do and in doing her teaching (instead of instruments). This is in line with the field of information architecture (Salaün 2012) where documents are developed by teachers from these resources, for performing their teaching. This new approach, combined with other approaches of the field (Gueudet et al. 2012), appears as a blossoming development of the instrumental approach, allowing to fully express the potentiality of this concept:

– The instrumentalization processes are strongly reinforced, a teacher collecting, modifying, and adjusting resources to build the material of his/her teaching; instrumentation and instrumentalization clearly developed as two interrelated processes.
– The social aspects are also strongly stimulated, the Internet offering a lot of opportunities to exchange and share resources; individual and social aspects of document geneses clearly appear as feeding each other.

We have underlined in this article some main dualities: instrumentation/instruction, artifact (vs. resource)/instrument (vs. document), and instrumentation/instrumentalization. They appear at the heart of each process of learning – and of human development.

**Acknowledgments** Thanks to Ghislaine Gueudet and Birgit Pepin for their reading of the preliminary version of this article and their inspiring comments.

## Cross-References

▶ Activity Theory in Mathematics Education
▶ Concept Development in Mathematics Education
▶ Information and Communication Technology (ICT) Affordances in Mathematics Education

▶ Instrumental and Relational Understanding in Mathematics Education
▶ Learning Environments in Mathematics Education
▶ Learning Practices in Digital Environments
▶ Teaching Practices in Digital Environments
▶ Theories of Learning Mathematics

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# Interactionist and Ethnomethodological Approaches in Mathematics Education

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## Keywords

Micro-sociology; Negotiation of meaning; Reflexivity; Accounting practice; Socio-mathematics norm; Interactional procedures; Argumentation; Interactional routines; Pattern of interaction; Interpretative classroom research

## Definition

Symbolic Interactionism and Ethnomethodology are sociological approaches that are based on the social psychology of George Herbert Mead and the phenomenological sociology of Alfred Schutz (Schutz 1932/67; Alfred Schütz is of German origin and his family name is originally written with the German umlaut "ü". In publications in German his name appears in its orgininal spelling). The empirical interest is the immediate concrete situation of the communicative exchange between individuals. Goffman calls this the "situational perspective," meaning a focussing on the occurrence to which an individual can be "alive to at a particular moment" (Goffman 1974, p. 8). These everyday episodes are governed by symbolic interaction: the meanings that people ascribe to things and events are developed and modified in an interactive process of negotiation of meaning based on the situational *interpretations* of the symbols used in their remarks (Blumer 1969).

Ethnomethodology focusses on the aspect by which means or "methods" the participants of a social situation accomplish their negotiations, how they achieve a "working consensus" about what is momentarily taken as shared. Characteristic for this approach is the identification of the "activities whereby members produce and manage settings of organized everyday affairs" as well as their "procedures for making those settings 'accountable' (Garfinkel 1967, p. 1). This identification is one of the basic ideas of ethnomethodology and firms under the concept of "ethnomethodological reflexivity" or "indexicality."

The general achievement of this type of research is the development of contextual theories which take into account the oral and processual, the specific and nonconformist, the local and domain-specific, and the historical and biographical. Abandoning decontextual theories is not meant to be an abandonment of research based on scientific standards. It is rather a shift to the empirically grounded development of "middle-range-theories" (Merton 1968, p. 50f).

The research methods of these approaches are characterized by two issues: they are *reconstructive* in the sense of redrawing the process of negotiation of meaning, and they are *interpretative* in the sense that they use hermeneutic methods of *interpreting* the interpretations of actors in a concrete situation that allow them to come to a working consensus with the other participants. Usually these methods are based on transcripts of audio or video recordings. Widely used is the technique of conversation analysis.

Both approaches are subsumed under what is called "micro-sociology," the foremost interest being the here and now as people interact with each other and create social reality. They accomplish this in their everyday affairs by talking with each other in symbolic ways about this reality, which in this sense is not (pre-)given but a result of their negation of meaning. Social reality comes into existence in a series of such "local productions." "They understand society to be something that is lived in the here and now, in the face-to-face and mediated interactions that connect persons to one another" (Denzin 1992, p. 22).

### Reception of Symbolic Interactionism and Ethnomethodology in Mathematics Education

In mathematics education usually these two sociological approaches are included under the concept of micro-sociology and/or rather unspecifically as the theoretical foundation for interpretative research. Historically, one can identify at least two sources that adapted these two sociological approaches in an attempt to overcome certain specific limitations of traditional psychologically oriented theories in mathematics education:

- Bauersfeld (1980) describes the limitation of attempts of curriculum implementation as far as they are based on a combination of subject matter theories and psychological assumptions about students' learning. This combined approach does not sufficiently take into account the dynamics of the everyday mathematics classroom life. Bauersfeld speaks of the "hidden dimensions" of the mathematics classroom. The "arena of interaction" with its patterns of interaction, routines, and interactive stereotypes creates a classroom reality that often is counterproductive to the well-meant intentions of teacher, schoolbook author, and/or curriculum developer. Most influential in this respect was the study of Mehan (1979). He describes a fundamental interaction pattern in teacher-guided lessons: initiation–reply–evaluation. Based on this initial work, several "patterns of interaction" had been reconstructed in everyday mathematics classroom situations, such as the "funnel pattern" by Bauersfeld (1980) or the "elicitation pattern" and various "thematic patterns."

- The second approach is an adaption of Steffe's "teaching experiment" (Steffe et al. 1983) to the conditions of a classroom situation with a larger group of students by Wood et al. (1993). Steffe's research design is a form of individual teaching of one researcher with one child. Wood et al. expand this approach to the regular classroom setting with 20 or more students and one teacher. The authors call this a "more naturalistic" access to mathematics learning situations (Wood et al. 1993, p. 8). They understand the learning of mathematics as an active process of problem solving whereby the constrains and contradictions of this process emerge in the classroom interaction. The enhancement of these interaction processes depends on "socio-mathematics norms" (Wood et al. 1993, p. 23), which also must be negotiated in these processes.

## Perspectives for Future Research

There are several perspectives for current or future research that are based on Symbolic Interactionism and/or Ethnomethodology. Taking the specific demands of mathematics education into account, usually the application of these two theories is intertwined with additional approaches: a subject matter-oriented curriculum theory, psychological theories of learning, or pedagogical theories of mathematics teacher education. The reference to Symbolic Interactionism and/or Ethnomethodology is more or less transparent. In the following these diverse research activities will be described with respect to:

- Sociological aspects of a theory of mathematics learning
- A combination and expansion of Symbolic Interactionism and Ethnomethodology with other socio-constructivist theories

### Sociological Aspects of a Theory of Mathematics Learning

From a mathematics education perspective, a major interest in applying these two micro-sociological approaches lies in the further elaboration of a theory of mathematics learning that constitutively takes into account the interactional aspects of the social conditions of mathematics teaching and learning situations. Primarily, this leads to research about typical patterns of interaction in mathematics classes as already mentioned above. Thus, the concept of learning evolves in a way that the sociological dimension of learning is more intensively stressed: learning is not (only) to be conceptualized as acquisition of knowledge, but it also can be understood as the individual's process of incrementally *participating in mathematics discourses* (Sfard 2008).

Explanation and justification and their specific demands are often named as major features of these discourses. Various studies about the "culture of argumentation" in mathematics teaching and learning situations have been conducted.

Within this framework of sociological aspects of a theory of mathematics learning, as a specific interest one can identify the use of the computer and Internet in mathematics classes. Also here symbolic interactionist and ethnomethodological research projects have been conducted or are still in process (Jungwirth 2005).

The fundamental research setting is the mathematics classroom. This research is complemented by studying mathematics learning situations in preschool and kindergarten, in families, and at the college level. Another research strand can be identified in the observation of specific groups of students in regular classroom situations focussing, for example, on small group activities or on second-language learners in mathematics classrooms.

### Combination and Expansion of Symbolic Interactionism and Ethnomethodology with Other Theoretical Approaches

The main research interest of symbolic interactionist and ethnomethodological research is concerned with the verbal aspects of social interaction leading to such general philosophical and linguistics questions as to the nature of language in mathematics and mathematics teaching/learning situations. An expansion of these theoretical aspects is found, for example, in the embedding of inscriptional aspects of mathematics communication or in the study of the aspect of gesture in such interaction processes.

Research adhering to the principles of Symbolic Interactionism and Ethnomethodology can be characterized as one that is based on a socio-constructivist position. Typical for these two approaches is the view of social reality as a series of local productions (see above). As such they are resistant to approaches that search for general theories (Denzin 1992, p. 22) that are more abstracted from the individual's context and environment. In this respect these two theories differ from cultural historical approaches that usually refer to the work of Vygotsky and Leont'ev. A current research endeavor is the integration of these two schools of socio-constructivism (for first attempts see Krummheuer 2012).

## Cross-References

▶ Argumentation in Mathematics Education
▶ Bilingual/Multilingual Issues in Learning Mathematics
▶ Collaborative Learning in Mathematics Education
▶ Constructivism in Mathematics Education
▶ Cultural Influences in Mathematics Education
▶ Discursive Approaches to Learning Mathematics
▶ Immigrant Students in Mathematics Education
▶ Informal Learning in Mathematics Education
▶ Language Background in Mathematics Education
▶ Semiotics in Mathematics Education
▶ Sociomathematical Norms in Mathematics Education

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## Interdisciplinary Approaches in Mathematics Education

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### Keywords

Discipline; Activity theory; Object/motive; Situated cognition

### Introduction

In the history of humanity, early forms of labor that provided for the satisfaction of basic needs – food, shelter, and clothing – gave rise to new, specialized forms through a progressive division of labor. Disciplines emerged – first stonemasons, farmers, and tailors and then mathematicians and mathematics teachers. Those who were highly skilled in one discipline were less so or had no skills in other disciplines. Eventually, theoretical disciplines emerged such as when some master craftsmen began to specialize in making building plans and others turned these plans into real buildings. Today, there is often very little communication between the disciplines, each of which forms a disciplinary "silo." The idea of interdisciplinarity is to combine multiple (academic) disciplines into one activity. Whereas this may appear to be simple and straightforward, in practice it turns out that those participating in an interdisciplinary endeavor often find it difficult to work with others across traditional disciplinary boundaries. Nevertheless, interdisciplinarity involving mathematics education has become of considerable interest to some mathematics educators (e.g., Sriraman and Freiman 2011).

### Definition

Interdisciplinarity denotes the fact, quality, or condition that pertains to two or more academic fields or branches of learning. Interdisciplinary projects tend to cross the traditional boundaries between academic disciplines.

### Interdisciplinarity and Mathematics (Education)

The very idea of an (academic) discipline embodies strength and weaknesses. On the one hand, discipline means orderly conduct that is the result of physical and mental discipline. Considerable discipline in the second sense of the word is required to be and become an outstanding practitioner in the former sense of the word. The strength of being disciplined (e.g., doing mathematics) is also the weakness. Those who are very disciplined in their ways of looking at problems also are very limited in the ways they can see a problem. The contradictions arising in and from interdisciplinary projects are in part linked to this limitation. To overcome the limitation of disciplinary approaches, there has been an increasing interest in establishing connections between different fields. In mathematics and mathematics teaching, interdisciplinarity often faces problems especially at the high school level because in other fields (e.g., biology, chemistry, or physics), mathematics is considered to be a mere service discipline rather than real mathematics. The curricular intentions in these subject matters and mathematics is different, which often leads to tensions of where to place the emphasis. In actual school practice, there tends to be very little work across disciplines and curriculum integration.

From the perspective of activity theory, the origin of these problems is easily understood (Roth and Lee 2007). This is so because activity theorists accept that "the production of ideas, conceptions, consciousness is initially immediately intertwined with material activity and the material intercourse of humans, language of real life" (Marx and Engels 1969, p. 26). Material activity and a focus on a particular object of activity involve different forms of relations between people. Because the relations between people ultimately become higher psychological functions (Vygotsky 1989), very different forms of knowing and understanding emerge within each discipline: consciousness and cognition are fundamentally situated (see ▶ Situated Cognition in Mathematics Education). That is, with each discipline, there are different forms of consciousness; and there are different collective *object/motives* (see ▶ Activity Theory in Mathematics Education) pursued in each discipline even if they work with precisely the same material objects. Interdisciplinarity requires new object/motives, which inherently differ from the object/motives that characterize the root disciplines. What is of interest in one discipline is not of interest to another. But, as the interdisciplinary design work for modern technology in the workplace shows (and current scientific practices more widely), these new interdisciplinary endeavors, while helping communication across the disciplines, create new objects and discourses that are different from the root disciplines (Ehn and Kyng 1991). Although school contexts differ from workplace settings, similar issues arise especially at the high school level where integration of mathematics with other school subjects tends to be rare. It does not come as a surprise, therefore, when as a result of interdisciplinary projects, mathematics teachers no longer find *their* mathematics just as other specialist teachers no longer find sufficient attention to their discipline in joint projects.

The differences between disciplines are beautifully illustrated in a classical case of the history of physics and mathematics. The Dirac δ function had, for mathematicians, strange properties: $\delta(x = 0) = \infty$; $\delta(x \neq 0) = 0$; $\int \delta(x)dx = 1$. Whereas it was useful in physics – because it could be used to model a very sharp pulse – it became a full mathematical object only over time. But physicists had a new tool that allowed them to deal with interesting phenomena such as the motion of waves in oscillators when stimulated by a sharp pulse. Although mathematicians had been interested in generalized functions before, the δ function became a fully fletched mathematical object only later when Laurent Schwartz developed the theory of distributions. The δ function then can be viewed as a distribution, the limit case of a Gaussian curve that is infinitely narrow and infinitely high:

$$\delta_a(x) = \lim_{a \to 0} \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2}.$$

Similarly, studies in workplace mathematics show that although the people working in a fish hatchery may make extensive use of mathematical processes and objects, they do not understand themselves as doing mathematics – they will describe themselves as raising fish. In their hands, mathematical entities are radically different than these are in the hands of mathematicians or mathematics teachers. This is so because the object/motive of a mathematician is mathematical; for a fish culturist, the object is to raise fish and mathematics is but a tool. Similarly, whereas a mathematician can find many patterns involved in the construction of a Sioux tent (Orey and Rosa 2012), the Sioux did not worry about mathematics but about having a shelter that withstands the intemperies of the prairies (see ▶ Ethnomathematics). Again, the object/motive of the mathematician is mathematical patterns and relations, whereas the object/motive of the Sioux is shelter from bad weather.

Interdisciplinarity in mathematics education turns out to be difficult, in part because the curricula specify very different goals for the subject areas that might be combined in one student project. Thus, for example, the calculus curriculum for grade 12 might specify – as it does in British Columbia – a prescribed learning outcome to be: define and evaluate the derivative at x = a as

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ and } \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

In the British Columbia physics curriculum at the same grade level, students learn about motion through the equations

$$x(t) = x_0 + v_0 t + \frac{a}{2} t^2 \text{ and } v(t) = v_0 + at$$

where $x$ is position, $v$ is velocity, $a$ is acceleration, and $t$ is time. The students in mathematics differentiate functions; the students in physics calculate problems given certain values of the constants and variables. But there is often very little overlap in the curriculum and little interdisciplinary inquiry – even though, in this case, it is possible to design curriculum in a way that students come to understand in a very qualitative way both physics and mathematics – and even other subjects, such as the arts. They may do so even before entering formal calculus courses, as the following example shows.

In a teaching experiment, the researchers provided tenth-grade students with a tool that allowed participants to exert forces that determined the acceleration of a car on a track; the forces where measured with force sensors (Whitacre et al. 2009). The forces and aspects of the car's motion – velocity and acceleration – could be plotted. The research shows that the students developed rich understandings based on the integration of their bodily experiences; the relation of forces, accelerations, and their bodily motions; and the properties of the resulting graphs. A convergence was observed between the different representations of one and the same phenomenon. The study also shows the connections to the arts, for example, how body motion during the act of painting comes to be expressed in the paintings of Jackson Pollock much as the body movements of the students came to be represented in the motion graphs. Whereas these students did not do "pure" and "typical" mathematics, one can argue that they developed embodied forms of sense that will allow them to better understand mathematical properties of functions and their derivatives. This was reported to be the case in another study, where students generated position-time, velocity-time, and acceleration-time graphs by moving carts connected to motion detectors (Roth 1993). They not only came to understand the relationship between their body motions and the graphs but also the relationships between special aspects of the graphs. For example, they began to note that the velocity graph crossed the abscissa ($v = 0$) precisely at the point where the position-time graph was at a maximum or a minimum; and they learned, with a great deal of surprise, that the acceleration was a maximum when the cars were turning rather than when these were near maximum speed. With some pointers on the part of their physics teacher, they came to realize that the slope of one graph was related to the absolute values of another and, therefore, that

$$\upsilon(t) = \frac{dx(t)}{dt} \text{ and } a(t) = \frac{d\upsilon(t)}{dt}$$

They had learned some calculus. The special issues of *Educational Studies in Mathematics*, which focused on gestures and multimodality, further underscore the role of embodied forms of mathematics (Nemirovsky et al. 2004; Radford et al. 2009, 2011).

## Perspectives

Historically, mathematical understandings have arisen from nonmathematical preoccupations in the world where increasing refinements of material entities eventually led to the development of ideal objects typical of mathematics (Husserl 1939). For example, the Greek were preoccupied with objects in and of their everyday lives, including those that they called *kúbos* (cube), *sphaira* (ball), or *kúlindros* (roller). As they become more and more skilled in perfecting these, they eventually developed the ideas of *ideal* cubes, spheres, and cylinders: mathematics was born. Similarly, the purpose of interdisciplinary endeavors involving mathematics may be the development of a rich set of experiences that underpin purely mathematical endeavors sometime later in the students' lives.

As shown in such endeavors as (a) the conferences on "Mathematics and Its Connections to the Arts and Sciences" (MACAS), (b) the "International Community of Teachers of Mathematical Modelling and Applications" (ICTMA), or (c) various groups interested in inquiry-based learning in mathematics (IBL), there are indeed endeavors to integrate mathematics with education in other disciplines. In fact, inquiry-based learning often encompasses mathematics and science and, thereby, practices interdisciplinarity. However, visits to schools in many countries show, however, that a more widespread implementation of interdisciplinary approaches in mathematics education still remains to be achieved. Because each discipline-related activity involves different forms of consciousness, interdisciplinarity will require rethinking mathematics education in terms of the new objects/motives of an interdisciplinary project; and these objects will be (very) different than the object/motives (goals) of traditional mathematics education. Achieving interdisciplinarity means redefining what mathematics education can be. But interdisciplinarity may simply turn into another disciplinary silo (Roth 2011).

## Cross-References

- ▶ Activity Theory in Mathematics Education
- ▶ Calculus Teaching and Learning
- ▶ Ethnomathematics
- ▶ History of Mathematics and Education
- ▶ Problem Solving in Mathematics Education

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# International Comparative Studies in Mathematics: An Overview


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## Keywords

International Comparative Studies; TIMSS, PISA, TEDS-M


## Characteristics

There has been a significant increase in international comparative studies (ICS) on achievement in mathematics in the last few decades. Particularly well-known amongst these ICS are those held under the auspices of the International Association for the Evaluation of Educational Achievement (IEA) and the Organization of Economic Co-operation and Development (OECD) that are briefly described later. This article looks briefly into the following questions: what are ICS? Why are ICS important? What are some issues with ICS? Where are we headed with ICS?

## What Are ICS?

The term comparative can be defined as studying things to find out how similar or different they are. In mathematics education there are several "things" that can possibly be compared internationally: students' achievement, teacher education, mathematics curricula, mathematics education policies and practices, and certainly pedagogical practices. While the "what" to be studied and compared internationally seems quite obvious, the "when" and the "how" to compare are not so. The "when" to compare brings forth a few points: At what point in time do we make the comparative study? What is the frequency with which we conduct such studies so as to have meaningful results? The "how" problematizes important methodological issues about the study of such complex phenomena in the diverse settings of individual participating countries or regions: Do we use qualitative or quantitative methods or a combination of both methods? Do we use cross-sectional or longitudinal studies?

Artigue and Winsløw (2010) have argued that:

> Comparative studies aim to identify and explain differences of homologous phenomena in two or more contexts. Comparative studies of mathematics teaching and learning are undertaken with a variety of purposes and methods, and their results and interpretations remain the subject of fierce debates, especially in the case of large-scale quantitative surveys such as PISA. (p. 2).

Comparative education is not really new and has existed for quite some time now (see Noah and Eckstein 1969; Shorrocks-Taylor 2000). Many of the recent ICS have been large-scale studies like the TIMSS and TEDS-M organized by the IEA and PISA organized by the OECD. These studies have generated large mass of data for further analysis. However, the conceptualization and the resources required for running the ICS have strong influences from the more affluent Western countries. For example, regarding TIMSS, Leung (2005) stated that the study design is still very much influenced by North American and Western European countries and that the test inevitably reflects the philosophy of these countries on mathematics education. The ICS have had various aims and have used a "wide diversity of approaches, perspectives and orientation" (Kaiser 1999, p. 9). Regarding the methodology used in these ICS, Eckstein (1988) has claimed that the approach is positivistic and that empirical and statistical methods generally used in the sciences form the basis of the studies. Several methodological issues come to the fore when we start to conceptualize ICS:

- How do we sample the content and the processes to be covered by the ICS survey?
- What kinds of items do we use in the survey?
- How do we construct the selected type of items that will cut across cultural and linguistic boundaries, test what they are supposed to test, and have the same level of difficulty?
- How and when do we administer these items to the selected sample of students? Do all students attempt all of the items?
- How do we sample the students from each country to participate in the study? What constitutes an adequate sample for a given country given the complexity of the student population?
- How do we ascertain that students sitting for the tests in these ICS take the test seriously and put in their best effort?

As international surveys, the ICS generally have six basic stages or dimensions: (1) the conceptual framework and research questions; (2) the design and methodology of the studies; (3) the sampling strategy; (4) the design of the instruments; (5) data collection, processing, and management; and (6) the analysis and reporting of the findings (see Loxley 1992 cited in Shorrocks-Taylor 2000, p. 15). These stages can be identified in the studies cited below.

## Some Examples of ICS

IEA organized the Third International Mathematics and Science Study (TIMSS) in 1995 which involved 45 countries. It was subsequently known as the Trends in International Mathematics and Science Study. TIMSS 1995 followed the earlier studies called First International

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Mathematics Study (FIMS) with 12 participating countries carried out in 1964 and the Second International Mathematics (SIMS) with 20 participating countries or region that was carried out in 1980–1982 (see Robitaille and Taylor 2002). Subsequently, TIMSS has been carried out in 1999 (at grade 8 level only), in 2003 and 2007 with the 2011 study under way at the time of writing. IEA also conducted the Teacher Education and Development Study in Mathematics (TEDS-M) in 2008 (see TEDS-M 2012; http://teds.educ.msu.edu/) which is a comparative study of the teacher preparation of primary and lower secondary teachers of mathematics in 17 countries. OECD has organized the Programme of International Student Achievement (PISA) every 3 years since 1997. The focus of PISA has been the assessment of the extent to which students can apply their knowledge to real-life situations at the end of compulsory education and the extent to which they are equipped for full participation in society (http://www.oecd.org/pisa/aboutpisa/). There have been other less-known ICS such as The Survey of Mathematics and Science Opportunities (Schmidt et al. 1996), The Curriculum Analysis Study (Schmidt et al. 1997), and The Videotape Study (Stigler and Hiebert 1999) and The Learner's Perspective Study (LPS) (Clarke et al. 2006).

## Why Are ICS Important?

There is an interest in finding out how mathematics is taught and learned elsewhere. Comparative studies in education are part of a long tradition, dating back to the ancient Greeks and encompassing many different approaches (Shorrocks-Taylor 2000). Mathematics education is a fairly recent field of study, coming to prominence only in the last 50 years or so. However, mathematics has always been taught, albeit to smaller select groups, in school curricula in many parts of world, in particular, in the Western world. In many newly independent states worldwide although the educational systems and mathematics curricula mimic those of the former colonial power, subtle differences exist in the way mathematics curricula are planned developed and implemented in schools. There are differences in policies surrounding many aspects of the teaching and learning of mathematics, for example, entry to various types of schools, compulsory education, teacher recruitment, teacher preparation, and professional development of teachers. Postlethwaite (1988) put forward four major aims of comparative education: (1) identifying what is happening elsewhere that might help improve our own system of education, (2) describing similarities and differences in educational phenomena between systems of education and interpreting why these exist, (3) estimating the relative effects of variables on outcomes, and identifying general principles concerning educational effects (p. xx).

Other reasons for conducting ICS promulgated by some authors include:

1. Comparative studies aim to identify and explain differences of homologous phenomena in two or more contexts (Artigue and Winsløw 2010).
2. Through comparative studies, we can observe the changes and innovations in each country's educational system, curriculum, contents of textbook, teaching-learning methods, teaching materials, and assessment methods (Shin 1997).
3. Perhaps the most obvious reason to study classrooms across cultures is that the effectiveness of schooling, as measured by academic achievement, differs across cultures (Stigler et al. 2000).
4. If we look for the goals of comparative education, history shows us that comparative education serves a variety of goals. It can deepen our understanding of our own education and society, be of assistance to policymakers and administrators, and be a valuable component of teacher education programs. These contributions can be made through work that is primarily descriptive as well as through work that seeks to be analytic or explanatory, through work that is limited to just one or a few nations, and through work that relies on nonquantitative as well as quantitative data and methods (Kaiser et al. 2002).

One idea that comes forth in these reasons for conducting for conducting ICS is that of distancing oneself as a researcher from one's own local practices and looking at these practices from a different lens to examine the implicit theories about the teaching and learning of mathematics (see Bodin 2005; Leung et al. 2006).

## Some Issues with ICS

Husén (1983) made this infamous comment that in ICS we are comparing the incomparables. To what extent are we comparing the incomparables? Several other questions can be raised about ICS. To what extent is the methodology employed in a particular ICS appropriate? To what extent is it possible to construct internationally equivalent instruments to collect similar data from different sociocultural contexts? To what extent does the study use an idealized curriculum for assessing students' achievement? Critics of ICS abound and some authors such as Holliday and Holliday (2003) have added that: "A much more important hurdle to overcome is the unique set of cultural factors situated in each country, such as differential national languages, social norms, cultural prides, ethical standards, political systems, educational goals, and school curricula" (p. 251). On the other hand, Keitel and Kilpatrick (1999) have asked: Who directs the ICS? Who pays for the ICS? Who controls the dissemination of the results? We may as well add: To what extent do the ICS portray real achievement levels in the participating countries? Are countries with high-performing students the new ideal models for curriculum, pedagogy and practice?

Other issues with ICS include the misuse of the outcome of such research. The media has often used catchy headlines focusing on the ranking of the countries rather than the subtle findings of the ICS such as TIMSS (Leung 2012). Others like Bracey (1997) have highlighted how the aggregate score does not tell the whole story and deplored how scores can be looked from the perspective of different cultural or ethnic groups. It seems unfortunate that the media, institutions, and even countries often times choose to focus on how favorable the results are to their own contexts. In addition, Clarke (2002) has claimed that international comparative research is open to misuse in at least three ways: (i) through the *imposition* on participating countries of a global curriculum against which their performance will be judged; (ii) through the *appropriation* of the research agenda by those countries most responsible for the conduct of the study, the design of the instruments, and the dissemination of the findings; and, (iii) through the *exploitation* of the results of such studies to disenfranchise communities, school systems, or the teaching profession through the implicit denigration of curricula or teaching practices that were never designed to achieve the goals of the global curriculum on which such studies appear predicated.

Another dimension worth mentioning is that of the choice of participating countries. Bishop (2006, p. 582) asked:

> If we seek to develop more cross-cultural research studies then the first major issue concerns which cultures should one choose? He added that equity issues should always be at the forefront and raised these questions: whose voices are heard? Who does the 'talking'? Which countries/cultures are under-represented in any particular study? Which countries/cultures are always being under- or mis-represented? And how can this issue of under-representation be dealt with? (p. 583).

## The Way Forward

The interest in international comparative studies which has existed for a long time is not going to dwindle any soon. In a shrinking global world where boundaries between the local and the international will get blurred, there is a likelihood of these comparative studies occurring more frequently and taking more complex forms. The "why," "what," "when," and "how" of these studies will certainly be revisited to address the criticisms leveled against current practices. As important is also the "who" providing the resources for carrying out these studies. Bishop (2006) has warned that it is rare for the financial supporters not to have an agenda of their own. In an era of globalization that focuses on promoting

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the knowledge-based economy for maintaining a competitive edge, countries rightfully look forward to have the best ideas about mathematics education that they could possibly be offering to their citizens. However, countries should not make direct links between mathematics achievement in schools and economic improvement. From this perspective, international comparisons are much misunderstood and abused (Ernest 1999). Ernest cautioned that:

The assumption that there is a direct link between economic and industrial performance and national teaching styles in mathematics is highly dubious. The further assumption that 'national teaching styles' in mathematics, if such a thing exists, can be transferred from one nation to another, is even more doubtful. Yet such assumptions underpin many of the educational policies of governments in the West. (p. viii).

Improvements need to be made to each of the six stages or dimensions of international surveys put forward by Loxley (1992 cited in Shorrocks-Taylor 2000, p. 15). These questions will need to be addressed again and again by those conducting and thinking of conducting ICS in the future: Are the conceptual framework and the research questions appropriate for the ICS? To what extent are the design and methodology of the ICS appropriate for the ICS? Is the sampling strategy adequately representing the population under investigation? How appropriate are the instruments used for collecting data? How will the data be collected, processed and managed? How will the data be analyzed and how will the findings be reported?

Hence going forward, ICS should be looked at more carefully. In particular, countries should consider questions such as: How important are ICS to their own contexts? What can they learn about their own practices? What kinds of practices from other countries can be adapted and used in their schools? What kinds of practices, if any, from other countries have to be avoided? For ICS to be more useful, a more inclusive kind of international survey has to be conducted that will consider the voices of all whether they are from rich or poor countries. The more affluent countries involved in the ICS should help their less affluent counterparts to make right choices following the publication of the results without any other covert agenda of their own. Accordingly, Clarke (2002) has suggested that international comparative research must be undertaken on a basis of mutual benefit to all participants and that we must guard against the cultural imperialism of an implicit global curriculum.

On the methodological level, there should be ongoing debates about the conceptualization of these studies. Kaiser (1999) has questioned the suitability of the approach of probabilistic test theory and regarding the results of ICS has put forward the idea to the scientific community to control how the results of the studies are used in political debates. These are important ideas to be carefully considered in future studies. On the other hand, Leung (2012) claimed that very rigorous methodologies are adopted in studies such as TIMSS and PISA, and hence within the limits imposed by the nature of these studies, they provide rather reliable results about student achievement in the participating countries. Adding to the discussion, Keitel and Kilpatrick (1999) have queried: "How can there be irrationality, when so many serious educators and scientists have worked so hard to produce orderly, scientific results?" Perhaps this quote from Leung sums it all:

> Results of international studies should serve as mirrors for us to better understand our system. In the process, we must bear in mind that education is a complex endeavor – we cannot expect international studies to produce answers for all our national problems in education. International studies provide rich dataset for individual countries to seek answers to their own issues. In so doing, we need wisdom, and not just data! (Leung 2012).

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## Intuition in Mathematics Education

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### Keywords

Intuition; Mathematics; Education; Concept image; Concept definition; Intuitive rules; Primary intuitions; Secondary intuitions

*"Innovation is often a triumph of intuition over logic"*
*Albert Einstein*

## Definition

Literature addressing a type of mathematical knowledge, characterized by immediacy, self-evidence, and intrinsic certainty.

## Characteristics, Approaches, and Role in Mathematics Education

The term *intuition* comes from the Latin word *intueri*, roughly translated as "to look inside" or "to contemplate." Diverse and controversial meanings and roles have been attributed to intuition in different domains, among them philosophy, psychology, religious studies, ethics, aesthetics, science, mathematics, and education. Intuition has been viewed as the highest form of knowledge, through which the very essence of things is revealed (e.g., Descartes 1967; Spinoza 1967); as a particular means of grasping truth (e.g., Bergson 1954); as the source of genuine, creative innovation (e.g., Hadamard 1945; Poincaré 1958); and as a first and necessary step for further education (e.g., Bruner 1965). Yet it has also been considered a source of misconceptions that should be eliminated (Hahn 1956). In mathematics education, debates on the role of intuition in the learning and teaching of mathematics are often embedded in the perennial discussions of "the appropriate balance between intuition and logic." Throughout history, prominent voices have regarded intuition and rigor as being at odds, while others have argued that these two dimensions play complementary roles in mathematics education (e.g., Klein 1953; Hahn 1956; Begle 1969; Freudenthal 1973; Thom 1973; Wittmann 1981; Howson 1984; Otte 1993; Bass 2005).

Efraim Fischbein has been instrumental in formulating intuition as a research domain in mathematics education. In his 1987 comprehensive book on intuition in science and mathematics, Fischbein offers a theoretical view of intuition, identifies and organizes his own previously published experimental findings on intuitive knowledge as well as other relevant findings, and proposes educational implications for the learning and teaching of mathematics and science. In this book and other publications, he identifies common intuitions in various areas of mathematics, among them combinatory and probabilistic intuitions (Fischbein 1975; Fischbein and Gazit 1984; Fischbein et al. 1991; Fischbein and Schnarch 1997), proof (Fischbein 1982), infinity (Fischbein et al. 1979; 2001), intuitive models of basic operations (Fischbein et al. 1985), geometry (Fischbein and Nachlieli 1998), irrational numbers (Fischbein et al. 1995), and algebraic expressions (Fischbein and Barash 1993).

Fischbein theorized that intuition is a type of cognition characterized by immediacy, self-evidence, intrinsic certainty, perseverance, coerciveness, implicitness, theory status, extrapolativeness, and globality. "Intuitive knowledge [is] a kind of knowledge which is not based on scientific empirical evidence or on rigorous logical arguments and, despite all this, one tends to accept it as certain and evident (1987, p. 26)."

Fischbein described various classifications of intuitions, including a distinction between *primary intuitions* and *secondary intuitions*. He claimed that primary intuitions arise spontaneously and their origins are rooted in our personal experience or prior knowledge. He further emphasized that knowledge that is acquired first shapes our primary intuitions. Due to the primacy effect (what we learn first is hardly forgotten and over-implemented), primary intuitions are usually very resistant. These intuitions frequently coexist with formal knowledge acquired through instruction. Fischbein provided numerous examples of mathematical intuitive reasoning from his own research, from other studies, and from the history of mathematics. One such example addresses the issue of comparing the number of elements in two infinite sets. A common intuitive response is that the number of elements in an infinite set is greater than the number of elements in each of its infinite proper subsets (a response based on our experience with finite sets). He outlined how this intuitive tendency has been described by mathematicians throughout the history of mathematics. For example, Hahn (1956, p. 1604) stated that "if we look for examples of enumerable infinite sets we arrive immediately at highly surprising results. The set of all positive even numbers is an enumerable infinite set and has the same cardinal number as the set of all the natural numbers, though we would be inclined to think that there are fewer even numbers than natural numbers."

Fischbein provided a comprehensive framework for analyzing learners' *mathematical performance* by addressing two additional components of mathematical knowledge: formal knowledge and algorithmic knowledge (e.g., Fischbein 1993). *Formal knowledge* is based on propositional thinking and refers to rigor and consistency in deductive construction. This type of knowledge is free of the constraints imposed by concrete or practical characteristics. *Algorithmic knowledge* is the ability to use theoretically justified procedures. Fischbein emphasized that each of the three components of mathematical knowledge (formal, algorithmic, intuitive) and their interrelations play a vital role in students' mathematics performance and that "the intuitive background manipulates and hinders the formal interpretation or the use of algorithmic procedures" (1993, p. 14). When referring to intuitive–algorithmic mixtures, Fischbein offered the notion of *algorithmic models*, pointing, for instance, to methods of reduction in processes of simplifying algebraic or trigonometric expressions. For example, the tendency of students to treat $(a + b)^5$ as $a^5 + b^5$ or log (x + t) as log x + log t was interpreted as an intuitive application of the distributive law (Fischbein 1993).

Three additional theoretical frameworks address learners' mathematical intuitions: System 1–System 2 (e.g., Kahneman 2002, 2011), concept image–concept definition (e.g., Tall and Vinner 1981), and conceptual change theory (e.g., Vosniadou and Verschaffel 2004). Intuitions are a pivotal motif in Kahneman's studies. He described two systems of the mind, System 1 (intuition) and System 2 (reasoning), proposing that System 1 thought processes operate automatically and quickly and are heavily influenced by context, biology, and past experience. This system assists in mapping and assimilating newly acquired stimuli into knowledge structures that are self-evidently accepted as valid. In contrast, System 2 thought processes are intentionally controlled, calling for justification via logic and analytical thinking. Several researchers in mathematics education have incorporated this framework as a means of interpreting their research findings on mathematical reasoning (e.g., Leron and Hazzan 2006).

Tall and Vinner (1981) coined the terms *concept image* and *concept definition*. Concept image comprises all the mental pictures and properties a person associates with a concept (i.e., intuitive and formal ideas), while concept definition addresses the concept's formal mathematical definition. For instance, the concept of tangent is usually introduced with reference to circles, implicitly insinuating that a tangent can meet a curve only at one point and should not cross the curve. This often becomes part of students' tangent image (or primary intuitions about a tangent) and hinders acquisition of related notions, such as inflection points (Vinner 1990). Tall and Vinner analyzed students' concept images of various advanced mathematical concepts, among them limits, continuity, and tangent (Tall 1980, 1992, 2001; Tall and Vinner 1981; Vinner 1990, 1991).

Mathematical intuitions are also addressed by the conceptual change approach, originally developed to explain students' difficulties in learning science. The term *conceptual change* characterizes the learning of new information that is in conflict with learners' *presuppositions*, i.e., prior intuitive knowledge. In such situations, a major reorganization of prior knowledge is required. In the last decade, several researchers have applied the notion of conceptual change in a series of studies in mathematics education (e.g., fractions, Stafylidou and Vosniadou 2004; rational numbers – Vamvakoussi and Vosniadou 2004; real numbers, Merenluoto and Lehtinen 2004; and algebra, Christou and Vosniadou 2012). Vamvakoussi and Vosniadou (2004) claimed that in the process of studying mathematics, students form *synthetic models* of mathematical notions. These synthetics models comprise a mix of primitive–intuitive and formal ideas regarding the notion, which are not necessarily compatible.

The terms *primary intuitions*, *System 1*, *concept image*, and *presuppositions* address preliminary, intuitive, and early mathematical ideas based upon daily experience. These ideas are often incompatible with the formal definitions of concepts. Because such ideas are resistant to traditional instruction, they are imposed on newly acquired mathematical notions. Fischbein, Tall and Vinner, and Vosniadou argued that blends of intuitions and formal knowledge are inevitable, and Tall and Vinner suggested that compartmentalization may be one reason for such intuitive–formal mixes that coexist in the learner's mind without sounding any alarms. Fischbein described the most favorable situation, namely, when formal knowledge turns into secondary intuitions. Nevertheless, the conceptual change framework emphasized the gradual and time-consuming nature of such changes and analyzed the *synthetic-model* stages, in which presuppositions and scientific knowledge coexist.

Fischbein, Kahneman, Tall and Vinner, and Vosniadou offer a content-oriented perspective of intuition, mainly addressing the impact of learners' prior knowledge on their mathematical performances. Another approach is suggested by the intuitive rules theory. This theory takes a task-oriented standpoint, addressing the impact of specific task characteristics on learners' responses to scientific and mathematical tasks (Stavy and Tirosh 1996, 2000; Tirosh and Stavy 1996; Tirosh et al. 2001; Tsamir 2007; Stavy et al. 2006). The main claim of this theory is that students tend to provide similar intuitive responses to various scientific, mathematical, and daily tasks that share some external features but are otherwise unrelated. The intuitive rules theory offers three major intuitive rules. Two of these rules (*more A–more B* and *same A–same B*) are identified in students' reactions to comparison tasks, and one (*everything can be divided*) is manifested in students' responses to processes of successive division. Here we refer briefly to the two comparison rules, whose impact can be seen in students' responses to a wide variety of situations.

The intuitive rule *more A–more B* was identified in students' reactions to comparison tasks in which two entities differ with respect to a certain salient quantity A ($A_1 > A_2$). In the task, students are asked to compare these entities with respect to another quantity, B, where $B_1$ is not necessarily greater than $B_2$. A common incorrect response to such tasks is as follows: "$B_1 > B_2$ because $A_1 > A_2$, or *more A–more B*." *More A–more B* responses have been observed in many tasks in science and mathematics, including classic Piagetian conservation tasks and tasks related to intensive quantities, number theory, algebra, geometry, infinity, and free fall. This tendency is evident in a wide range of ages. Klartag and Tsamir (2000), for instance, found that high school students tended to claim that for any function f (x), if $f(x_1) > f(x_2)$, then $f'(x_1) > f'(x_2)$, or *more A* (value of f(x))–*more B* (value of f ' (x)).

The intuitive rule *same A–same B* has also been identified in students' reactions to comparison situations in which two entities are equal for a noticeable quantity A ($A_1 = A_2$) but differ for another quantity B ($B_1 \neq B_2$). When asked to compare $B_1$ and $B_2$, students often respond "$B_1 = B_2$ because $A_1 = A_2$, or *same A–same B*." *Same A–same B* responses have been identified in various domains, including geometry, percentages, ratio, and proportion. Tsamir (2007), for instance, reported that university students tended to claim that hexagons with equal sides have equal angles, that is, *same A* (sides)–*same B* (angles).

In conclusion, a major objective of mathematics education should be to encourage students to use critical thinking (e.g., NCTM 1989). Yet, encouraging students to critically examine their own processes should be done carefully and cautiously so as not to discourage basic and intuitive mechanisms of thought. Various instructional methods have been suggested for handling this delicate situation, among them teaching by analogy, conflict teaching, calling attention to relevant variables, raising students' awareness of the role of intuition in their thinking processes, developing metacognitive abilities, experiencing practical activities, and introducing the formal meaning and formal content of concepts as early as possible. Nevertheless, the feasibility and impact of these methods in specific situations still need to be explored.

## Cross-References

- ▶ Epistemological Obstacles in Mathematics Education
- ▶ Misconceptions and Alternative Conceptions in Mathematics Education

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## Language Background in Mathematics Education

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### Keywords

Language background; Bilingual learners; Second language learners; Bilingual mathematics classrooms; Multilingual mathematics classrooms

### Definition

Language is ubiquitous – it is everywhere a part of human life. Language is also highly diverse and constantly changing. This diversity includes the languages spoken by different peoples around the world. This diversity also includes social variation within languages, for example, in relation to social class, gender, or race. Language also varies in relation to different activities: the languages of mathematics, of science, of sport, of religious practice, and so on. Finally, language varies in relation to different modes of communication, such as in speech, symbols, written prose, or texting. The term *language background* can be used to refer to the particular set of national, social, and professional language varieties in which any individual or group of people has experience and expertise. In this entry, language background refers more specifically to the different languages that learners and teachers use.

What does language background have to do with mathematics education? The teaching and learning of mathematics depend fundamentally on language. Mathematics classrooms, for example, may feature discussion among students, lectures by the teacher, printed curriculum materials or textbooks, and writing on a blackboard or on a screen. If mathematics education is reliant on language, however, a question arises: Do students' or teachers' language backgrounds have any impact on their learning or teaching or understanding of mathematics?

### The Issue of Language Background in Mathematics Education

As a focus for research, the above question did not receive much attention until the 1970s. In 1974, however, a regional symposium addressed the topic of linguistics and mathematics education. The final report (UNESCO 1974) highlights a number of issues that have formed the basis of much subsequent research. These issues include:

- The challenges of learning mathematics in a second language or in bilingual or multilingual settings
- The influence of the structure of different languages on mathematical thinking

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- The challenges of developing mathematical registers in languages in which mathematics was not previously taught as a formal subject
- The impact of language policy on mathematics education
- The effects of differences between formal mathematical language and everyday language

Happily, the 1980s saw the beginning of more sustained attention to language issues in mathematics education, including attention to language background. This entry summarizes the main trends and findings in this research.

## Theoretical Perspectives on Language Background in Mathematics Education

The broad trajectory of research on language background in mathematics education falls into perhaps three main phases. In the first phase, most research focused on students' attainment in mathematics, as measured by national examinations or standardized tests. The general goal of this research was to establish whether students' language backgrounds had any impact on their attainment in mathematics. The second phase was characterized by an interest in classroom processes, looking at, for example, how students participated in mathematics lessons and how teachers adjusted their teaching of mathematics in response to students' language backgrounds. The third and most recent phase has highlighted the political role of language and has sought to examine the connections between language status and students' participation and attainment in mathematics education. The later phases have not replaced the earlier ones, although they have recast them in some respects. For example, there is now greater awareness that students' attainment (phase 1) is influenced by classroom processes (phase 2) and the political nature of language (phase 3).

The three phases of research have been accompanied by shifts in theoretical orientations. In the first phase, most research was from a broadly cognitivist perspective, in which language background was suspected to be an influence on how students thought about mathematics. This thinking was assumed to be measurable, often through standardized tests. Researchers' perspectives on language were largely influenced by psycholinguistics (see Moschkovich 2007); language was assumed to be separate from thought and thinking was treated as an entirely internal, mental process.

In the second phase, researchers began to adopt discursive perspectives, which looked at mathematical learning and thinking as social processes mediated by interaction with others (e.g., teachers or other students). From this perspective, learning mathematics was a process of enculturation into mathematical practices, including discursive practices (e.g., ways of explaining, proving, or defining mathematical concepts). Thinking was examined by analysis of students' or teachers' talk, particularly in classroom settings, rather than by scores on a test or a clinical interview. Researchers' perspectives on language in this phase were drawn more from sociolinguistics (see Moschkovich 2007), a branch of linguistics that examines use and variation in language in relation to the context and the speakers. This kind of perspective includes a critique of a narrowly individualist view of language, often associated with deficit models of bilingualism or multilingualism. Bilingual language use has, in the past, been seen as degenerate and as a barrier to learning. Research drawing on sociolinguistics shows that this is not the case.

The third phase is still evolving but features an attempt to relate individual outcomes to broader political dimensions of language. This work challenges a narrow view of mathematics as a western, largely male, white, middle-class domain. Such work often draws on sociological theories of language to explain how mathematics education stratifies students according to their language backgrounds. From this perspective, language is as much a social force as a tool for thinking. Another strand in this phase has emerged within ethnomathematics, which looks at language structure to develop mathematics curricula that challenge the western, postcolonial bias in many curricula (see Barton 2008).

## Language Background and Attainment in Mathematics

The question that is perhaps of most concern to teachers, parents, and policymakers is the question of whether language background affects students' attainment in mathematics. Research shows fairly clearly that there *can* be a correlation between students' language background and their attainment, depending on a number of factors. Indeed, there has long been evidence that students learning through a second language underperform in mathematics (e.g., in the USA, by Secada 1992). This kind of work is based on fairly crude research designs, however, that crucially do not always take sufficient account of students' language proficiency in *all* the languages they use and which generally fail to assess students' mathematical thinking in their first language or a mixture of their two languages. It is also difficult to untangle effects on learning that are due to language background from effects that are due to socioeconomic circumstances, racial discrimination, and other factors (if, indeed, such factors can ever be untangled).

More carefully designed research has, however, demonstrated a subtle relationship between language background and mathematical attainment. This work is based on a specific theory called the Threshold Hypothesis, developed by Cummins (e.g., 2000), an expert in bilingual education. This hypothesis suggests that students' academic attainment is related to the languages that they speak in the following way:

- Bilingual students who reach a high level of academic language use in at least two languages outperform monolingual students.
- Bilingual students who reach a high level of academic language use in one language have comparable levels of attainment to monolingual students.
- Bilingual students who do not reach a high level of academic language use in any language underperform relative to monolingual students (see Cummins 2000).

This hypothesis has been used as a basis for research specifically focused on mathematics attainment. Results have fairly consistently shown that students with low levels of academic language proficiency underperform in mathematics, compared with students who reach a high level of academic language use in at least one language. There is also reasonable evidence that students who develop high levels of academic language use in two languages do, on average, outperform monolingual students in mathematics. This work has been conducted with students of immigrant backgrounds in the UK, in Australia, and in multilingual Papua New Guinea (these studies are reviewed in Barwell 2009). These findings are indirectly supported by separate studies based on data from international comparisons of mathematics attainment, such as the TIMMS and PISA studies (e.g., Howie 2003), and by findings from immersion education programs that show enhanced mathematics performance (e.g., Bournot-Trites and Reeder 2001). Hence this general relationship seems to hold across a variety of different settings.

These findings suggest that in many situations, as much attention needs to be paid to students' language development as to their mathematical learning. For example, in some circumstances, there may be cognitive advantages for students who are learning in a second language to continue to develop their home language to a high level.



## Language Background and Learning and Teaching Mathematics

The majority of research relating to language background in mathematics education has probably been devoted to examining mathematics classroom processes in a wide variety of settings. These processes include students' interaction with each other and with their teacher, students' interpretation of various mathematical tasks, and teachers' strategies in relation to students' language backgrounds. This section is designed to give a general overview of this work as well as illustrate the range of language backgrounds that have been examined.

### Second Language Education Settings

Second language education refers to education in the language of wider society, where some students are learning the classroom language as second or additional language. Second language contexts often include students from migrant backgrounds or from aboriginal backgrounds in mainstream education systems. Terms used in English include English as a second language (ESL), English as an additional language (EAL) or English language learner (ELL). Mathematics education research in second language contexts has been conducted in the UK, the USA, Australia, Canada, Spain, the Czech Republic, Germany, Italy, and Sweden (for some examples, see Cocking and Mestre 2008; Barwell 2009).

This work has shown how students often make use of their home language or L1 to interpret problems and for some mathematical thinking. This use of L1 may occur even where their home languages are not supported or encouraged in the classroom; students simply use their home language privately. Students often are particularly challenged by text-rich problems, such as word problems. The language of word problems is complex and quite specific to mathematics education. Students may struggle to make sense of the context of the problem, as well as the unusual grammar and syntax (something that monolingual students also frequently experience). Nevertheless, research suggests that where students start from meaningful situations, they are able to interpret word problems successfully.

## Bilingual Education Settings

Bilingual education refers to programs in which two languages are used in the teaching and learning of mathematics. Students may have a bilingual mathematics teacher or two different teachers, either together or separately who speak different languages. The aim of many such programs is to “transition” from proficiency in one language to proficiency in the other, while maintaining work in curriculum subjects like mathematics. Bilingual education settings for mathematics have particularly been researched in the USA, where Spanish-English programs are quite common. Some research has also been conducted in Wales (Welsh-English) and New Zealand (Maori-English) (for examples, see Barwell 2009; Téllez et al. 2011).

Research in bilingual education settings for mathematics education has examined how students draw on multiple language resources to make sense of mathematics. These resources include aspects of both languages. This kind of work demonstrates how bilingualism does not have to be a problem or barrier to learning mathematics; quite the opposite, bilingualism gives students a wide repertoire of different meanings and ideas to draw on as they learn mathematics.

Research has also identified productive teaching strategies for bilingual mathematics classrooms. These strategies include the maintenance of a focus on mathematical meaning rather than students’ particular use of language. That is, successful teachers seem to pay careful attention to students’ mathematical ideas and to work with them to ensure they clearly understand them.

## Plurilingual Societies

Plurilingual societies refers to societies in which many languages are recognized and used, such as countries in South and Southeast Asia and much of Africa. (Arguably all societies are multilingual, but many do not recognize the fact.) In such societies, a small subset of languages is used for schooling; these languages may include local languages, regional languages, or former colonial languages. In South Africa, for example, English is the most widely used language of schooling. Research on mathematics education in plurilingual societies has largely been conducted in South and southern Africa (see Adler 2001; Setati 2005; Setati and Barwell 2008). Some work is also beginning to emerge from India and Pakistan.

Research in plurilingual settings for mathematics education has highlighted the complex set of challenges that face teachers, learners, and parents alike. In plurilingual societies, the use of multiple languages is widespread and is likely to occur in mathematics classrooms. Such practices are often frowned on and teachers may struggle with various dilemmas that arise. For example, is it better to allow students to use their home language in order to express their mathematical thinking fluently, or to encourage them to use the language of schooling, which may inhibit their mathematical thinking (Adler 2001)?

As in bilingual education settings, research has shown how learners and teachers can draw on their multiple languages to learn mathematics. In many respects "switching" between languages can be used productively; indeed teachers can encourage deliberate use of such switching to enhance their students' learning of mathematics.

Research in plurilingual settings has also highlighted the effects of language politics on mathematics education. In South Africa, for example, research has shown how many students and teachers accept that learning and teaching mathematics in English is more difficult than when using their home languages. But they still prefer to use English because it is seen as a more valuable language in terms of the access to jobs and higher education it is perceived to provide (Setati 2008). Similar trends are also apparent in much of Asia.

### Immersion Education Settings

Immersion education refers to the use of a target language to teach across the curriculum in order that students become proficient in that language. Immersion education is common in Canada, in Switzerland, and in many parts of the world where it is used to teach students a prestigious "foreign" language, such as English or Chinese.

There has not been much research into mathematics learning and teaching in immersion settings. Some studies have demonstrated the efficacy of immersion education for teaching mathematics (e.g., Bournot-Trites and Reeder 2001) in terms of students' mathematical attainment, but there has been little research on classroom processes.

## Language Background in Mathematics Teacher Education

There has been little research on language background in mathematics teacher education. Issues under this heading include the preparation of mathematics teachers to respond to students' language backgrounds, such as the specific strategies that might be needed to teach mathematics in the different settings discussed in the preceding section. Mathematics teacher education also, of course, takes place in these different settings.

One study in plurilingual Malawi by Chitera (2009) showed how mathematics teacher education tended to reinforce certain assumptions about language and mathematics, often contrary to the national language policy for education. Mathematics teacher educators tended to view multilingualism as a problem and did not seem to have adequate preparation in implementing a policy which promoted the use of students' home language in teaching mathematics.

## Language Background and Researchers in Mathematics Education

Mathematics education as a research domain also makes use of language and researchers come from a wide range of language backgrounds. The politics of language are to some extent self-evident in the structure of the research community. In particular, English is the predominant language of this community; the leading international journals and conferences all prefer English (as does this encyclopedia), with Spanish, French, and Portuguese as distant acceptable secondary languages. There are thousands more languages in the world that are entirely absent from mathematics education research discourse. The preference for English makes things easier for English-speaking researchers (predominantly from, or working in, the UK, USA, Canada, Australia, and New Zealand) and more challenging for everyone else. It also, however, privileges certain ways of thinking about mathematics, teaching and learning, while rendering invisible other alternatives (Barwell 2003; Barton 2008).

## Future Directions

This area of research continues to develop. There is a need for a stronger theorization of the interaction between language background and mathematics learning and teaching. The critical perspectives emerging in the third phase of research described above are likely to be an important source of such a theorization.

By its nature, the issue of language background in mathematics education is of interest around the world and has, indeed, been researched around the world. Nevertheless, it would be valuable to see research in a wider range of geographical settings as well as in a wider range of linguistically distinct settings.

Finally, there is a continuing need to find ways to support mathematics teachers as they are increasingly faced with language diversity in their classrooms. Such diversity can be a great opportunity for teachers and learners of mathematics, but ways of harnessing this potential are not simple.

## Cross-References


▶ Bilingual/Multilingual Issues in Learning Mathematics
▶ Cultural Diversity in Mathematics Education
▶ Cultural Influences in Mathematics Education
▶ Equity and Access in Mathematics Education
▶ Immigrant Students in Mathematics Education
▶ Inclusive Mathematics Classrooms
▶ Indigenous Students in Mathematics Education
▶ Urban Mathematics Education


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# Language Disorders, Special Needs and Mathematics Learning


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## Keywords

Developmental aphasia; Developmental language disorders; Inclusion; Individual differences; Specific language impairment; Special needs


## Definition

Language disorders are shown by children whose oral language skills, such as producing speech and understanding what others say, are significantly impaired relative to their peers. They are at risk for poorer educational achievement in mathematics just as in other curriculum subjects. This is not surprising when one considers the general importance of communication in schooling, the role of oral language in classroom mathematical investigations, and more specific connections such as the fundamental contribution made by knowledge of the number-word sequence to developing understanding of symbolic notation.

## Characteristics

Language disorders can be the consequence of physical problems (such as hearing loss, visual impairment, or accidental injury), impoverished experience, or general learning disabilities such as are common in children with conditions such as autism, Down syndrome, fragile X, Williams syndrome, Apert syndrome, and cerebral palsy (Bishop 1997; Dockrell and Messer 1999). Nevertheless, some children show language disorders when there is no reason to suppose their difficulties result from these above-mentioned causes. These children have been described as having *developmental aphasia*, *developmental language disorders*, or *specific language impairment*. Their conditions have been recognized in conventional classification schemes used by doctors and psychiatrists, such as the World Health Organization's *International Classification of Diseases and Disorders* and the American Psychiatric Association's *Diagnostic and Statistical Manual*.

Behavioral genetics studies have shown that identical twins are more alike than nonidentical twins and that there is substantial overlap between the genetic variance underlying language impairment and that underlying reading and arithmetic difficulties. The evidence of genetic influences does not imply that the environment is irrelevant: such studies do not support a strong genetic determinism (Plomin and Dale 2000; Plomin and Kovas 2005; Resnik and Vorhaus 2006).

Children enter the world of number through learning to count and mastering the number-word sequence of their language. Counting provides the basis for computation, and a grasp of spoken number is presumed for developing understanding of the Hindu-Arabic notation for representing number. Behavioral studies find that children with language disorders are more likely to show delays in mastering the number-word sequence and the natural number system (Donlan 2007).

As much of elementary mathematics depends on competence with the natural number system, these delays have substantial consequences. Proficiency in both mental and written computation, even with single-digit numbers, is compromised in children with *specific language impairment* (Donlan 2007). Nevertheless, there are considerable individual differences in these children: some progress comparably to their typically developing peers, while others show attainment in line with their linguistic development which is several years below their chronological age. The reasons for this variation are not understood: for example, it may reflect variation in the effectiveness of support they receive at home and at school or variation in other individual characteristics such as motivation, memory functioning, and visuospatial abilities.

There is still much to be learnt about children with language disorders: studies of the mathematical progress of adolescents with language disorders are very rare. Advice for the teaching of children with language disorders is available in book form (e.g., Hutt 1986) and from several organizations with online presence, such as Afasic (http://www.afasicengland.org.uk/), I CAN (http://www.ican.org.uk/en.aspx), and The Communication Trust (http://www.thecommunicationtrust.org.uk/schools.aspx).

## Cross-References


- ▶ Deaf Children, Special Needs, and Mathematics Learning
- ▶ Down Syndrome, Special Needs, and Mathematics Learning
- ▶ Inclusive Mathematics Classrooms
- ▶ Language Disorders, Special Needs and Mathematics Learning

► Learner-Centered Teaching in Mathematics Education
► Learning Difficulties, Special Needs, and Mathematics Learning
► Mathematical Ability
► Word Problems in Mathematics Education

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# Learner-Centered Teaching in Mathematics Education

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## Keywords

Inquiry mathematics; Constructivist; Discovery; Problem solving; Standards-based instruction; NCTM

## Definition

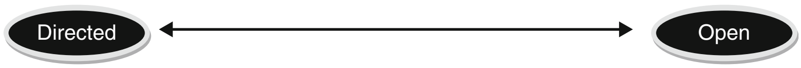
Learner-centered teaching is an approach to mathematics instruction that places heavy emphasis on the students taking responsibility for problem solving and inquiry. The teacher is viewed as a facilitator by posing problems and guiding students as they work with partners toward creating a solution.

## Characteristics

### Intellectual Autonomy

Many researchers have contended that one of the most important contributions that education can make in individuals' lives is to their development of autonomy (Piaget 1948/1973). Autonomy is defined as the determination to be self-governing, to make rules for oneself rather than rely on the rules of others to make one's decisions (heteronomy). Kamii (1982) suggests that autonomy is the ability to think for oneself and make decisions independently of the promise of rewards or punishments. In relation to education, Richards (1991) distinguishes between two types of traditions in the mathematics education of children, what he terms *school mathematics* and *inquiry mathematics*. School mathematics is what is typically thought of as a teacher-directed environment in which learning mathematics is a process of memorizing rules and procedures that are modeled by a teacher and solving routine problems that often have little significance to the real world until mastery of the teacher's solution methods is attained. Heteronomy is fostered here as students learn to replicate what the teacher has shown them, often with little connection to how they make sense of the world. Mathematics is seen as transmitted from the teacher to passive students with little opportunity to negotiate the meaning of their actions.

An inquiry or student-centered tradition, on the other hand, is one in which students are actively engaged in genuine problem-solving activities. Students are given open-ended problem situations and work with each other to create multiple, meaningful solutions that are elaborated, debated, and




**Learner-Centered Teaching in Mathematics Education, Fig. 1**

validated in the public discourse created by the students and their teacher. Together, the participants create a community of learners that engage in practices similar to actual mathematicians. Rather than a transmission of skills from one person to another, the metaphor here is one of negotiating meaning among participants under the guidance of an instructor. In the inquiry tradition, students are heavily encouraged to develop autonomy as they try to invent meaningful strategies for the problems they are solving.

The term *student-centered teaching* has been most notably associated with John Dewey's work (Dewey 1938) and is known today by many names: discovery (Anthony 1973), problem-based (Barrows and Tamblyn 1980), student-centered (Chung and Walsh 2000), constructivist (Jonassen 1991), teaching for understanding (Hiebert et al. 1997), standards-based instruction (Tarr et al. 2008), and experiential (Kolb and Fry 1975) to name a few. While there is no universal definition of *student-centered teaching*, in general, these traditions argue for placing students at the center of problem solving in some capacity, with teachers taking a less dominant role.

At the heart of these approaches is the idea that students should learn to reason critically about mathematics in more than just a skill-based manner. S*tudent-centered teaching*, however, has grown so prominent in both research and teaching venues over the decades that many differences have emerged, rendering one, unified approach difficult to describe. The differences lie mainly on how directed the inquiry investigation is, who motivates the inquiry, and what can be thought of on a continuum from directed to open inquiry (Fig. 1).

In more directed approaches, the teacher poses a situation for inquiry, guides students' investigations, and directs students' learning and summarizing. At the other end of the continuum, the inquiries are completely student-initiated and the teachers' lessons are designed around what the students wish to explore. Kirschner et al. (2006) argue that minimal guidance during instruction does not work and unfortunately seem to lump most student-centered traditions into this "minimally guided" category. However, many student-centered approaches incorporate some forms of guidance into their program, and the results have shown that this approach can produce higher gains in achievement than the more teacher-centered tradition (Tarr et al. 2008).

## History

Since the publication of the National Council of Teachers of Mathematics *Curriculum and Evaluation Standards for School Mathematics* in 1989, there has been a significant push toward student-centered teaching in mathematics. The 1989 NCTM Standards argued for a radical reconstruction of classroom mathematics with more emphasis placed on students' representation, communication, and mathematical processes. Since then, the NCTM has revised its recommendations in the *Principles and Standards* (2000) as well as other key documents promoting student-centered mathematics instruction (*Curriculum Focal Points*; see www.nctm.org). While these recommendations have been made by a prominent national organization comprised of mathematics educators and researchers, the student-centered approach has garnered even more attention and traction with the adoption of the Common Core State Standards (CCSS 2011) by a majority of the United States. Not only does the Common Core set out the mathematical content to be taught, but more importantly, it outlines eight Mathematical Practices that are consistent with student-centered teaching and are to be engendered in all students, including communicating viable arguments, critiquing the reasoning of others, and problem solving. The publication of these important documents ensures that student-centered teaching is not going away any time soon.

## Characteristics of Student-Centered Classrooms

There are some basic characteristics of student-centered classrooms that transcend the open/direct dichotomy now plaguing various implementations of this approach. If a principal were to enter a student-centered classroom, she might expect to see certain characteristics that focus on problem solving, classroom environment, collaboration, mathematical discourse, and tools/manipulatives.

### Problem Solving

A student-centered classroom can be distinguished from a teacher-centered one in that the students are doing the problem solving rather than the teacher. In a more directed approach, the teacher has modeled how to solve and make sense out of a problem situation, usually with a manipulative, and the students are working together or independently to create their own solutions. In less directed classrooms, the children are posed problems without being guided by the teacher, and asked to create their own, personally meaningful solutions. In either case, it is the students who are solving problems, using critical thinking skills and reasoning to develop their solutions. Creating genuine problem-solving environments begins first with worthwhile, open-ended mathematical tasks. Rich tasks that elicit more than one way to solve a problem and/or more than one correct answer have potential to support students in their problem-solving endeavors.

### Classroom Environment

Student-centered teaching is most often associated with a certain set of social norms for creating a safe, engaging classroom environment. Social norms refer to the expectations that the teacher and students have for one another during mathematical discussions. Yackel and Cobb (1994) have documented at least four social norms that support student-centered instruction: Students are expected to (1) explain and justify their solutions and methods, (2) attempt to make sense of others' explanations, (3) indicate agreement or disagreement, and (4) ask clarifying questions when the need arises. The teacher's role is to help set these expectations and to maintain them once they have become established in the classroom. An example of some dialogue that might take place in a student-centered environment can be seen in an excerpt from a middle-school classroom (12–14-year-olds who are studying integer operations). The task is to fill in the blank with a meaningful operation in the problem 10,000 _________ = 12,000.

T: What is the other easy one?

Dusty: Minusing debt of 2000 [T writes – (−2000)]

T: Anybody else got that one on their paper? Do you agree with this one Brad or did you just put it because Dusty said?

Brad: I agree.

Charlie: I do not agree.

T: You do not agree? Okay, talk about it Charlie.

Charlie: Because you are minusing. . .never mind I agree.

T: You do. You just changed your mind. Why do you agree now?

Charlie: Minusing debt is like she owed $2000 and then she did not have to pay it so she went up (excerpt from Akyuz 2010).

In this example, the teacher presses students to indicate whether they agree or disagree with Dusty's solution. The teacher must create a safe environment that allows students to indicate their disagreement without fear and must feel comfortable expressing when they are wrong, like Charlie.

### Collaboration

Another hallmark of a student-centered classroom is that a large portion of the problem solving is done in collaboration with peers. When given a problem to solve, students are often directed to pair with a partner or work with others in prespecified teams that range from two to six students. Collaboration is paramount to supporting students' learning because research shows that people learn mathematics deeper as they explain it to their peers, and students who are having difficulty with a math concept can draw on their peers' explanations for support. Additionally, teams of students often invent much more sophisticated strategies than they might have alone. Collaboration, however, is not without its controversy, especially when it comes to assessment of individuals' learning, so the teacher must plan collaboration strategically in her classroom.

Mathematical Discourse

One of the crucial aspects of student-centered instruction involves using student discourse in whole class discussion to bring out important mathematical ideas. In a traditional setting, the teacher controls what is being said and can ensure that the lecture includes the intended mathematics. However, with student-led discussion, the teacher has to carefully guide students toward discussing the mathematics that is intended. For example, one of the goals of a seventh-grade teacher was to engage students in a discussion in which ordering integers correctly on a vertical number line was the main topic. As a first step, the teacher chose a problem that all of her students could work in some meaningful way: Paris' net worth is −\$20,000, and Nicole's net worth is −\$22,000. Who is worth more and by how much? Students had about 5 min to work this and another similar problem and the teacher called on Nathan to show his reasoning to the class.

Nathan created a vertical black and red number line that had been introduced in a previous class period (Fig. 2).

T: Tell us why you did put Nicole there and why you put Paris there. Is that a logical question to ask?

Nathan: I do not know why.

T: He says that he does not know. Okay Flora, say it a little louder.

Flora: He should put Nicole into red.

T: Do you know why she says that? Say it again Flora.

Flora: Nicole should be in the red.

T: Can you change it, Nathan? Go ahead. That is helpful Flora (Fig. 3).

Gabe: Because positive numbers are in the black and the red is negative.


Nathan's first attempt.

**Learner-Centered Teaching in Mathematics Education, Fig. 2**


Nathan's second attempt.

**Learner-Centered Teaching in Mathematics Education, Fig. 3**

T: Gabe says in the black is positive in the red is negative. That might be helpful keep in mind. What do you want to say Adam?

Adam: He has to switch 20,000 and 22,000. 20,000 is supposed to be before 22,000.

T: Did you hear that Charlie? Do you agree with that? Does it matter guys? Why?

Adam: Paris is closer to zero.

Charlie: Because Nicole owes more so she has to be in the red more.

T: She has to be more in the red. Nathan? He says because Nicole owes more. How do you know she owes more?

Charlie: It has already said it. Negative 22,000 and negative 20,000.

T: Dusty, what do you want to say?

Dusty: It should be opposite of going up to zero.

Mark: I think we should put the less number in front of the higher number.

T: In front of it, like this [puts 20,000 above 22,000 on the vertical number line]. Why?

Mark: Because −20,000 is closer to 0.

T: You guys keep saying that, what do you mean? Marsha?

Marsha: Yes, there is a reflection, if you like flip it [the top half of the number line] upside down.

Charlie: Because −20,000 is being closer to out of debt than −22,000.

Brad: The reason 22,000 should be farther down is because it is further down in the hole. Like you owe more than the other person (excerpt from Stephan and Akyuz 2012).

In this example, the teacher uses the contributions, both correct and incorrect ones, to guide the discussion in which ordering of integers on a number line is the topic of conversation. The teacher purposely chose Nathan to begin the discussion as she was aware from her observations of Nathan during small-group time that he was confused about the order. She knew that his solution would create debate in class and cause several students to offer counter solutions. It is important to note that the teacher did not just accept students' "correct" ordering and move on, hoping Nathan would change his mind. In order to give Nathan good reason for changing his opinion, the teacher pushed students to give justifications for their ordering. Strong mathematical reasoning came to the forefront and, as a consequence, several images emerged (e.g., "in the hole," reflection lines, closer to out of debt). As a result of this high-level, engaging discourse, the intended mathematical ideas came from the students.

### Tools/Manipulatives

Student-centered approaches utilize tools, including manipulatives, notations, and symbols, as an integral part of teaching. Researchers in education have shown that tools can be powerful instruments for supporting students' mathematical development (Bowers et al. 1999; Stephan et al. 2001). Thompson and Lambdin (1994), however, caution that simply using manipulatives in a classroom does not necessarily improve student learning. Teachers must be very thoughtful about which manipulative best supports the concept that is to be developed. Thompson and Lambdin also argue that not only is the appropriate tool necessary but also that the teacher's instruction with the tool is equally important. Depending on how guided the inquiry is, tools and notations can be introduced at the onset of instruction (heavily guided inquiry) or *after/alongside* students' problem solving (more open inquiry) as a means of helping students better organize and structure their thinking. In guided student-centered methods, tools are introduced at the *beginning* of a concept and students' are guided to decode their meaning in order to act meaningfully with it. More guided student-centered teachers teach students the steps for using the tool and ask questions to help students interpret their actions with the tool meaningfully, i.e., directly instruct how to use the tool. In contrast, the tools from a less directed, student-centered approach are introduced to students in a planned, bottom-up manner as a way to help students organize or better structure their mathematical activity. Rather than hand students a tool and tell them how to use this new device, the teacher asks for student strategies so that the reason for a new tool would be based upon their ideas.

## Critiques

Critics of the student-centered approach often site a lack of emphasis on teaching basic skills as one of the primary weaknesses. Additionally, without guidance, opponents question how students ever come to "discover" the concepts that are necessary for success in higher-level mathematics. Others argue that, while manipulatives and real-world contexts can serve as a source of motivation for students' mathematical activity, too many students do not develop the abstract reasoning associated with higher-level mathematical thinking. In contrast, NCTM officials released a statement that basic skills are a major component of student-centered approaches but emphasized that students' development of skills and facts should arise from critical thinking rather than memorization so that mathematics has meaning. Proponents of student-centered instruction also argue that real-world contexts and manipulatives are crucial for making meaning of mathematics but that students should use those experiences to create abstract meaning in mathematics.

## Cross-References


- ▶ Collaborative Learning in Mathematics Education
- ▶ Manipulatives in Mathematics Education
- ▶ Mathematical Proof, Argumentation, and Reasoning
- ▶ Problem Solving in Mathematics Education
- ▶ Teacher-Centered Teaching in Mathematics Education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# Learning Difficulties, Special Needs, and Mathematics Learning


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## Keywords

Mathematics learning difficulties; Mathematical reasoning; Arithmetic; Genetic syndromes and mathematical learning

## Introduction

"Learning difficulties" and "special educational needs" are terms that have been connected with different groups of children. Three of these groups are considered here initially, but the focus of this article is on the third one.

First, some children find learning many things more difficult than other children, and this includes learning mathematics. These children are assumed to have an intellectual disability due to genetic causes. Children with Down syndrome (DS) or with Williams syndrome (WS) exemplify the finding that genetically based intellectual disability also results in difficulty in learning mathematics. But there is a long way between genes and phenotypes in educational achievement, and one must be cautious about generalizations. Research shows very wide variation in the measured intelligence of children and adults with DS (estimates of their intellectual quotient [IQ] vary between about 30 and 70; average IQ in the non-affected population is 100) as well as those with WS (estimates of IQ vary between about 40 and 112. Research on the development of numerical cognition of individuals with DS and WS shows differences in the profiles of the two groups. Infants with DS perform less well than those with WS in numerosity recognition tasks (i.e., tasks that measure infants' reactions to displays with varying number of objects up to 4), but adults with DS achieve more in numerical cognition than those with WS, even when they are of comparable intellectual levels. Evidence from other genetic syndromes shows more specific effects on mathematics learning. Turner syndrome and fragile X syndrome are genetic disorders that affect girls and are associated with mathematical disability, although these syndromes do not typically result in general intellectual disability.

A second group to be considered relates to the finding that difficulties in mathematics can be a consequence of brain injury. The connection between different neurological circuits in the parietal lobe and mathematical activities has been investigated extensively, and some researchers suggest that damage to these brain circuits causes difficulties in mathematics. Consequently, some children may have difficulty in learning mathematics due to brain injury. Children who have genetic disorders or brain injuries have been included among children with learning difficulties and special educational needs. However, the term learning disability, rather than learning difficulty, is considered more appropriate in reference to these groups, due to its connection to the word ability and in view of the causes of the children's learning problems. This article focuses on the third group of children, whose measured intellectual ability is in the normal range but who find learning mathematics quite difficult.

Estimates of how common mathematics learning difficulty is vary depending on the method used in the study. The most reliable method is a cohort study, in which all the children born within a particular geographical region during a specified period are assessed and the results are scrutinized. Using a large cohort study and the American Psychiatric Association definition, which requires a discrepancy between performance in intelligence tests and in mathematics assessments, Barbaresi et al. (2005) estimated that 5.9–9.8 % of children and adolescents experience a substantial difficulty in some area of mathematics. However, the rate increased to 13.8 % if all children who experience difficulty in learning mathematics are considered and not only those for whom the difficulty is unexpected.

There are two main issues to be considered in the analysis of mathematics learning difficulty. The first is the nature of the mathematical skills affected, and the second is the specificity of the learning difficulty. Each of these issues is considered in turn with a focus on primary school mathematics learning.

## Characteristics

### Two Sorts of Mathematical Skills to Be Learned

The aim of mathematics instruction in primary school is to provide a basis for people to think mathematically even if they will not pursue a career that requires deeper mathematics knowledge. In order to think mathematically, people need to learn to represent quantities, relations, and space using numbers and other mathematical tools, such as algebra, graphs, and calculators, which are commonplace in today's society. A crucial distinction is made between quantities and relations, on the one hand, and numbers, on the other hand. Numbers are elements in a conventional system of signs and are used to represent quantities and relations between quantities. But numbers and quantities are not the same thing. "Quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them" (Thompson 1993, p. 165). For example, if you know that Robert is taller than Stephen and that Stephen is taller than Patrick, you conclude by reasoning about the quantities that Robert is taller than Patrick, although you do not know their heights.

The distinction between reasoning about quantities and knowledge of numbers supports the identification of two types of skill, which are both important in mathematics learning: mathematical reasoning and arithmetic skill (Nunes et al. 2011a). When a child solves a mathematical problem in school, the child is faced with two sorts of cognitive demand: the child must know which calculations to use to solve the problem and how to calculate. Some problems demand more reasoning than others because they demand cognitive transformations of the information before one can choose the operation to solve the problem. A comparison between problems A and B exemplifies this.

(a) Emma has a box with 14 CDs. Harmony gives her 19 CDs. How many CDs does Emma have now?
(b) The postman has some letters to deliver. He delivers 12 at one house. Now he has 39 letters in his bag. How many letters did he have before?

Both problems can be solved by an addition, but the first problem requires little transformation of the information: Emma gets more CDs and the problem is solved by an addition. In contrast, in the second problem, the postman delivers some letter and has fewer letters in his bag, but the problem is solved by an addition, which is the inverse of subtraction, because the starting number of letters is missing. Several studies have demonstrated that it is significantly more difficult to solve problems like B than like A. In a recent study with 7–8-year-olds, the children were asked to solve problems of these two types using a calculator, in order to circumvent difficulties with arithmetic. The level of success in type A problems was 86 % and in type B problems 37 %.

The distinction between these two sorts of skill, mathematical reasoning and arithmetic, has influenced how researchers define mathematics learning difficulty. One group of researchers defines mathematics learning difficulties as an inability to learn number relations (e.g., order of magnitude in the number system, addition and multiplication bonds) and to calculate quickly and accurately. The second group finds this a limited definition and argues that learning difficulties should be defined with relation to problems with mathematical reasoning as well as number skills: some children may know how to calculate but not know when to use which arithmetic operation.

The two groups agree on a research strategy that can be used to achieve a better understanding of mathematics learning difficulties. Both groups seek to predict which children will do better and which will do less well in mathematics. In these predictive or longitudinal studies, children are assessed at an earlier age on the factors that are hypothesized to be connected to mathematics achievement. At a later age, the same children are assessed in mathematics. If the factors measured earlier on do in fact predict the children's later achievement after the right controls have been taken into account, the study helps us understand mathematics learning difficulties better. Both groups agree that it is necessary to control for the children's performance in intelligence tests because mathematics learning difficulty is defined as an unexpected difficulty in learning mathematics. In spite of the similarity in the use of predictive studies, these two groups of researchers differ with respect to the assessments that they use as criteria for success or difficulty in learning mathematics.

Researchers who define mathematics learning on the basis of arithmetic skills seek to predict children's success on standardized tests, such as the Wechsler Individual Achievement Test (WIAT), which assess number discrimination, counting, number production, knowledge of basic addition and subtraction, multi-digit addition and subtraction, and multiplication and division but do not include items that require reasoning. The predictors that they use in their research are typically earlier forms of the same sort of number knowledge and cognitive processes related to memory, which are considered important for learning number relations and calculation rules. Geary et al. (2009), for example, used as predictors in their research children's early counting skill, speed and accuracy in identifying the number in sets, their ability to order numbers by magnitude, and their ability to recall addition facts. They also assessed the children's working memory (i.e., the ability to keep information in mind and work on the information at the same time). They assessed the children on these measures when they were in kindergarten. The children's success in mathematics learning was assessed when they were in first grade, using the WIAT. Because of the focus on arithmetic, children are not allowed to use calculators during this assessment. Geary and colleagues found that measures of number knowledge and working memory obtained when the children were in kindergarten predicted the children's performance in the WIAT mathematics measure, after controlling for the differences in the children's intelligence.

Researchers who think of mathematics learning more widely seek to predict the children's achievement in broader assessments. In England, for example, children are given state-designed standardized tests, called Key Stage tests (KS), which measure the children's mathematical learning. By the time children are in their sixth year in school, the KS mathematics tests include mental and written arithmetic as well as knowledge of decimals, problem solving, geometric reasoning, measurement of space and time, identification of number patterns in sequences of figures, and line and bar graph reading. The children are allowed to use a calculator for some parts of the assessment, but not for those that measure arithmetic knowledge. Nunes et al. (2011) hypothesized that arithmetic skills and mathematical reasoning would both predict children's achievement in these KS tests, and they used a large cohort study to investigate this hypothesis. They assessed the children's arithmetic skills by means of a standardized test in which the children were asked to solve arithmetic problems that required little reasoning; they assessed the children's mathematical reasoning by asking them to solve problems that required processing information about quantities before deciding which calculation to use but involved very simple arithmetic. The children were given these measures when they were between 8 and 9 years. They took the KS tests when they were 11 and again at age 14. Nunes and colleagues found that both the measure of arithmetic and the measure of mathematical reasoning predicted the children's performance in the KS tests, after controlling for individual differences in intelligence. It was also found that the reasoning measure was the better predictor of the two.

In summary, researchers start from different conceptions of mathematics learning difficulty and therefore use different measures of mathematics learning. Some researchers focus exclusively on number and arithmetic skills and exclude reasoning about quantities and relations from their analysis, whereas others include both types of knowledge. This theoretical divergence also leads to a discrepancy in the explanations for mathematics learning difficulties.

## The Specificity of Mathematics Learning Difficulties

Researchers differ in the way they define mathematics learning difficulties, but they agree that it is important to find out whether children's difficulties are specific to mathematics or result from more general cognitive processing mechanisms. There are different methods to investigate the specificity of learning difficulties. One is called comorbidity study: children who have difficulties in mathematics are screened for other learning difficulties, such as reading problems. The second is to analyze whether the factors that predict mathematics difficulties also predict other difficulties, such as English (or more generally, mother tongue) achievement.

The two large cohort studies mentioned earlier on in this paper used either of these methods in the investigation of specificity. Barbaresi and colleagues (2005) used the comorbidity method. They reported that many children who showed difficulty in learning mathematics did not show reading problems, therefore supporting the specificity of mathematics learning difficulties. However, the rates differed depending on the definition used. When the discrepancy definition was used, 56.7 % of the children who showed difficulty in mathematics did not show difficulty in reading, but this rate fell to 35 % if the discrepancy between intellectual and mathematical skills was not used in the identification of mathematics learning difficulty. Thus, when intellectual ability is not controlled for in the definition of mathematics learning difficulty, there is greater comorbidity between mathematics and reading problems.

The second approach, which assessed the specificity of the predictors of mathematics learning, was used in the cohort study by Nunes et al. (2011). They reported that arithmetic skill and mathematical reasoning are strong predictors of mathematical achievement but have little relation to the children's achievement in English KS tests. In contrast, intelligence measures predicted results in the KS tests for mathematics and for English. Thus, both methods used in cohort studies found evidence for the significance of general cognitive processes and for the specificity of mathematics learning difficulties.

## Further Research

Although some progress has been made in the investigation of the specificity of mathematics learning difficulties, there is an urgent need for further research. The issue of specificity needs to be investigated within mathematics learning itself. The tools used in mathematics, such as numerical and algebraic representation systems, calculation procedures, calculators, and computer programs, are increasingly more varied both in the same culture and across cultures. These tools clearly place different sorts of cognitive demands on learners, but very little is known about the continuities and discontinuities in learning when children use these different tools. It is quite possible that some children can do better in mathematics if they use one mathematical tool than another. For example, in the domain of simple calculation, there is evidence that some children are significantly better at oral calculation than at using written calculation procedures (Nunes et al. 1993) and that some are able to perform calculations quickly and accurately with the abacus but they do not perform as well without it. Mathematics educators recognize the empowering role of mathematical tools – for example, students learn algebra because it is expected to increase the power of their mathematical reasoning – but research has not yet started to consider how the use of different tools could impact the definition of mathematics learning difficulties. Therefore, this article closes with a question: could an earlier mastery of calculators and computers as tools for calculation change the definition of mathematics learning difficulty, or is the access to such tools dependent on children's knowledge of arithmetic?



## Cross-References


▶ Autism, Special Needs, and Mathematics Learning
▶ Blind Students, Special Needs, and Mathematics Learning
▶ Concept Development in Mathematics Education
▶ Deaf Children, Special Needs, and Mathematics Learning
▶ Down Syndrome, Special Needs, and Mathematics Learning
▶ Inclusive Mathematics Classrooms
▶ Language Disorders, Special Needs and Mathematics Learning


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## Learning Environments in Mathematics Education

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### Keywords

Scaffolding; Computer scaffolded learning; Computer-supported collaborative learning; Mathematical communication

### Computer Scaffolded Learning

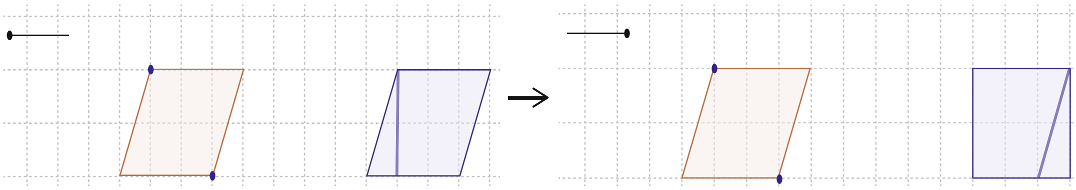
Scaffolding refers to adults helping a child in a process of tutorial interactions (Wood et al. 1976). The original definition can be generalized as capable people helping a novice, for instance, parents, tutors, or capable peers. However, when the novice who is scaffolded becomes capable, the scaffolds should fade in order to pass control back to the student.

In terms of Vygotsky's theory (1978), capable people as a form of scaffolding can help students to develop their potentials that they cannot reach alone, which is well known as the zone of proximal development. In other words, although low-ability students lack enough prior knowledge, they can complete a task if supported appropriately. Furthermore, Bloom (1984) found that if students were taught one-to-one by a human tutor, they could perform two standard deviations better than those taught in a conventional classroom. The finding suggested that capable people could effectively scaffold low-ability students and improve their performance.

Previous educators have found that there is a positive correlation between a student's prior knowledge and academic performance (Alexander and Jetton 2000; Dochy et al. 1999). The finding suggests that low-ability students need support to work from their prior knowledge when they learn new knowledge. Scaffolding is accordingly widely used as an appropriate tutoring strategy to solving the problem nowadays, because it can bridge and expand a student's capability by linking his/her prior knowledge and new knowledge (Wood et al. 1976; Wood and Wood 1996). Furthermore, owing to the additional and appropriate support, it is regarded as a core tutoring strategy to help students carry out a task.

Nowadays, learning technologies have prompted many changes in the design of scaffolding. Furthermore, in a computer-supported learning environment, the forms of scaffolding have shifted and have been extended from interaction with capable people to the support of artifacts, resources, and environments. The research of computer-based scaffolds focuses on cognitive and interface designs (Sharma and Hannafin 2007). The former emphasizes making cognitive processes visible to students. For example, procedural scaffolds provide explicit tasks and their sequences for achieving a goal (Quintana et al. 2002). The latter emphasizes using accurate and efficient representations of scaffolds. For example, embedded contextual scaffolds provide hyperlinks to supportive resources as well as contradictory evidence in order to facilitate students' critical thinking (Saye and Brush 2002).

Here is an example of using computer-based scaffolding of dynamic geometry software for supporting the exploratory learning of the mathematical topic "area of closed shapes." In general, students commonly have three types of



**Learning Environments in Mathematics Education, Fig. 1** Using a computer-based scaffold for supporting students to develop the concept of area conservation

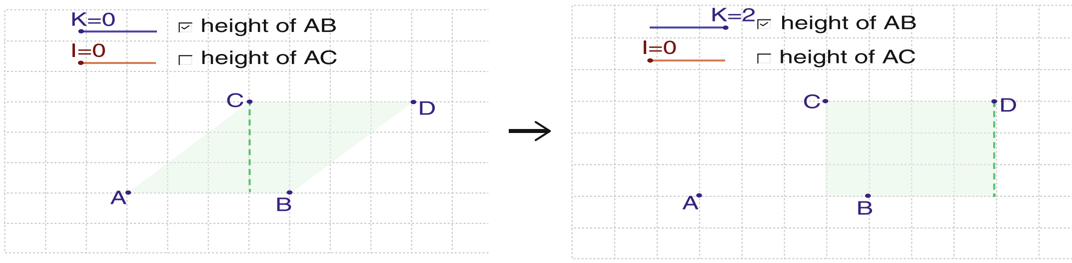
difficulties in learning this mathematical topic (Kospentaris et al. 2011; Naidoo and Naidoo 2007; Yu and Tawfeeq 2011). The first type of difficulty is the lack of the concept of area conservation, with a misunderstanding that the area of a shape is not the same before dissection and after re-combination. The second type of difficulty is the failure to identify a base and its corresponding height for area calculation. The third type of difficulty is the misconception that only regular closed shapes such as squares and rectangles have measurable area and corresponding mathematical formulas for area calculation; other irregular closed shapes have none.

The example discussed here uses GeoGebra, a computer-supported geometry software, to support students to explore mathematical formulas for calculating the area of closed shapes. GeoGebra has a graphics view interface with a dynamic coordinate plane which accurately and efficiently represents geometric objects. The dynamic functions of this and other dynamic geometry software (DGS) packages support users to flexibly manipulate, such as move, duplicate, and rotate, the geometric objects displayed, for a clear visualization of cognitive processes behind the actions on the geometric objects (Aydin and Monaghan 2011; Hohenwarter et al. 2009; Taylor et al. 2007). Mathematics teachers can use DGS packages to design interactive learning tools for exploratory learning which address students' three common difficulties in learning the area calculation of closed shapes.

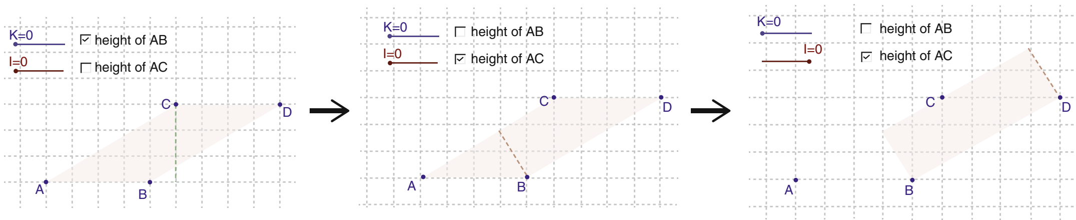
Figure 1 shows the use of a GeoGebra-based interactive learning tool for addressing the first type of learning difficulty. The interface of this interactive learning tool displays a parallelogram. Teachers in this exploratory learning activity ask students to use the dynamic function of shape movement to move the triangle dissected from the parallelogram for the final display of a rectangle. The accurate and efficient graphical support provided in this exploratory learning activity helps students to understand the concept of area conservation, through visualizing the cognitive process that after dissection the original parallelogram has the same area as the re-combined rectangle after shape movement. This also promotes students' association of irregular closed shapes with regular ones, and then their induction of the mathematical formula "base × height" for calculating the area of parallelograms.

Figures 2 and 3 show the use of two GeoGebra-based interactive learning tools for addressing the second type of learning difficulty. The interface of the first interactive learning tool (see Fig. 2) displays a parallelogram of which the perpendicular line starting from the upper left vertex (Vertex C) locates between the two vertices of the opposite side (Vertex A and Vertex B). The interface of the second interactive learning tool (see Fig. 3) displays a parallelogram of which the perpendicular line starting from the upper left vertex (Vertex C) locates outside of the two vertices of the opposite side (Vertex A and Vertex B).

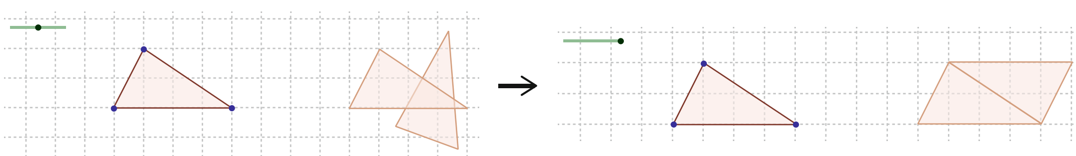
Teachers in this exploratory learning activity first ask students to explore the first interactive learning tool, selecting the option for displaying the "height of AB," and use the dynamic function of shape movement to move the triangle dissected from the parallelogram along the line AB for the final display of a rectangle. Subsequently, teachers ask students to explore the second interactive learning tool and continue the option for displaying the "height of AB." Students will then find that no triangle is dissected from the




**Learning Environments in Mathematics Education, Fig. 2** Using a computer-based scaffold to identify the height corresponding to the designated base (line AB)




**Learning Environments in Mathematics Education, Fig. 3** Using a computer-based scaffold to identify the height corresponding to the designated base (line AC)

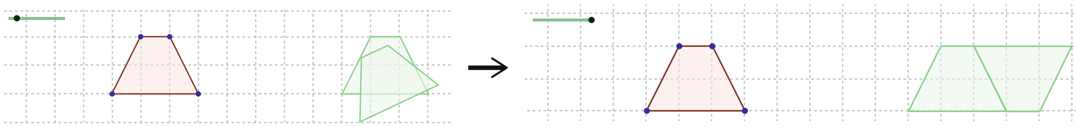


**Learning Environments in Mathematics Education, Fig. 4** Using a computer-based scaffold to explore the mathematical formula for calculating the area of triangles

parallelogram for movement along the line AB. Teachers can soon ask students to select the option for displaying the "height of AC." Students this time will find that they are able to move a triangle dissected from the parallelogram along the line AC for the final display of a rectangle. The graphical support in this exploratory learning activity accurately and efficiently represents the geometric objects in response to students' manipulations, which promotes students' realization that every parallelogram has two sets of base and the corresponding height for area calculation. This also helps students to visualize the underlying process in identifying the height corresponding to the designated base, and therefore promotes their induction of the relationship between a base and its corresponding height of parallelograms for area calculation.

Figures 4 and 5 show the use of two GeoGebra-based interactive learning tools for addressing the third type of learning difficulty. The interface of the first interactive learning tool (see Fig. 4) displays a duplicable triangle. The interface of the other interactive learning tool (see Fig. 5) displays a duplicable trapezoid.

Teachers in this exploratory learning activity ask students to use the dynamic function of shape duplication to duplicate one triangle and one trapezoid, and then use the dynamic function of shape rotation to rotate the duplicated shapes for the final display of a parallelogram. The accurate and efficient graphical support in response to



**Learning Environments in Mathematics Education, Fig. 5** Using a computer-based scaffold to explore the mathematical formula for calculating the area of trapezoids

students' manipulations of geometric objects enables students to visualize the cognitive process of associating irregular closed shapes with regular ones. Students can then find that irregular closed shapes like triangles and trapezoids, the same as with other regular closed shapes like squares and rectangles, have measurable area that can be calculated by mathematical formulas. Teachers can subsequently remind students to recall prior knowledge about the mathematical formula for area calculation of parallelograms, in order to promote their gradual induction that the mathematical formulas for calculating the area of triangles and trapezoids are "base × height/2" and "(upper base + lower base) × height/2" respectively.

## Computer-Supported Collaborative Learning

Even if children have not learnt mathematics, they live in a world with numbers and shapes. Mathematics helps people to understand the world by simplifying complex problems, solving them reasonably, and conveying the solution to other people persuasively. For this reason, mathematical communication emphasizes that students express their mathematical thinking coherently to peers and teachers. Furthermore, mathematical communication can be achieved by verbal and written forms (Hiebert 1992; Silver and Smith 1996). More specifically, students should use mathematical language to explore and express mathematical concepts and ideas in their own ways (Baroody 2000; Ginsburg et al. 1999; NCTM 2000; Rubenstein and Thompson 2002; Whitin and Whitin 2003). By doing so, students can broaden and deepen their conceptual understanding through making mathematical connections within mathematics and between mathematics and other domains (Brown and Borko 1992; NCTM 1991).

The National Council of Teachers of Mathematics (NCTM) describes the importance of mathematical communication: "communication is an essential part of mathematics and mathematics education (NCTM 2000, p. 60)." Mathematical communication involves adaptive reasoning (Kilpatrick et al. 2001, p. 170) and even argumentation (Andriessen 2006). In terms of adaptive reasoning, students have to acquire the ability to think logically, to explain a mathematical concept or procedure, and to justify their own or others' assertions. Adaptive reasoning also relates to the usage of representation (English 1997). The ability to use appropriate representation can facilitate conceptual understanding, and problem solving. In terms of argumentation, students have to elaborate what they think, and to debate with sufficient evidence (Toulmin 1958). When students attempt to build arguments, they aim to produce their mathematical ideas. For doing so, they may direct themselves to learn new concepts and procedures.

In order to facilitate the ability to communicate mathematically, students should be given opportunities, encouragement, and scaffolds to engage in oral communication in classrooms (NCTM 2000; Whitin and Whitin 2003). Previous research has identified several approaches to the facilitation of mathematical communication, which are introduced as follows.

Self-explanation (or think aloud) is a domain-general learning strategy (Chi et al. 1994), which emphasizes the linkage between prior knowledge and new one (Chi and van Lehn 1991). Previous research has shown that successful problem-solvers can generate more explanation (Chi et al. 1989).

Compared with self-explanation, peer-explanation is an interactive explanation strategy, which can be applied in a natural and social learning environment. Among various peer-explanation pedagogies, peer instruction is a widely adopted and effective pedagogy, which allows students to explain their own ideas for reducing misconceptions (Mazur 1997).

Furthermore, students may benefit from tutoring others (Cohen et al. 1982; Rohrbeck et al. 2003) as well as preparing teaching materials (Ching et al. 2005). Additionally, peer-teaching facilitates spontaneous and appropriate use of diagrams in order to solve mathematics word problems (Uesaka and Manalo 2007, 2011).

## Cross-References


- ▶ Collaborative Learning in Mathematics Education
- ▶ Information and Communication Technology (ICT) Affordances in Mathematics Education
- ▶ Learning Environments in Mathematics Education
- ▶ Learning Practices in Digital Environments
- ▶ Manipulatives in Mathematics Education
- ▶ Mathematical Representations
- ▶ Scaffolding in Mathematics Education
- ▶ Visualization and Learning in Mathematics Education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Yu PWD, Tawfeeq DA (2011) Can a kite be a triangle? Bidirectional discourse and student inquiry in a middle school interactive geometric lesson. N Engl Math J 43:7–20

# Learning Practices in Digital Environments

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## Keywords

Digital representations; Dynamic geometry; Coaction; Border objects; Executable representations

## Definition

To examine how new affordances of digital representations enable students and teachers' access to core mathematical ideas and develop deeper thinking and mathematical expressivity.

## Characteristics

The inherited corpus of mathematical knowledge produced with pre-digital technologies is large and stable. This stability has generated a kind of Platonic illusion as if this knowledge were independent of human beings.

Today, digital environments are becoming infrastructural for education, and inevitably they are confronted with the Platonic vision of knowledge that demands mathematical objects to be not just stable but immutable.

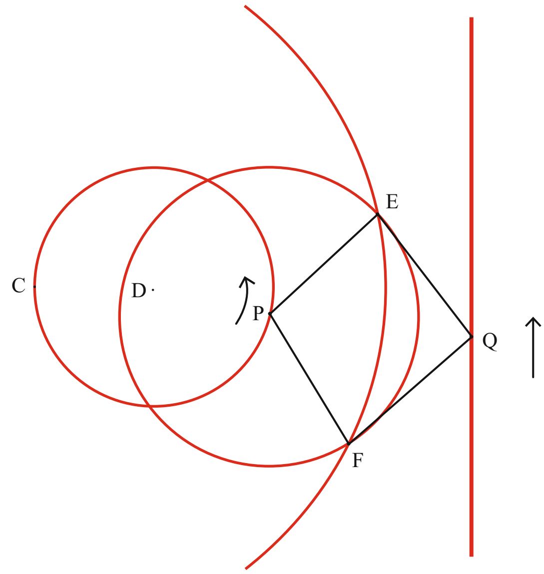
Learning practices in digital environments should take this confrontation seriously. In fact, as digital representations are executable, the environment reacts to the actions of the learner, and thus, the representation of the object is transformed. It has been aptly explained by Duval (2006), how the only way to access a mathematical object is by means of a semiotic representation. At the same time, the object does not have a pre-semiotic life as Plato had wished.

The fact that digital representations are executable enables the learner to manipulate the object in ways that are not possible in the traditional static environments. Executable representations constitute an affordance of the digital environments. Today, learning demands the design of transition strategies to transform the corpus of knowledge coming from the static environment and make it feasible in digital environments. Curriculum designers cannot ignore that the presence of executable representations translates into an enhanced expressivity on the side of learners. For instance, the graph of a function is not anymore the static picture seen in paper books but an enlivened object with the power to ignite new ways of exploration: The learner *coacts* (Moreno and Hegedus 2009) with the environment. Related work (Trouche and Drijvers 2010) investigates handheld technologies to distinguish between instrumentation – how tools affect and shape the thinking of the user – and instrumentalization, where the tool is shaped by the user.

Vygotsky (1981) explained that human action is mediated by tools and how the inclusion of a tool in a learning process modifies the nature of the process itself. The initial encounter with a digital environment can *amplify* what the learner already knows, by making explicit some features of the knowledge that appears as hidden in a static representation.

We will illustrate this position with a famous mechanical linkage – the Peaucellier Inversor. This machine transforms circular motion of the point P (see Fig. 1) into the straight-line motion of point Q. Point P moves around the circle with center D that contains the center C of the largest circle in the next figure. At the same time, point P is the center of the circle that intersects the largest circle at E and F. Point Q is the fourth vertex of the parallelogram.

When it appeared in 1864, the inversor went almost unnoticed. It is interesting to observe that when JJ Sylvester (in 1874) delivered a lecture on mechanical conversion of motion, one of the attendants to the lecture, Lord Kelvin, exclaimed when he saw the mechanical inversor in action:




**Learning Practices in Digital Environments, Fig. 1** A dynamic form of the Peaucellier Inversor

*It is the most beautiful thing I have ever seen in my life*!

Our experience with teachers (as learners) is that from their first encounter with the inversor, in its dynamic geometry embodiment, they are fascinated as well. They live a new kind of experience with geometry far from their former experience with static geometry. They learn precisely what a *theorem in motion* means. A new way of thinking emerges.

The learner establishes, gradually, a more profound relationship with the digital environment. With time, from being guided by the environment, the learner overcomes the resistance inherent in the environment reaching a new level of dexterity that makes it possible, for the learner, to guide the environment. This bilateral relationship (from being guided to guiding) implies, for instance, that the learner takes profit from the affordances of the environment to enhance her problem-solving strategies (Verillon and Rabardel 1995). None of this takes place in a social vacuum. The plasticity of the symbiotic relationship between learners and environments is sensitive to the presence of other learners (and teachers) and the ways they coact with the given environment and share their points of view.

Thus, the collective guiding of the environment eventually takes it to a higher structural level, as if the environment were crystallizing its own zone of proximal development.

Coaction is enhanced in such contexts where the environment is structured by executable visual representations as well as by the presence of a public space where the work of every learner can be viewed and analyzed (Hegedus and Moreno-Armella 2010). In such a space, the presentation can be controlled, and the teacher can ask questions about expectations before a set of graphs or motions are displayed as in SimCalc MathWorlds® (see http://www.kaputcenter.umassd.edu/simcalc/).

Actions displayed in the digital environments entail various forms of expressivity. Participants aim to explain "what they see," and consequently they express themselves in terms of gestures as well as speech. Coaction extends into the social space between the user (learner) and the whole set of contributions from all the participants. The action is not owned – in fact, agency is a plastic collaboration between the user and environment, both are actors and reactors. This occurs thanks to the infrastructural affordances provided by the environment.

Again, there is an "invisible hand" that can guide both the conceptual structure of the task and the flow of argumentation in the classroom. Coaction becomes a relationship between a learner, other learners, and the executable space within the technological environment.

Now, let us consider what are called *border objects* (Moreno-Armella and Hegedus 2009) that are essential for coaction to occur. They are digital-dynamic embodiments of mathematical objects that are defined initially within a paper-and-pencil environment and that can be meaningfully explored within the new environment. This kind of embodiment is not the same as a change of semiotic representation within the same medium – the static medium, for instance. In fact, a semiotic digital representation of a border object possesses a new quality that is not present in paper-and-pencil semiotic representations: the executability of the representation. This is a refraction into a different medium where the refracted object acquires a new operational field due to the executable nature of its new semiotic representation. This quality transforms the interaction that a learner can have with the mathematics, now embedded in the digital medium. For instance, when the learner finds a familiar object, a triangle let us say, and she drags a vertex, the medium reacts to her action producing a new triangle – revealing the plasticity of the object as it does not lose its identity as a triangle. This behavior is enabled by the executability of the digital representation of the border object. This reaction stimulates a new action from the hands of the learner.

The border object possesses some points, like the vertex, that are infrastructural. These points are called hot spots. It is the existence of hot spots in the object that creates the dynamic for coaction. These hot spots are points that can be used to construct mathematical figures, e.g., join two points with a segment or construct a piecewise graph, and then used to dynamically change the construction, as in the case of dragging the vertex of a triangle. In digital media – such as dynamic geometry environments (e.g., Cabri II Plus or Geometer's Sketchpad®) or SimCalc MathWorlds® – hot spots are key infrastructural pieces.

When students explore mathematics in a digital medium, where hot spots are present and where mathematics is embedded, they can experience mathematics through a qualitatively different semiotic mediator – that is, the new digital medium. The emergent knowledge from this digital medium is different from the knowledge emerging from a paper-and-pencil medium because the mediator is not epistemologically neutral. The nature of the knowledge is inextricably linked to the mediating artifact. This is where the border objects can guide us in the design of new models to explore mathematical thinking in classroom environments.

## Cross-References

▶ Learning Environments in Mathematics Education
▶ Technology and Curricula in Mathematics Education

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# Learning Study in Mathematics Education

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## Keywords

Learning; Teaching; Variation theory; Object of learning; Teachers' collaborative work

## Characteristics

Learning Study is an arrangement for teachers' collaborative work (with or without a researcher) aimed at enhancing student learning of a particular topic (Runesson 2008). By carefully and systematically studying their classroom teaching and students' learning, teachers explore what the students must learn in order to develop a certain capability. Learning Study is premised on an explicit learning theory and centers on students' learning problems. It was first conducted in Hong Kong 1999 and has since been developed in other parts of the world (e.g., Sweden, Brunei, and the UK). The Learning Study was developed from the background of several studies of classroom learning and differences in learning outcomes when the same topic was taught by different teachers to different classes. In these studies it was found that how the topic taught was handled, in terms of those aspects or features that were brought out in the lesson, was reflected in students' learning (Marton and Tsui 2004).

## The Cyclic Process and the Object of Inquiry

Just like the Japanese Lesson Study cycles (Stigler and Hiebert 1999), it is a process entailing planning a research lesson (or a sequence of lessons), which is taught, observed, evaluated, and modified in further lessons. The process starts with a group of teachers choosing and deciding about the object of learning, usually something they know is hard to learn and to teach. Next they design a pretest and give it to the students to find out about their learning problems. On the basis of this, they plan the first lesson(s) in the cycle, and one of the teachers implements the research lesson(s). This is documented by video recording, and after the lesson(s) a posttest is given to the students. The teachers meet again in a post-lesson session to analyze the recorded lesson and the results on the posttest. They reflect on the students' performance and the enactments of the lesson. If needed, they revise the plan, and another teacher implements the revised lesson in her class. This continues in a number of cycles until all the teachers in the group have conducted one lesson. So, in Learning Study, teachers try to find out why students fail to learn something specific and try to solve this problem. The failure is not sought in inadequacy of the learner, neither in the teaching arrangements nor methods used. Instead it is the relation learning – teaching that is the object of inquiry.

## Guided by a Learning Theory

One significant characteristic of the Learning Study is that in the iterative process of planning and revising, the teachers are guided by a learning theory – Variation Theory (Marton and Tsui 2004) – which helps them to focus on the object of learning and its critical features. The object of learning refers to the capability that the learners are expected to develop. It has a specific aspect – what is to be learned (e.g., Pythagoras' theorem, division with a decimal number) – and a general aspect, which refers to the way the learner masters that which is learned (e.g., explain, calculate, understand). Variation Theory states that how something is understood, seen, or experienced is a function of those features that are attended to at the same time. So, differences in ways of understanding are due to differences in the discernment of the features of what is learned. For every object of learning, there are some features that must be attended to at the same time by the learner; they are critical. Students might not focus on those features, or not focus on them simultaneously and their relationship, and thus not learn what is expected. In Learning Study the teachers try to identify features that are critical for a specific group of learners.

## Exploring the Object of Learning

To find out what the critical features are, it is necessary to go deeply into exploring the object of learning. In the process the teachers try to understand what it means to know something in mathematics by asking questions like: "What does it imply to understand that decimal numbers are dense?" and "What must be learned to understand this?" Teachers can use many sources to find the answers to these questions: literature review or their own and colleagues' teaching experience. The main source, however, is students' learning, how they experience that which is learned. One theoretical point of departure in Learning Study is that students' learning problems can arise from the teacher taking the critical features for granted; therefore, it is necessary to explore students' learning also. That is why some tasks are given to the students before and after the lesson, either as a written "test" or by interviewing the students before and after the lesson(s). Gaining information about what features the students do not discern must also be done by carefully observing students' responses in the video-recorded lesson. The aim is to get a deeper understanding of what features the students have failed to grasp. For instance, in a Learning Study about subtracting negative numbers, it was found on the posttest that the students had great problems with calculating, for example, $-3-\ (-5) = ?$. From a deep analysis of the lesson and students' learning, it was found that they did not realize that $-18$ is a smaller number than 3 (i.e., they had not learned the magnitude of integers). So, this was found to be one (of several) critical feature of being able to calculate with negative numbers.

Although the teacher has an idea of what the critical features may be for a particular group of students, she may not be able to bring them out in the classroom in a way that makes them learnable for the students. Here Variation Theory can help the teacher by being a guiding principle when designing learning possibilities. From a Variation Theory perspective, the relationship between learning and teaching is not seen as one of cause and effect. Teaching can only bring out possibilities for learning by helping the learners to discern the critical features. One fundament in Variation Theory is that a feature can be discerned only when it is experienced as a dimension of variation. Bowden and Marton (1998) state that something that varies against a stable background is likely to be discerned. So, applying Variation Theory when designing for learning implies creating a pattern of variation and invariance of those aspects that are critical for learning. For instance, in a Learning Study where the object of learning was to realize that decimal numbers are dense, it was found that seeing the decimal number as a part-whole relationship was a critical feature for learning. The students must learn that 0.97 is a point on the number line as well as hundreds in relation to one whole. To make it possible to learn this, the lesson was designed from a Variation Theory perspective so the same rational number was represented in different ways (e.g., 0.97 = 97/100 = 97 %). When the number was kept invariant whereas the representation varied, the students learned better compared to the lesson when only one representation was presented (i.e., did not vary) (Runesson and Kullberg 2010). Those students that had encountered representation of rational numbers as a dimension of variation were better at explaining why there are infinite decimal numbers in an interval.

Another example of how the principle in Variation Theory can be used is a Learning Study about angles (Runesson and Kullberg 2010). From the pretest it was found that the students (grades 4 and 5) thought that the size of an angle has to do with the lengths of the arms. Hence, if two angles with the same size, but with different length of the arms were compared, they thought the angle with the longest arms was the biggest. Guided by Variation Theory, the teachers designed tasks that would help them to focus on the amount of the rotation between the arms and disregard the length of the arms (i.e., the critical feature). For instance, in one of the (several different) tasks, two angles were compared. The students had to decide which was the biggest angle, a smaller angle (e.g., 30°) with the longest arms or a bigger angle (e.g., 60°) with the shortest arms.

## Teachers' Learning and Students' Learning: A Parallel Process

Besides enhancing student learning, Learning Study contributes to teachers' learning also. They learn about the object of learning from their teaching and from the learners. Their experiences are preferably documented so they can be accessible to other teachers. The documentation of a Learning Study is not a lesson plan in a general sense of the word. Instead it is a documentation of the critical features identified and a description of the pattern of variation and invariance that was found being effective in bringing them out in the lesson.

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# Lesson Study in Mathematics Education

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## Keywords

Lesson study; Professional development; Action research; Classroom instruction

## Definition

Lesson study, originated in Japan, is a common element in approaches to professional developments whereby a group of teachers collaborate to study the subject matter, instruction, and how students think and understand in the classroom. The original term of lesson study, "jugyo kenkyu" in Japanese, literally means the study of lesson.

## Historical Developments and Contexts

Lesson study is a Japanese approach to improve teaching and learning mathematics through a particular form of activity by a group of teachers. It provides teachers with key learning opportunities in working collaboratively with their colleagues to study subject matter, students' thinking and learning, and how to change classroom instruction.

The origin of lesson study can be traced back to the late 1890s, when teachers at elementary schools affiliated to the normal schools started to study lessons by observing and examining them critically (Inagaki 1995). The group of teachers started to have study meetings on newly proposed teaching methods. The original way of observing and examining lessons has spread out nationwide with some major refinements and improvements.

Teachers shared two types of methods to learn about new teaching approaches, called 'criticism lesson' and 'model lesson'. 'Criticism lesson' included a particular function of studying lessons, carefully examining the effectiveness of teaching, and publicly discussing ways to improve teaching and learning. The term 'research lesson', or kenkyu-jyugyo, might come from this particular function of lesson study with its major focus on producing a new idea, or testing a hypothesis in the form of an operationalised teaching method or teaching materials. On the other hand, 'model lesson' included another function of studying lessons; demonstrating or showcasing exemplary lessons, or presenting new approaches for teaching. For this purpose, the lesson should be carefully planned and based on research conducted by a teacher or a group of teachers. Participants can observe and discuss actual lessons with a hypothesis, instead of simply reading papers that describe the results of the study. The two different functions of lesson study – 'criticism lesson' and 'model lesson' – can be the original model of a variety of lesson study practiced around the county.

Lesson study takes place in various contexts (Shimizu 2002). Preservice teacher-training programs at universities and colleges, for example, include lesson study as a crucial and challenging part in the final week of student teaching practice. In-service teachers also have opportunities to participating in it, that is held within their school, outside their school but in the same school district or city, prefecture, and even at the national level for a couple of objectives. Teachers at university-affiliated schools that have a mission to developing a new approach to teaching often open their lesson study meeting for demonstrating an approach or new teaching materials they developed.

## Key Elements

The activity of lesson study includes planning and implementing the "research lesson" as a core of the whole activity, followed by post-lesson discussion and reflection by participants. A lesson plan plays a key role as a medium for the teachers to share and discuss the ideas to be examined through the process of lesson study.

Lesson study is a problem-solving process whereby a group of teachers work on a problem related to a certain theme. The theme can be related to examining the ways for teaching a new content or for using new teaching materials in relation to the revision of national curriculum guidelines or to assessing students' learning of a certain difficult topic in mathematics such as common fractions or ratio.

The first step of lesson study is defining the problem. In some cases, teachers themselves pose a problem to solve, such as how to introduce a concept of common fraction or what is the effective way to motivate students to learn mathematics. Second, planning lesson follows after the problem is defined. The group of teachers collaboratively develop a lesson plan. A lesson plan typically includes analyses of the task to be presented and of the mathematical connections both between the current topic and previous topics (and forthcoming ones in some cases) and within the topic, anticipation for students' approaches to the task, and planning of instructional activities based on them. The third step is a research lesson in which a teacher teaches the planned lesson with observation by colleagues. In most cases, a detailed record of teacher and students utterances is taken by the observers for the discussion in a post-lesson discussion. Evaluation of the lesson follows in a post-lesson discussion focusing on the issues such as the role of the implemented tasks, students' response to the tasks, and appropriateness of teachers' questionings. Based on the evaluation of the lesson, a revised lesson plan is developed to try the lesson again. These entire process forms a cycle of lesson study.

## The Role of Outside Experts

In lesson study, an outside expert is often invited as an advisor who facilitates and makes comments on the improvement of lesson in the post-lesson discussion (Fernandez and Yoshida 2004). The expert may be an experienced teacher, a supervisor, a principal of a different school, or a professor from the nearby university. In some cases, not only inviting the expert as a commentator of the discussion on site, the group of teachers may meet with him/her several times prior to conducting the research lesson to discuss issues such as reshaping the objective of the lesson, clarifying the role of the task to be posed in the classroom, and anticipating students' response to the task. In this context, outside expert can be a collaborator who shares responsibility for the quality of lesson with the teachers, not just an authority who directs the team of teachers.

## Lesson Study Adopted as a Model of Professional Development in Other Countries

After researchers in the USA introduced lesson study to the mathematics education community during the late 1990s, the term "lesson study" spreads among researchers and educators in the USA and later around the world (Hart et al. 2011). One of the most influential books that discusses about lesson study is *The Teaching Gap* (Stigler and Hiebert 1999). Then, schools and teachers in different countries have been trying to implement lesson study into their own education systems. The central question to the possibilities of "adoption" of the approach to other place is raised from a perspective on teaching as a cultural activity.

Improvement of teaching and learning through lesson study over a long period of time can take place, in Japanese education system, within the context in which clear learning goals for students are shared among teachers in relation to the national curriculum standards as well as teachers' voluntary hard efforts with the support of administrators. There are challenges to be resolved in practice and research possibilities to be explored in each context.

## Cross-References

▶ Communities of Inquiry in Mathematics Teacher Education
▶ Learning Study in Mathematics Education
▶ Mathematical Knowledge for Teaching
▶ Mathematics Teacher as Learner
▶ Models of In-Service Mathematics Teacher Education Professional Development
▶ Professional Learning Communities in Mathematics Education
▶ Reflective Practitioner in Mathematics Education

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# Logic in Mathematics Education

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## Keywords

Propositional calculus; Predicate calculus; Language; Proof; Reasoning; Semantics; Syntax; Truth; Validity


## Definition

Logic is a domain that was developed among the ancient Greeks and was first formalized by Aristotle in the Organon, where we already find the essential features of contemporaneous first-order logic: the importance of quantified statements and the interplay between syntax and semantics on the one hand, between truth in an interpretation and logical validity on the other hand. So, although Aristotle system was limited and insufficient for mathematics, most authors who developed the modern first-order logic (i.e., Frege, Russell, Wittgenstein, Tarski, Quine) implicitly or explicitly refered to Aristotle. Of course, logic is a field with a very large spectrum of aspects and use; so we will not try here to give a general definition, but restrict to a definition suitable for mathematics education. Following Durand-Guerrier et al. (2012), we define logic as:

> (…) the discipline that deals with both the semantic and syntactic aspects of the organization of mathematical discourse with the aim of deducing results that follow necessarily from a set of premises. (op. cit. p. 370)

## The Role of Logic in Mathematics Education

In the period of modern mathematics in the 1960s and 1970s, logic and naïve set theory were briefly part of the high school curriculum in some countries (e.g., in France). Since the 1980s, the question of the role of logic in mathematics education is highly controversial. In particular, there are discussions among authors if logic should be opposed to or considered as complementary of intuition. A rather common position among mathematics educators (and also mathematicians) against the teaching of logic is that practicing mathematics should develop the logical competencies required for mathematical activity. More over research have shown that teaching logic for itself does not necessarily improve mathematical compentencies. As a consequence, in a number of countries, the teaching of logic is nowadays developed in departments of computer sciences, often with links to discrete mathematics.

However, it seems rather clear that logic is closely intertwined with mathematical activity in two main aspects: the first one concerns mathematical language, and the second one concerns mathematical proof, argumentation, and reasoning.

## Logic and Mathematical Language

A first aspect concerns the role of logical categories in conceptualization process (Vergnaud 2009). These categories are *proposition* (a linguistic entity either true or false), *predicate* (*propositional function*) that models either a property (one place predicate) or a relationship (two or more places predicate), and *argument* that can be assigned to a placeholder in a predicate. Below are some examples of such categories.

*Propositions*: 23 is a prime number (true). For all Cauchy sequence in the rational number set, there exists a rational number which is the limit of the sequence (false). Symmetry preserves distance (true).

*Predicates*: To be a prime number (one place), to be convergent (for a sequence, a series; one place), to be an axis of symmetry of (two places).

*Arguments*: Integers, real numbers, sequences, convergence (of sequence, series), line, symmetry etc.

It is indeed a remarkable feature of mathematics that the process of conceptualization goes ahead along with a process of nominalization, such that properties at a given level are likely to become arguments at a more advanced level.

The second important issue concerns formalization. Indeed, the main interest of predicate calculus is to provide formal languages aiming to get rid of ambiguities that are inherent in natural languages and constitute a large part of their richness. In educational contexts, ambiguities are likely to lead to deep misunderstanding between teachers and students, or among students. As shown by research, such misunderstandings are often related with quantification matters (Durand-Guerrier et al. 2012). In this respect predicate calculus offers a sound resource for conceptual clarification. However, although formalizing statements is often useful to examine their truth-value or to engage in a proving process, even advanced students might fail to master properly such tasks (i.e., Selden and Selden 1995), and opposite with what could be expected, the introduction of formalized language to undergraduates appears for many students as an insuperable obstacle (Chellougui 2009). In educational contexts where students learn mathematics in a non-motherhood language, beyond differences in lexics or notations, differences in syntactical (grammatical) structures such as negation or quantifiaction are likely to impact strongly the understanding of the mathematical discourse in classroom. This last point needs international research development.

Although the empirical results from various research attest that these logical aspects of mathematical language are source of difficulties for many students (Durand-Guerrier et al. 2012), mathematics educators tend to underestimate these difficulties, a shared opinion being that clarity is a clue feature of mathematical language. Then, there is a tendency to neglect the importance of a specific work on the logical structure of mathematical statements.

## The Role of Logic in Mathematical Proof, Argumentation, and Reasoning

In an educational perspective, it could seem obvious to consider the role of logic in proof and proving in mathematics (Epp, 2003). But this position has been weakened by results in psychological research on reasoning, in particular around the famous Wason selection task, that seemed to show the irrelevance of formal logic to actual human reasoning. Nevertheless, as Stenning and Van Lambalgen (2008) state, this is a consequence of an interpretation of the given results from a strictly syntactic point of view. As soon as semantic aspects are considered as part of logic, the inadequacy of logic for modeling reasoning has to be reconsidered (Durand-Guerrier et al. 2012).

As developed in Quine (1982), logic is the theory of form and inference. This point has been clearly established by Tarski for quantified logic. Tarski (1994) provided a semantic definition of truth for formalized languages and developed a methodology for deductive sciences, introducing a model-theoretic point of view so that, according to Sinaceur (2001), logic can be considered as an effective epistemology to understand mathematical activity.

The relevance of this approach for mathematics education is developed in Durand-Guerrier (2008). In fact, considering proof, argumentation, and reasoning, as well as problem solving, it is quite clear that being able to recognize whether an inference is valid (i.e., is associated to a logical theorem) or not is crucial, as illustrated in the two following examples.

The first example is about *valid and not valid inferences involving implication in propositional calculus*. In propositional calculus, the truth tables provide means to prove that a formula is a logical theorem (a tautology, i.e., a statement that takes the value true for all combinations of values of its components) or that it is not. As shown by Quine (1982), this is closely related to inference rules in interpretation. The two tautologies

"$(p \cdot (p \Rightarrow q)) \Rightarrow q$" and "$(-q \cdot (p \Rightarrow q)) \Rightarrow -p$" are respectively associated with the inference rules named modus ponens "*A*; and "If *A*, then *B*"; hence *B*" and modus tollens "not *B*; and "If *A*, then *B*"; hence not *A*." The two implicative formulas "$(q \cdot (p \Rightarrow q)) \Rightarrow p$" and "$(-p \cdot (p \Rightarrow q)) \Rightarrow -q$" are not tautologies (it is possible that the premises are true and the consequence false); therefore, it is neither possible to deduce *A* from *B* and "If A, then B" nor to deduce not *B* from not *A* and "If *A*, then *B*." It is important to notice that this is at the core of the distinction between an implication and its converse, and hence between implication and equivalence.

The second example is about *a not valid inference rule involving multiple quantifiers*. It is well known that the following rule – "For all *x*, there exits *y* such that $P(x, y)$"; and "For all *x*, there exists *y* such that $Q(x, y)$"; hence "For all *x*, there exists *y* such that $P(x, y)$ and $Q(x, y)$" – is not a valid inference rule. Indeed, it is easy to find counterexamples where the premises are true while the conclusion is false: given an interpretation, once a generic element *a* has been considered, the first (resp. the second) premise allows considering an element *b* (resp. *c*) such as $P(a, b)$ (resp. $Q(a, c)$) is true; *b* and *c* are a priori different. However, in a large number of mathematical contexts, it is possible once having considered such elements *b* and *c* to build a third element satisfying both premises and hence the conclusion (e.g., in ordered sets in some cases, the maximum of *b* and *c* satisfies both premises), so that teachers can decide to delete this step of reasoning. As a consequence, students, *forgetting* that the rule is not valid, can use it in cases where it is not possible to find an element satisfying both premises. This can lead them either to prove a false statement or to provide an incorrect proof for a true statement (Durand-Guerrier and Arsac 2005).

To control validity of written text, natural deduction (i.e., Copi 1954) offers tools allowing students and teachers to become aware of the necessity of and to be able to control the validity of the inference rules used (Durand-Guerrier et al. 2012). However concerning the production and the control of arguments exchanged during classroom sessions, in particular during *situations of validation*, dialogical models are more suitable as shown by Barrier (2011) who introduced semantics and pragmatics aspects as developed in Semantics Games Theory (Hintikka and Sandu 1997).

## Conclusion

Examining the role of logic in mathematics education brings argument for the value of integrating logical instruction in mathematics curricula. The question of how to do this in order to foster the development of mathematics conceptualization and the development of proof and reasoning competencies remains largely opened. Some first paths are given in Durand-Guerrier et al. (2012), but further research taking in consideration the variety of educational and linguistic contexts are needed.



## Cross-References


▶ Discrete Mathematics Teaching and Learning
▶ Intuition in Mathematics Education
▶ Mathematical Language
▶ Mathematical Proof, Argumentation, and Reasoning
▶ Problem Solving in Mathematics Education


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## Manipulatives in Mathematics Education

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### Keywords

Concrete manipulatives; Virtual manipulatives; Historic-cultural manipulatives; "Artificial" manipulatives

### Definition

Mathematical manipulatives are artifacts used in mathematics education: they are handled by students in order to explore, acquire, or investigate mathematical concepts or processes and to perform problem-solving activities drawing on perceptual (visual, tactile, or, more generally, sensory) evidence.

### Characteristics

#### Manipulatives and Mathematics Education

One can distinguish several kinds of manipulatives used in schools and education. Two classifications that emerge from the literature may be suggested, referring to either the quality of interaction user-manipulative or the origin of the manipulative: concrete versus virtual manipulatives and historic-cultural versus "artificial" manipulatives.

*Concrete manipulatives* are physical artifacts that can be concretely handled by students and offer a large and deep set of sensory experience.

*Virtual manipulatives* are digital artifacts that resemble physical objects and can be manipulated, usually with a mouse, in a similar way as their authentic, concrete counterparts.

*Historic-cultural manipulatives* are concrete artifacts that have been created in the longstanding history of mathematics to either explore or solve specific problems, both from inside and from outside mathematics.

"*Artificial*" *manipulatives* are artifacts that have been designed by educators with specific educational aims.

The following table lists some examples according to the combination of the two classifications above.

| | Concrete | Virtual |
|---|---|---|
| **Historic-cultural** | Different kinds of abaci; Napier's bones; measuring tools such as graded rulers and protractors; polyhedrons; mathematical machines; topological puzzles; geometrical puzzles; dices and knucklebones; ancient board games | Suanpan the Chinese abacus, virtual copies of mathematical machines |

(*continued*)

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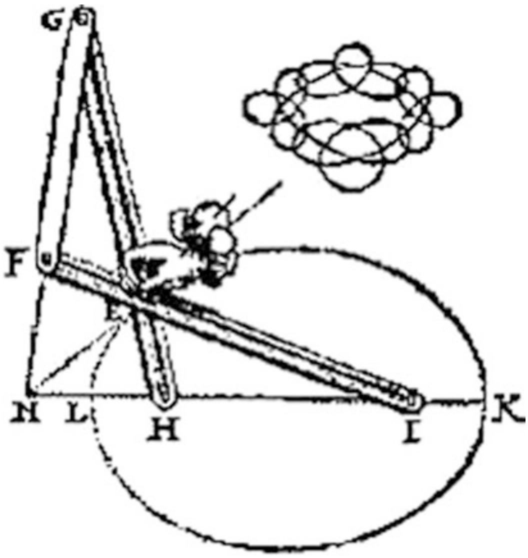
**Manipulatives in Mathematics Education, Fig. 1** Schoty and Froebel gifts

| | Concrete | Virtual |
|---|---|---|
| "**Artificial**" | Froebel's gifts, Montessori's materials, Cuisenaire rods, Dienes's materials, multibase blocks, fraction strips and circles, bee-bot | Library of virtual manipulatives |

Historic-cultural manipulatives refer to mathematical meanings, as they have paved the way towards today's mathematics (some examples are discussed in a further section). Artificial manipulatives are the outcomes of an opposite path: an ingenuous educator invented, for specific educational purposes, a new way to embody an established mathematical concept into an object or a game. At the beginning this choice might be considered artificial (and this is the reason of using this term in the classification above). A famous example is given by Dienes who explains the root of multibase blocks and the teachers' resistance to this introduction, perceived as completely artificial. One might object that the difference between the historic-cultural and artificial ones is fuzzy. Is one allowed to consider Froebel's gifts artificial and the Slavonic abacus historic-cultural? Not exactly, if one considers that both artifacts date back to the same period and have been designed for educational purposes. The Slavonic abacus was carried to France around 1820 from Russia by Poncelet who transformed the Russian abacus for educational purposes. Froebel gifts were designed around 1840 for activity in the kindergarten. In the proposed classification, the Slavonic abacus is considered a historic-cultural one, because of the strict relationship with other kinds of abaci, while Froebel gifts are considered the ancestors of other artificial manipulatives produced later by educators like Montessori, Cuisenaire, and Dienes (Fig. 1).

Both are examples of the inclination to give value in Europe to active involvement of mathematics students during the nineteenth century (see Bartolini Bussi et al. 2010) and represent the background where the International Commission on Mathematical Instruction (ICMI) started to work with a big emphasis on active methods and laboratory activities.

The distinction between concrete and virtual manipulatives deserves some observation. A whole library of virtual manipulatives is available on the web. In this library, there are digital "objects" (mostly in the form of Java applets) representing many artificial manipulatives and allowing to act on them in a way similar to the action on their concrete counterparts. There are also websites, where digital copies of historic-cultural manipulatives are available. In all cases the user-manipulative interaction is limited to mouse piloting and looking at effects. Systematic research on virtual manipulatives and on comparison between concrete and virtual manipulatives is still at the very beginning. Virtual manipulatives are easily available (wherever a computer




**Manipulatives in Mathematics Education, Fig. 2** van Schooten ellipsograph

laboratory is located), are time and space saving, and are motivating, because of the appeal they exert on students accustomed to digital devices. However, if compared with concrete manipulatives, virtual manipulatives seem to highlight mainly visual experience, skipping reference to tactile and other sensory experience. The new touch-screen technology with the possibility of touching simultaneously different points on the screen seems to open new possibilities (see, for instance, Mak-trace, by Anna Baccaglini-Frank (Baccaglini-Frank et al. (2012)) and TouchCounts by Nathalie Sinclair).

A few studies have been carried out about the comparison between concrete and virtual manipulatives. For instance, Hunt et al. (2011) report the findings of a three-year study with prospective middle-grade mathematics teachers enrolled in Clayton State University. Perceived advantages and disadvantages of concrete versus virtual manipulatives are compared after a full course where both kinds of manipulatives for Number Concepts had been used. Concrete manipulatives appeared to be more effective for building preservice teachers' and students' conceptual understanding. The virtual manipulatives were used to reinforce those concepts. The usefulness of using both concrete and virtual manipulatives is emphasized by Maschietto and Bartolini Bussi (2011). Both a concrete and a virtual copy of the same manipulative (i.e., the van Schooten ellipsograph by antiparallelogram – Fig. 2) are analyzed, comparing classroom tasks and tasks for teachers about the textual description with "realistic" drawings.

### Critical Issues

The first critical issue concerns the students' autonomy in using manipulatives. In the western tradition, since the time of Montessori, the use of manipulatives was mainly aimed at spontaneous activity within a well-prepared environment: adults organize the environment where students (usually aged between 3 and 10–12) are free to select activity. This trend has to be historically contextualized as a reaction against the lecture-based school, criticized also by Dewey (1907). Yet there are studies (e.g., Uttal et al. 1997; McNeil and Jarvin 2007) which have a more critical approach to manipulatives. The effectiveness of manipulatives over more traditional methods is analyzed, claiming that the sharp distinction between concrete and symbolic forms of mathematical expression is not useful. There is no guarantee that students will establish the necessary connections between manipulatives and more traditional mathematical expressions. In particular this issue calls into play the importance of instruction (or teaching) about manipulatives and the connection between manipulatives and symbols.

The second critical issue concerns the students' age. Most research about manipulatives has been carried out at preschool and primary school level, highlighting the usefulness of manipulatives at a certain age only (e.g., Kamii et al. 2001). In most guides for teachers, the use of manipulatives is especially aimed at either primary school students or students with special needs. Curtain-Phillips complains about the scarce use of manipulatives in US high schools, quoting, as an exception, Marilyn Burns who used manipulative materials at all levels for 30 years. Moreover, she quotes the attention of the National Council of Teachers of Mathematics (NCTM) that has encouraged the use of manipulatives at all grade levels, in every decade, since 1940. She asks an interesting

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question: why are high school teachers reluctant to use this type of resources? One reason might be the nearly unique emphasis on artificial manipulatives that have been created with the declared aim to embody an abstract mathematical concept into a concrete (or virtual) object. If this is the shared approach, the effect is that they are used with either young children or students with special needs, who are expected to need more time for concrete-enactive exploration. Nührenbörger and Steinbring (2008) contrast this position emphasizing that manipulatives are symbolic representations in which mathematical relationships, structures, and patterns are contained and can be actively interpreted, exchanged within the discursive context, and checked with regard to plausibility (see also Uttal et al. 1997). The "theoretical ambiguity" of manipulatives is to be considered a central theme in mathematics lessons. This very ambiguity makes manipulatives suitable to all school levels, up to university, as a context where fundamental processes, as defining, conjecturing, arguing, and proving, are fostered. This requires a very strong and deep analysis of manipulatives, from theoretical and epistemological points of view, and a study of the consequence of this analysis in teachers' design of tasks and interventions in the mathematics classroom. To cope with this problem, in our research team (Bartolini Bussi and Mariotti 2008), we have developed the framework of semiotic mediation after a Vygotskian approach. In the following section, we outline this framework together with some examples, mainly taken from the historic tradition.

## A Comprehensive Theoretical Approach to Manipulatives: Semiotic Mediation After a Vygotskian Approach

Vygotsky studied the role of artifacts (including language) in the cognitive development and suggested a list of possible examples: "various systems for counting; mnemonic techniques; algebraic symbol systems; works of art; writing; schemes, diagrams, maps, and mechanical drawings; all sorts of conventional signs, etc." (Vygotsky 1981, p. 137). Manipulatives might be included in this list. The introduction of an artifact in a classroom does not automatically determine the way it is used and conceived of by the students and may create the condition for generating the production of different voices. In short, the manipulatives are polysemic, and they may create the condition for generating the production of different voices (*polyphony*). This position is consistent with Nührenbörger and Steinbring's theoretical ambiguity mentioned above (2008). The teacher mediates mathematical meanings, using the artifact as a tool of semiotic mediation. Without teacher's intervention, there might be a fracture between concrete learners' activity on the manipulative and the mathematical culture, hence no learner's construction of mathematical meanings. In this framework the theoretical construct of the *semiotic potential of an artifact* is central: i.e., the double semiotic link which may occur between an artifact and the personal meanings emerging from its use to accomplish a task and at the same time the mathematical meanings evoked by its use and recognizable as mathematics by an expert (Bartolini Bussi and Mariotti 2008).

## Some Examples of Manipulatives and Tasks

This section presents the semiotic potential of some manipulatives, known as Mathematical Machines. *A geometrical machine is a tool that forces a point to follow a trajectory or to be transformed according to a given law. An arithmetical machine is a tool that allows the user to perform at least one of the following actions*: *counting*, *reckoning*, *and representing numbers*. They are concretely handled and explored by students at very different school levels, including university. In most cases also virtual copies exist as either available resources (see the right frame at www.macchinematematiche.org) or outcomes of suitable tasks for students themselves (Bartolini Bussi and Mariotti 2008). The historic-cultural feature of these manipulatives allows to create a classroom context where history of mathematics is effectively used to foster students' construction of mathematical meanings (Maschietto and Bartolini Bussi 2011). Each example contains a short description of the manipulative, an exemplary task and the mathematical meaning, as intended by the teacher.


**Manipulatives in Mathematics Education, Fig. 3** Shuxue ISBN 7-107-14-632-7

## Counting Stick

Counting sticks, dating back to ancient China, are thin bamboo or plastic sticks. The sticks are counted, grouped, and bundled (and tied with ribbons or rubber bands) into tens for counting up to hundred; ten bundles are grouped and bundled into hundreds and so on.

Figure 3a–b is taken from a Chinese textbook: the oral numerals beyond ten are introduced by grouping and tying ten sticks (left, 1st grade) and a "difficult" subtraction is realized by untying and ungrouping a bundle (right, 1st grade).

**Tasks**: To guess numerals between 10 and 20 in the first case and to calculate 36–8 in the second case.

In this case the triangle of semiotic potential hints at:

**Mathematical knowledge**: Grouping/regrouping.

There is a perfect correspondence between the two opposite actions: tying/untying and grouping/ungrouping. The former refers to the concrete action with sticks and bundles; the latter refers to a mathematical action with units and tens. It is likely that primary students' descriptions refer to the concrete action (in one class, 1st graders invented the Italian neologism "elasticare," i.e., "rubbering"). It is not difficult for the teacher to guide the transition from the wording of the concrete action towards the wording of a mathematical action. In this way also the need (as perceived by teachers) to use "borrowing" from tens to units is overcome (see Ma 1999, p. 1ff. for a discussion of this issue).

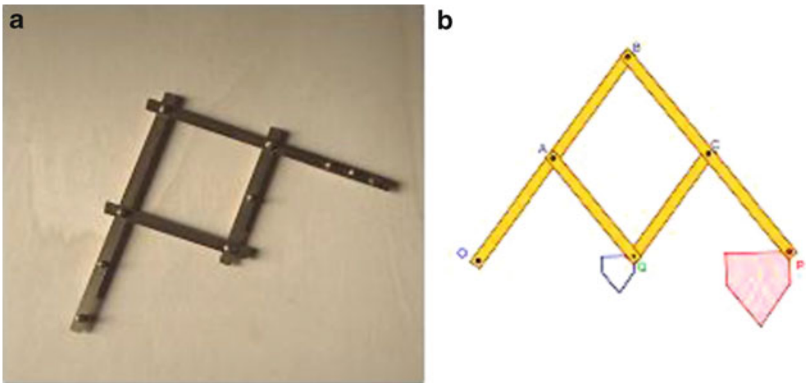
**Manipulatives in Mathematics Education, Fig. 4** Pascaline "zero + 1"

## Pascaline

The pascaline is a mechanical calculator (see Fig. 4) (Bartolini Bussi and Boni 2009).

The name of the instrument hints at the design of a mechanical calculator by Blaise Pascal (for details, see Bartolini Bussi et al. 2010). An exemplary task is the following: **Task**: Represent the number 23 and explain how you made it. Different pieces of **mathematical knowledge** may be involved to answer the task, for instance:

- The generation of whichever natural number by iteration of the function "+1" (one step ahead for the right bottom wheel)
- The decomposition of a 2-digit number (23) into 2 tens and 3 units



**Manipulatives in Mathematics Education, Fig. 5** (**a**–**b**) Scheiner's pantograph http://www.macchinematematiche.org/index.php?option=com_content&view=article&id=112&Itemid=195

The first mathematical action may be carried out on the pascaline by iterating 23 times the function "+1"; the second mathematical action may be carried out by iterating the function "+1" 3 times on the right bottom wheel and 2 times on the central bottom wheel.

### Pair of Compasses and Other Curve Drawers

The compass (pair of compasses) is the oldest geometrical machine; it is a technical drawing instrument that can be used for inscribing circles or arcs. It is used also as a tool to measure distances, in particular on maps. The compass objectifies, by means of its structure and its functional use, the defining elements of the circle (center and radius) and reflects a clear definition of the circle as a closed curve such that all its points are equidistant from an inside common point (Bartolini Bussi et al. 2007).

**Tasks**: How is the pair of compasses made? What does it draw? Why does it do that?

**Mathematical knowledge**: From primary school the compass can be used and analyzed in order to learn concepts and to understand how it embodies some mathematical laws (Chassapis 1999). The same can be done in the upper grades (up to teacher education programs, Martignone 2011), after the exploration of the compass structure and movements, student can become theoretically aware about how the mathematical law is developed by compass and then they can use this instrument to solve problems and to produce proofs in Euclidean geometry.

Even if the compass is the most famous curve drawer, over the centuries many different types of curve drawers have been designed and used as tools for studying mathematics and for solving problems (see http://www.museo.unimo.it/labmat/usa1.htm). The oldest linkages date back to the Alexandrian and Arabic cultures, but it is in seventeenth century, thanks to the work of Descartes (1637), that these machines obtained a wide theoretical importance and played a fundamental role in creating new symbolic languages (see http://kmoddl.library.cornell.edu/linkages/).

### Pantographs

Over the century the pantographs were described in different types of documents, such as mathematical texts and technical treatises for architects and painters. In particular, in nineteenth century, when the theory of geometrical transformations became fundamental in mathematics, they were designed and studied by many scientists. A famous linkage is the Scheiner's pantograph: a parallelogram linkage, one of whose joints has its movement duplicated by an attached bar. This has been used for centuries to copy and/or enlarge drawings. Since the end of the sixteenth century, this type of machines was used by painters even if it was improved and described by Scheiner in 1631 (Fig. 5a–b).

**Tasks**: Students can study how the machine is made, how the different components move, what are the constraints, and the variables modeling the structure by means of Euclidean geometry.

**Mathematical knowledge**: The Scheiner's pantograph can be used for introducing the concept of dilation (homothety) and/or for developing argumentation processes about why the machine does a dilatation.

Finally, it should be emphasized that these ancient technologies, whose use and study date back to past centuries, have modern developments, for example, modeling the robot arms. Also in mathematics, the study of linkages has been recently revived. In the twentieth century, ideas growing from Kempe's work were further generalized by Denis Jordan, Michael Kapovich, Henry King, John Millson, Warren Smith, Marcel Steiner, and others (Demaine and O'Rourke 2007).

### Open Questions

There is no best educational choice between different kinds of manipulatives. Rather the choice depends on different factors (what is available, what fits better the students' culture and expectations, and so on) and, above all, on teachers' system of beliefs and view on mathematics. There is never a "natural" access to the embodied mathematics, as no artifact is transparent in its embodied mathematical meaning (Ball 1992; Meira 1998): a suitable context and set of tasks are always required. There are many reasons to support the use of manipulatives in the mathematics classrooms, but the short review of literature above shows that there is still a place for developing studies about:

- Manipulatives: to analyze limits and potentialities of different kinds of manipulatives (concrete vs. virtual; historic-cultural vs. artificial) from an epistemological, cognitive, and didactical perspective
- Classroom practice: to design, test, and analyze tasks about manipulatives at different school levels and in different cultural traditions
- Teacher education and development: to design, test, and analyze tasks for teachers about the use of manipulatives in the mathematics classroom

## Cross-References

▶ Activity Theory in Mathematics Education
▶ Semiotics in Mathematics Education
▶ Teaching Practices in Digital Environments

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# Mathematical Ability

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## Keywords

Development of mathematical ability; Evaluating mathematical ability; High ability; Individual differences; Low ability; Mathematical reasoning; Spectrum of mathematical abilities

## Definitions

Mathematical ability is a human construct, which may be defined cognitively or pragmatically, depending on the purpose of definitions. Cognitive definitions are used when relating to this construct from a theoretical perspective; mathematical ability can then be defined as the ability to obtain, process, and retain mathematical information (Krutetskii 1976; Vilkomir and O'Donoghue 2009) or as the capacity to learn and master new mathematical ideas and skills (Koshy et al. 2009). Pragmatic definitions are usually used when looking at this construct from a perspective of evaluation (e.g., when the focus is on identifying learners' potential or assessing learning outcomes). From this perspective, it can be defined as the ability to perform mathematical tasks and to effectively solve given mathematical problems. Such definitions are general in nature and are commonly unpacked into several components, which are not necessarily exclusive to one definition or another. Thus, we speak of an assemblage of mathematical *abilities* rather than a single ability. One of the most acknowledged and widely accepted theories in this respect is that of Krutetskii (1976), who suggested that mathematical ability is comprised of the following abilities: use formal language and operate within formal structures of connections, generalize, think in a logic-sequential manner, perform shortcuts ("curtailments") while solving problems, switch thinking directions, move flexibly between mental processes, and recall previously acquired concepts and generalizations.

## Characteristics

### The Evolvement of Mathematical Abilities

Mathematical abilities develop in correspondence with the development of rational and logical thinking. According to Piaget's theory of cognitive development (Piaget and Inhelder 1958), logical thinking skills are limited in the first two developmental stages of normative childhood, the sensorimotor stage and the preoperational stage. This means that although young children, who have acquired the use of language (around the age of 2–3), are able to link numbers to objects and may have some understanding of the concepts of numbers and counting, they still cannot comprehend logical notions such as reversible actions or transitivity until they reach the concrete-operational stage, around the age of 7–8. At this stage, a child can comprehend, for example, that the distance from point A to point B is the same as the distance from point B to point A and that if $x \leq y$ and $y \leq z$, then $x \leq z$. During the concrete-operational stage (ages 7–8 to 11–12), a considerable growth in mathematical abilities is enabled due to the acquisition of two additional logical operations: seriation, defined as the ability to order objects according to increasing or decreasing values, and classification, which is grouping objects by a common characteristic (Ojose 2008). Yet, the abstract thinking necessary for grasping and constructing mathematical ideas evolves during the formal-operational stage, around the ages of 11–12 to 14–15. At this stage, according to Piagetian theory, adolescents are able to reason using symbols, make inductive and deductive inferences, form hypotheses, and generalize and evaluate logical arguments.

Piaget's theory was criticized, among other things, for underestimating the abilities of young children while overestimating the abilities of adolescents (Ojose 2008). However, Piaget himself emphasized that the stages in his theory do not necessarily occur in the ages specified. That is, some children will advance more quickly and reach a certain cognitive stage at a relatively early age; others may not arrive at this stage until much later in their lives. The speed of development and the degree to which the last formal-operational stage is realized depend on various personal and environmental attributes. This view corresponds with Vygotsky's theory (Vygotsky 1978) which emphasizes the crucial role that social interactions and adult guidance, available in children's environment, play in their cognitive development. Thus, as a result of variations in individuals' circumstances and available mathematical experiences, we find that the spectrum of mathematical abilities in a specific age group is of a wide magnitude.

## Characterizing Different Students on the Spectrum of Mathematical Abilities

Researchers have endeavored to characterize students located close to both ends of the mathematical ability spectrum: on the one hand mathematically gifted and highly able students and on the other hand students who are lacking in their mathematical abilities, compared with their peers.

The aforementioned classical work of Krutetskii (1976) concentrated on the higher end of mathematical abilities. Krutetskii used a wide-ranging set of mathematical problems and an in-depth analysis of children's answers, in an attempt to pinpoint the components of mathematical ability in general and higher ability in particular. Based on his investigations, Krutetskii referred to four groups of children: extremely able, able, average, and low. He inferred that extremely able children are characterized by what he termed as a "mathematical cast of mind." This term designates the tendency to perceive the surrounding environment through lenses of mathematical and logical relationships, to be highly interested in solving challenging mathematical problems, and to keep high levels of concentration during mathematical activities. Interpreting Krutetskii's theory, Vilkomir and O'Donoghue (2009) suggest that a mathematical cast of mind stimulates all other components of mathematical ability to be developed to the highest level, if the student is provided with the necessary environment and instruction.

At the other end of the spectrum, we find learners with low mathematical abilities. Although these learners typically perform poorly in school mathematics, the inverse is not necessarily true. In other words, the presumption that poor mathematical performance of students is indicative of their low mathematical abilities is problematic; a range of social, behavioral, and cultural circumstances can result in low achievements in school mathematics (Secada 1992). In addition, students may develop a negative mathematical self-schema that reduces their motivation to succeed in mathematics, regardless of their overall abilities (Karsenty 2004). Nevertheless, characteristics of low mathematical abilities are available in the literature. Overcoming the abovementioned pitfall may be achieved through careful consideration of a child performance in a supportive environment, under a personal guidance of a trusted adult. Thus, we find that the main features of low mathematical abilities are difficulties in establishing connections between mathematical elements of a problem; inability to generalize mathematical material according to essential attributes, even with help and after a number of practice exercises; lack of capability to deduce one thing from another and find the common principle of series of numbers even with assistance; avoidance from using symbolic notations; and short-lived memory for mathematical procedures (Karsenty et al. 2007; Vilkomir and O'Donoghue 2009). In extreme cases of low mathematical abilities, the term mathematical disability (MD) is used. Research on MD is commonly conducted on subjects with notable deficiencies in basic arithmetic skills and includes explorations of the disability known as dyscalculia. MD is not an uncommon disorder (estimations range between 3 % and 8 % of the school-age population) and is mainly attributed to cognitive, neuropsychological, and genetic origins (Geary 1993).

### Mathematical Abilities and General Intelligence

Despite the popular view that links mathematical ability with intelligence, the relation between these two constructs remains elusive. The original intelligence test developed by Binet and Simon in the early 1900s emphasized mostly verbal reasoning and did not include a mathematical component, except for simple counting. The later version, known as the Stanford-Binet test, which was composed by Terman in 1916 (and is still used today, after several revisions along the years), includes a quantitative reasoning part. Terman assumed that mathematical abilities play some role in determining general intelligence, yet he did not conduct empirical studies to support this argument. Later theories of intelligence also suggested that there is a quantitative element in models describing intelligence. For instance, Thurstone (1935) stated that number facility is one of the seven components of which human intelligence is comprised; Wechsler (1939) included mental arithmetic problems in his widely used IQ tests. There is some evidence that fluid intelligence, defined as general reasoning and problem-solving abilities independent from specific knowledge and culture, is positively correlated with the ability to solve realistic mathematical word problems (Xin and Zhang 2009). However, since mathematical ability stretches far beyond number sense and successful encountering of arithmetic or word problems, we cannot construe on the basis of existing data that intelligence and mathematical ability are mutually related.

Multidimensional theories of intelligence offer a different view on this issue. Gardner, in his seminal work first presented in his book "Frames of Mind" in 1983, suggested that there are several distinct intelligences, one of which is the logical-mathematical intelligence. Gardner argued that traditional models of intelligence, such as Terman's, combine together human capacities that do not necessarily correlate with one another. Thus, a person with high mathematical abilities, as described, for instance, by Krutetskii, will be defined by Gardner's Multiple Intelligences theory as having high logical-mathematical intelligence; this definition does not necessarily imply that this person's score in a conventional IQ test will be superior.

### Measuring and Evaluating Students' Mathematical Ability

Following the above, it became clear to researchers that a standard IQ test is not an appropriate tool for evaluating the mathematical ability of students, especially for the purpose of identifying extremely able ones (Carter and Contos 1987). Instead, one of the most prevalent means for this purpose is known as *aptitude tests*. Aptitude tests are aimed at measuring a specific ability or talent and are often used to predict the likelihood of success in certain areas or occupations (e.g., foreign language learning, military service, or, in this case, mathematics). Among the many existing aptitude tests, a widely known one is the SAT (an acronym which originally stood for Scholastic Aptitude Test), designed by the College Board in USA for predicting academic success. The SAT includes three parts, one of which is the SAT-M, referring to mathematics. Julian Stanley, founder of SMPY (the Study of Mathematically Precocious Youth) at Johns Hopkins University, found that SAT-M is an efficient means for identifying mathematically gifted students at junior high school age (Stanley et al. 1974). However, the use of aptitude tests like SAT-M for the purpose of measuring mathematical ability was criticized by several scholars as inadequate. For instance, Lester and Schroeder (1983) claimed that multiple-choice, standardized tests, such as SAT-M, provide no information about students' ability to solve nonroutine mathematical problems, and moreover, they cannot reveal the nature and quality of students' mathematical reasoning. These tests focus on a narrow interpretation of mathematical ability, ignoring important problem-solving behaviors that are indicative of this ability. Krutetskii (1976) attacked the credibility of psychometric items for measuring mathematical ability, claiming that (a) a single assessment event is highly affected by the subject's anxiety or fatigue, (b) training and exercise influence the rate of success, and (c) psychometric means concentrate on quantitative rather than qualitative aspects of mathematical ability, i.e., they focus on final outcomes instead of thinking processes, thus missing the central meaning of this construct. Despite criticisms, the current predominant method for assessing students' mathematical ability is still different versions of multiple-choice aptitude tests, most likely due to considerations of time and budget resources. Nevertheless, efforts are being conducted to develop low-cost assessment tools that follow the qualitative approach characteristic of the work of Krutetskii and others (e.g., Vilkomir and O'Donoghue 2009).

## Cross-References


- ▶ Competency Frameworks in Mathematics Education
- ▶ External Assessment in Mathematics Education
- ▶ Giftedness and High Ability in Mathematics
- ▶ Learning Difficulties, Special Needs, and Mathematics Learning
- ▶ Zone of Proximal Development in Mathematics Education


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# Mathematical Approaches

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## Keywords

Contributing discipline; Didactical analysis; Didactic transposition; Mathematical analysis; Mathematical approach; Mathematics education community; Mathematical structure

## Introduction

Research in mathematics education is interdisciplinary. According to Higginson (1980), mathematics, philosophy, psychology, and sociology are *contributing disciplines* to mathematics education (similar to what Michael Otte called *Bezugsdisziplinen*; Otte et al. 1974, p. 20). Linguistics and semiotics could be added. Framing of research, by means of theories or methods from these, amounts to different *approaches*, mathematics itself being one obvious choice. According to one view, mathematics education as a research field belongs to mathematics: at the second International Congress on Mathematical Education (ICME) in Exeter, Zofia Krygowska suggested that mathematics education should be classified as "a part of mathematics with a status similar to that of analysis or topology" (Howson 1973, p. 48). Another view sees mathematics education as an autonomous science (*didactics of mathematics* as Hans Georg Steiner in 1968 called the new discipline he wanted to establish; see Furinghetti et al. 2008, p. 132), strongly linked to mathematics, as expressed at ICME1 in Lyon 1969: "The theory of mathematical education is becoming a science in its own right, with its own problems both of mathematical and pedagogical content. The new science should be given a place in the mathematical departments of Universities or Research Institutes, with appropriate qualifications available" (quoted in Furinghetti et al. 2008, p. 132). However, in many countries, mathematics education research has an institutional placement mainly in educational departments.

## Definition

*Mathematical approaches* in mathematics education take the characteristics and inner structures of mathematics as a discipline (i.e., the logic of the subject) as its main reference point in curriculum and research studies. These characteristics, however, might be questioned. Studies include philosophical, historical, and didactical analyses of mathematical content and of how it is selected, adapted, or transformed in the process of recontextualization by requirements due to educational constraints, as well as the consequences entailed by these transformations on didactic decisions and processes.

## Developments

The field of mathematics education research historically emerged from the scientific disciplines of mathematics and of psychology (Kilpatrick 1992). On an international level, through the activities promoted by ICMI (International Commission on Mathematical Instruction) during the first half of the twentieth century, with their focus on comparing issues of mathematical content in curricula from different parts of the world, with little consideration of research on teaching and learning (Kilpatrick 1992), the approach to secondary and tertiary mathematics education was predominantly mathematical. During the same period, however, in primary mathematics education, the approaches were commonly psychologically or philosophically oriented. Independently, the use of concrete materials in schools is widely developed (Furinghetti et al. 2013). While this situation led to a decrease of ICMI's influence on mathematics education, through and after the *New Math* movement in the 1960s, ICMI regained its voice with support of OEEC/OECD, UNESCO, and through the collaboration of mathematicians with mathematics educators, mainly through CIEAEM, concerned with the full complexity of teaching and learning at all school levels (Furinghetti et al. 2008). The mathematical approach underpinning the reform was warranted not only by the aim to update curricula with modern developments in mathematics but also by Piagetian psychology pointing to "similarities" between mental and mathematical structures (Furinghetti et al. 2008). The aim of the New Math to be a *mathematics for all* was counteracted by its emphasis on general mathematical structures and fundamental concepts. This type of mathematical approach was strongly criticized, most notably by Hans Freudenthal who used the term *anti-didactic inversion* for a static axiomatic ready-made version of mathematics presented to students. An influential similar critique was offered by René Thom (1973, p. 202), who suggested that mathematics education should be founded on meaning rather than rigor.

The eventual failure of the New Math pointed to the need of establishing mathematics education as a discipline "in its own rights" and a wider scope for the work of ICMI. In retrospective, the first ICME congress in 1969 can be said to mark the creation of an autonomous mathematics education community (during a period when several institutions and journals specialized in mathematics education were founded; see, e.g., Furinghetti et al. 2013) and a loosening of the strong link to the community of mathematicians with implications for the "status" of mathematical approaches. With this wider scope, besides mathematical and psychological approaches, a variety of approaches for the study of phenomena within the field was needed, especially with reference to social dimensions.

This development highlights different interpretations of *mathematical approach*. While the New Math was the outcome of a deliberate and research-based program prepared in collaboration, the type of "mathematical approaches" of later movements in the USA, such as *Back to Basics* in the 1970s and even more so the *Math Wars* in the 1990s, is better described as ideologically based reactions to what was seen by some individuals and interest groups as *fuzzy* mathematics. The return to the skill-oriented curriculum advocated failed to take into account not only reported high dropout rates and research showing how it disadvantages underprivileged social groups but also research that highlights the complexity in teaching and learning processes (Goldin 2003; Schoenfeld 2004). In the more research-oriented mathematical approaches that developed in Europe during the same period, it was shown how both the character and learning of mathematics at school are institutionally conditioned.

## Characteristics

The following quote gives an argument for taking a mathematical approach to research: "The mathematical science in its real development is therefore the central focus of the mathematics educators, because the separation of creative activity and learning – taking into account the fundamental difference between research and learning – is unfruitful and does not allow to adequately capture the learning nor to properly guide the learning process" (auth. transl., Jahnke et al. 1974, p. 5). To develop mathematical knowledge, the learner must engage in creative mathematical activities. Another rationale for a focus on mathematics itself in didactical research draws on the observation that mathematics "lives" differently in different institutions and is transformed (recontextualized) when moved. In a mathematics classroom, different ideologies influence what kind of mathematical knowledge is proposed as legitimate, requiring from both, the teacher and the researcher, an awareness of the structure of the knowledge produced. The often cited claim by René Thom (1973, p. 204) that "whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics," also applies to research in mathematics education. This can be seen as an argument for the necessity of keeping an awareness of how mathematics is viewed in all approaches to research in mathematical education.

In the following, some examples of theorizing in the field of mathematics education that employ a mathematical approach will be discussed, with a focus on the role given to mathematics as a main point of reference.

## Stoffdidaktik

The *Stoffdidaktik* (subject matter didactics, content-oriented analysis) tradition in German-speaking countries, originally with main focus on secondary school mathematics and teacher education, has its modern roots in the efforts by Felix Klein, during the first decades of the twentieth century, to structure *elementary mathematics from an advanced standpoint* and to include lectures on the didactics of mathematics in the education of future teachers. While his classic book (Klein 1908) served the aim to teach (future) teachers to think mathematically, the aim of the lectures was to teach (future) teachers to think didactically (Vollrath 1994).

According to Tietze (1994), "Stoffdidaktik mainly deals with the subject matter under the aspects of mathematical analysis and of transforming mathematical theories into school mathematics" (p. 42). This approach in mathematics education operates through an explicit didactic transposition of (academic) mathematics for the purpose of making it accessible to students at specific educational levels. Some key principles used in this process, constituted by a mathematical analysis and selection of the content to be taught, are *elementarizing*, *exactifying*, *simplifying*, and *visualizing* (Tietze 1994). Students' problems to cope with, for example, definitions in mathematics, are in this approach seen as based in the complex logical structure of the definitions, which then must be analyzed by way of these principles in order to prepare their teaching.

An example of such analyses is Padberg's (1995) work on fractions, a textbook for teacher education outlining four central aspects (*Größenkonzept*, *Äquivalenzklassenkonzept*, *Gleichungskonzept*, *Operatorkonzept*) and two basic ideas (*Grundvorstellungen*; see below), elaborating on accessible metaphorical descriptions of the concepts but also including a chapter on the mathematical foundation of fractions, presenting an axiomatic characterization of the topic aimed to provide background knowledge for the teacher. Such *mathematical background theories* in mathematics education have commonly been introduced and used within *Stoffdidaktik*. For geometry, Vollrath (1988, pp. 121–127) identifies five (historical) phases of background theories: Euclid's elements (from early times, perfected by Hilbert), transformation geometry (from the early 1800s; e.g., Möbius, later Klein), different axiomatic theories as competing background theories (from early 1900s), an axiomatic theory developed by didacticians from practice of teaching (from 1960s, to decrease the gap from theoretical mathematics to teaching practice; e.g., Steiner 1966), and "The totality of geometric knowledge, including the ideas, connections, applications, and evaluations." As an early example of this kind of mathematical approach, Steiner (1969) outlines *a mathematical analysis* of the relation of rational numbers to measurement and interpretation as operators, with the aim to characterize possibilities for teaching. He calls his procedure a *didactical analysis* (p. 371).

A specific focus for the transposition work is on so-called *fundamental ideas* (*Fundamentale Ideen*). According to Schwill (1993), for an idea to be fundamental, it must appear within different topics of mathematics (*Horizontalkriterium*) and at different levels of the curriculum (*Vertikalkriterium*), be recovered in the historical development of mathematics (*Zeitkriterium*), and be anchored in everyday life activities (*Sinnkriterium*). Using the term *universal ideas*, Schreiber (1983) in a similar vein presents the requirements of *comprehensiveness*, *profusion*, and *meaningfulness*. As an example, Riemann integration is not a fundamental idea but a specific application of the fundamental idea of exhaustion. Another example is *reversibility*. Historically, already Whitehead (1913) suggested that school mathematics should emphasize main universally significant general ideas rather than drown in details that may not lead to access to big ideas or provide necessary connections to everyday knowledge. In line with this and with explicit reference to Jerome Bruner's principle that teaching should be oriented towards the structure of science, much work in *Stoffdidaktik* consist of analyses of fundamental ideas in different areas of mathematics. For the teaching of fundamental ideas, Schwill (1993) suggests Bruner's *spiral principle* to be used, in terms of *extendibility*, *prefiguration of notions*, and *anticipated learning*. It still remains unclear; however, at what level of abstraction, fundamental ideas are located.

As basis for teaching a mathematical concept, meta-knowledge about the concept is seen as necessary and has to be addressed in teacher education. A theory of concept teaching (e.g., Vollrath 1984) needs to build on the evaluation of mathematical concepts and their hierarchical structure, their historical development, and the *principle of complementarity* (Otte and Steinbring 1977) that concepts should offer both knowledge and use.

Research methods of early work within *Stoffdidaktik* were mainly the same as those of mathematics (Griesel 1974). In Griesel (1969), for example, an axiomatically based mathematical theory for a system of quantities is outlined. It has been pointed out by Griesel, however, that without also empirically investigating the outcomes from such analyses in teaching and learning, the analytical work would not be justified. *Stoffdidaktik* later widened to consider not only academic mathematics along with its epistemology and history but also factors relating to the learner of mathematics. In this context the notion of *Grundvorstellungen* became widely used (e.g., vom Hofe 1995), that is, the basic meanings and representations students should develop about mathematical concepts and their use within and outside mathematics. Conceptualized both as mental objects and as a prescriptive didactical constructs for prototypical metaphorical situations, the epistemological status of *Grundvorstellungen* remains debated.

Outside German-speaking countries, mathematics-oriented didactical research has dominated mathematics education, for instance, in the Baltic countries (Lepik 2009). One example of a non-European work employing the approach is Carraher (1993), where a ratio and operator model of rational numbers is developed. There are also regional and international periodic journals for teachers, mathematicians, and mathematics educators that publish mathematical and didactical analyses of elementary topics for school and undergraduate mathematics.

## An Epistemological Program

Mathematics also serves as a basic reference point for the "French school" in mathematics education research referred to as an *epistemological program* (Gascon 2003), including the theory of didactical situations (TDS) developed by Guy Brousseau and the anthropological theory of didactics (ATD) developed by Yves Chevallard. What constitutes mathematical knowledge is here seen as relative to the institution where it is practiced and thus, in research, needs to be questioned regarding its structure and content as practiced. In studies of the diffusion of mathematical knowledge within an institution, it is therefore necessary for the researcher to construct a *reference epistemological model* of the corresponding body of mathematical knowledge (Bosch and Gascon 2006), in order to avoid a bias of the institution studied.

Brousseau (1997) proposes *didactical situations* as epistemological models of mathematical knowledge, both for setting up the target knowledge and for developing it in classroom activity. For the researcher, such models are employed mainly for the analysis of didactical phenomena emerging in the process of instruction. They are also used for *didactical engineering* (e.g., Artigue 1994), where they are analyzed in terms of possible constraints of epistemological, cognitive, or didactical nature (Artigue 1994, p. 32). By investigating the historical development of the mathematical knowledge at issue, as well as its current use, the epistemological constraints can be analyzed. In particular the functionality of the knowledge to be taught is seen as a key component of a *fundamental didactical situation*, constituting a milieu that promotes the student's use of the knowledge. An idea is here to "restore" the epistemological conditions that were at hand where the knowledge originated but have disappeared in curriculum processes such as decontextualization and sequentialization of knowledge.

In ATD mathematics is seen as a human activity within institutions (as social organizations), with collective practices that form how the participants think and define their goals. It includes a focus on how mathematical knowledge, having a preexistence outside the educational institution, is *transposed* by institutional constraints when moved into it. The structure of the mathematical knowledge and work is modeled by *praxeologies* (or mathematical organizations) that provide a holistic description of the relations between different aspects of the institutional mathematical practice, in terms of types of tasks and techniques for dealing with these tasks, and those technologies and overall theoretical structures that justify the practice. In didactical research, the characteristics of praxeologies are analyzed in terms of aspects, such as connectedness and levels of generality, and issues linked to the *didactic transposition*, in order to identify possible constraints that are being imposed on students' knowledge development. According to ATD, "phenomena of didactic transposition are at the very core of any didactic problem" (Bosch and Gascon 2006, p. 58). To develop a target mathematical praxeology for classroom teaching, a didactical praxeology needs to be set up. Here one finds a strong emphasis on the functionality of the mathematical knowledge studied (its raison d'être), to avoid a *monumentalistic* noncritical selection of traditional school mathematics topics, often described as alien to the reality of the students.

## Realistic Mathematics Education

*Realistic mathematics education* (RME) views mathematics as an emerging activity: "The learner should reinvent mathematising rather than mathematics, abstracting rather than abstractions, schematising rather than schemes, algorithmising rather than algorithms, verbalising rather than language" (Freudenthal 1991, p X). While keeping mathematics as a main reference point, researchers within RME take on didactical, phenomenological, epistemological, and historical-cultural analyses as bases for curricular design (see ▶ Didactical Phenomenology (Freudenthal)). Activities of *horizontal mathematization* aim to link mathematical concepts and methods to real situations, while *vertical mathematization* takes place entirely within mathematics. An example of work within RME employing a strong mathematical approach is found in Freudenthal (1983), with its elaborated analyses of mathematical concepts and methods and efforts to root the meanings of those mathematical structures in everyday experiences and language.

## Mathematical Knowledge for Teaching

Empirical quantitative research on the amount of mathematical studies needed for a successful or effective teaching of mathematics at different school levels has not been able to settle the issue. Rather, the character of teachers' knowledge and the overall approach to teaching seem to matter more (Ma 1999; Boaler 2002; Hill et al. 2005). With reference to the distinction between subject matter knowledge and *pedagogical content knowledge* (PCK), during the last decades, descriptions and measurements of what has been named *mathematical knowledge for teaching* (MKT) for use in preservice and in-service teacher education have been developed. This mathematically based approach to mathematics education sets out to characterize the mathematical knowledge that teachers need to effectively teach mathematics and to investigate relations between teaching and learning. MKT stays close to the PCK construct while applying and further detailing the latter in order to grasp the specificities of school mathematics. The approach has much in common with the didactical analyses of mathematical content developed much earlier within *Stoffdidaktik*, though with more focus on primary mathematics. However, as the approach is less systematic and without reference to different possible mathematical background theories, the level of analysis remains unclear. The scope of the empirical research includes efforts to both develop and measure MKT for groups of teachers and its relation to student achievement (e.g., Hill et al. 2005).

## Some Further Aspects of Mathematical Approaches

In university mathematics, educational issues identified in beginning courses (such as calculus and linear algebra), especially in the context of the transition from secondary school to university, have commonly been addressed by a mathematical approach by ways of analyses of mathematical structures and processes in the courses. However, in line with the widened scope of mathematics education research since the time of New Math, de Guzman et al. (1998) suggest epistemological and cognitive, sociological and cultural, as well as didactical approaches to study the transition problem. Beside cognitivistic (still constituting the dominating approach), sociological, and discursive approaches, today more recent mathematical approaches (such as the epistemological program) are common for investigating university mathematics education (see, e.g., Artigue et al. 2007).

The importance and relevance of the history of mathematics for mathematics education has long been emphasized in the mathematics education community (e.g., the report from the ICME working group on history in Athen and Kunle 1976, pp. 303–307). In this context, both the didactical analyses of the historical material and the ways of using these in teaching practice often employ a mathematical approach. The claim of a parallel between the historical development and individual learning of mathematical concepts (the *phylogeny-ontogeny parallel*) has been one of the arguments for this approach, while others relate the use of history to motivational and cultural-historical issues or introduce historical outlines as a tool for teaching mathematics (Athen and Kunle 1976).

The examples of theoretical perspectives presented above employ different mathematical approaches to mathematics education as an overarching approach in the research. However, also within other approaches (psychological, social, etc.), mathematical aspects often come into focus. As an example, the APOS framework (e.g., Cottrill et al. 1996) takes a psychological approach to model and study the development of students' conceptual knowledge. However, as a basis for the construction of a *genetic decomposition* of the taught mathematical concept, a mathematical analysis of its structure and historical development is undertaken.

There are several influential mathematics educators whose work cannot be subsumed under the theoretical perspectives considered above, but who have sought to understand and improve mathematics instruction by means of analyzing mathematical processes and structures, often with a focus on developing teaching aids and didactical suggestions. Emma Castelnuovo, Zoltan Dienes, Caleb Gattegno, and George Polya, among others, could be mentioned here.

## Unresolved Issues

The community of mathematics education tends to become disintegrated by its diversity of theoretical approaches used in research with a knowledge structure fragmented into what Jablonka and Bergsten (2010) call *branches*. If mathematics education research strives to enhance the understanding of mathematics teaching and learning, including its social, political, and economic conditions and consequences, only a productive interaction of research approaches is likely to move the field forward. Unresolved issues are often due to institutionalized separation of researchers taking distinct approaches, as, for example, epitomized in bemoaning a loss of the focus on mathematics, which need to be resolved through theory (for theory networking, see Prediger et al. 2008). This would, for example, include integrating approaches that focus on mathematical knowledge structures with discursive and sociological approaches. Further, for producing unbiased policy advice, it is necessary to integrate research outcomes on students' and teachers' engagement with mathematics, including cognitive, emotional, language-related, and social dimensions of teaching and learning in classrooms. Such work has been attempted in a range of initiatives and working groups, as, for example, at the conferences of ERME (Prediger et al. 2008). In discussions of goals of mathematics education, mathematical approaches

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combined with sociological theorizing become pertinent to analyses of the use and exchange values of (school) mathematics for students.

## Cross-References

- ► Anthropological Approaches in Mathematics Education, French Perspectives
- ► Critical Thinking in Mathematics Education
- ► Didactic Situations in Mathematics Education
- ► Didactic Transposition in Mathematics Education
- ► Didactical Phenomenology (Freudenthal)
- ► Mathematical Knowledge for Teaching
- ► Pedagogical Content Knowledge in Mathematics Education
- ► Psychological Approaches in Mathematics Education
- ► Recontextualization in Mathematics Education
- ► Stoffdidaktik in Mathematics Education

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# Mathematical Games in Learning and Teaching

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## Keywords

Games; Computer games; Visualization; Motivation; Programming; Learner-centered teaching

## Definition

Literature examining the contribution of mathematical games in the learning and teaching of mathematics.

## Characteristics

Piaget, Bruner, and Dienes suggest that games have a very important part to play in the learning of mathematics (Ernest 1986). In the last four decades, games have been proposed by a number of researchers as a potential learning tool in the mathematics classroom, and there are quite a few researchers who make claims about their efficacy in the learning and teaching of mathematics (e.g., Ernest 1986; Gee 2007; Kafai 1995). Some authors take a step further; Papert (1980) was among the first who suggested that students could learn mathematics effectively not only by playing (video) games but also by designing their own computer games, using, for instance, authoring programming tools like Scratch and ToonTalk (Kafai 1995; Mousoulides and Philippou 2005).

By synthesizing definitions by Harvey and Bright (1985, p. ii) and Oldfield (1991, p. 41), a task or activity can be defined as a pedagogical appropriate mathematical game when it meets the following criteria: has specific mathematical cognitive objectives; students use mathematical knowledge to achieve content-specific goals and outcomes in order to win the game; is enjoyable and with potential to engage students; is governed by a definite set of rules and has a clear underlying structure; involves a challenge against either a task or an opponent(s) and interactivity between opponents; includes elements of knowledge, skills, strategy, and luck; and has a specific objective and a distinct finishing point.

While mathematical games have been the core of discussion of researchers since the late 1960s (e.g., Gardner 1970), the inclusion of games for the teaching and learning of school mathematics, among other subject areas, has been in the core of discussion in the 1990s (Provenzo 1991). An example of this perspective appears in Lim-Teo's (1991) work, who claimed that "there is certainly a place for games in the teaching of Mathematics … teacher to creatively modify and use games to enhance the effective teaching of Mathematics" (p. 53). At the same time, Ernest (1986) raised a question that is still cutting: "Can mathematics be taught effectively by using games?" (p. 3).

The answer to Ernest's question is not easy yet straightforward. The main pedagogical aim of using games in mathematics classrooms is to enhance the learning and teaching of mathematics through developing students' mathematical knowledge, including spatial reasoning, mathematical abstraction, higher level thinking, decision making, and problem solving (Ernest 1986; Bragg 2012). Further, mathematical games help the teaching and learning of mathematics through the advantage of providing meaningful situations to students and by increasing learning (independent and at different levels) through rich interaction between players. There are positive results, suggesting that the appropriate mathematics games might improve mathematics achievement. A meta-analysis conducted by Vogel and colleagues (2006) concluded that mathematical games appear to be more effective than other instructional approaches on students' cognitive developments. The positive impact of mathematical games is further enhanced by technology. Digital mathematical games provide, for instance, a powerful environment for visualization of difficult mathematical concepts, linkage between different representations, and direct manipulation of mathematical objects (Presmeg 2006). However, Vogel et al. (2006), among others, exemplify that the positive relation between mathematics games and higher achievement is not the case in all studies that have been conducted in the field.

Games for learning mathematics are also beneficial for a number of other, frequently-cited, arguments, including benefits like students' motivation, active engagement and discussion (Skemp 1993), improved attitudes towards mathematics and social skills, learning and understanding of complex problem solving, and collaboration and teamwork among learners (Kaptelin and Cole 2002). Among these benefits of using mathematical games, the most cited one is active engagement. Papert (1980) expressed the opinion that learning happens best when students are engaged in demanding and challenging activities. In line with Papert, Ernest (1986) claimed that the nature of games demands children's active involvement, "making them more receptive to learning, and of course increasing their motivation" (p. 3). Various studies in both digital and non-digital mathematical games have shown that students are highly engaged with working in a game environment and that this milieu creates an appropriate venue for teaching and learning mathematics (e.g., Devlin 2011).

Research has highlighted various factors that should be taken into consideration as to acknowledge mathematical games as an appropriate and successful vehicle for the learning and teaching of mathematics. Games should not be faced in isolation of broader mathematical programs and approaches. Clear instructional objectives and pedagogies have to accompany the use of games, while at the same time these pedagogies should consider peer interaction, teacher-facilitator role, the access to and the use of technological tools, and the use of rich problem-solving contexts.

## Cross-References


- ▶ Learner-Centered Teaching in Mathematics Education
- ▶ Motivation in Mathematics Learning

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# Mathematical Knowledge for Teaching

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## Keywords

Mathematical knowledge for teaching; Professional knowledge; Lee Shulman; Deborah Ball

## Characteristics

It was Shulman's Presidential Address at the 1985 annual meeting of the American Educational Research Association, and its publication the following year, that placed content knowledge in and for teaching firmly on the educational research, policy, and practice agenda. Shulman developed this focus from a critique of research on teaching at the time, arguing that attention is needed as much to "the content aspects of teaching as we have recently devoted to the elements of teaching process" (1986, p. 8) and elaborated three components that in concert comprise the professional knowledge base of teaching: subject matter knowledge (SMK), pedagogic content knowledge (PCK), and curriculum knowledge.

Shulman's description of a teacher's SMK marked out its specificity:

> We expect that the subject matter content understanding of the teacher be at least equal to that of his or her lay colleague, the mere subject matter major. The teacher need not only understand that something is so; the teacher must further understand why it is so . . . (and) why a given topic is particularly central to a discipline. . . (Shulman 1986, 1987 p. 9)

and its substantive and syntactic structures:

> The substantive structures are the variety of ways in which the basic concepts and principles of the discipline are organized to incorporate its facts. The syntactic structure of a discipline is the set of ways in which truth or falsehood, validity or

> invalidity, are established... Teachers must not only be capable of defining for students the accepted truths in a domain. They must also be able to explain why a particular proposition is deemed warranted . . . how it relates to other propositions, both within the discipline and without, both in theory and in practice. (P. 9)
>
> SMK was also distinct from PCK, "a form of content knowledge that embodies the aspects of content most germane to its teachability" (p. 9).

The past 20 years has seen ongoing research into mathematical knowledge for teaching (MKT), often focused on both SMK and PCK. In recognition of the deep connection between these two elements of the professional knowledge base for teaching and contestation over the boundary between them, there are separate but related entries on SMK and PCK in this encyclopedia. Work on SMK has focused on defining and theorizing the nature of this specialized knowledge, its measurement, its significance for pedagogy, and the implications for teacher education.

## Defining, Theorizing, and Measuring SMK

Deborah Ball and colleagues at the University of Michigan engaged in detailed studies of teaching practice with the goal of developing Shulman's work empirically, analytically, and theoretically and with the desire to understand how components teachers' professional knowledge are associated with student achievement gains. They focused simultaneously on defining distinct forms of MKT and on developing related measures. They distinguish common, specialized, and horizon content knowledge as forms of SMK and knowledge of mathematics and students, mathematics and teaching, and mathematics and curriculum as forms of PCK (Ball et al. 2008). They described specialized content knowledge (SCK) as mathematical knowledge that is unique to the work of teaching and distinct from the common content knowledge (CCK) which is needed and used by teachers and non-teachers alike.

Multiple choice items developed as measures for each of these components of MKT were administered to elementary teachers, together with data on the learning gains across a year of children in these teachers' classes. Their analysis showed significant associations between teachers' content knowledge measures (their CCK and SCK) and students' learning gains (Hill et al. 2005). In a following study comprising five case studies of teaching and associated quantitative data, Hill et al. (2008) also report a strong positive association between levels of MKT and the mathematical quality of instruction, but note a range of factors (e.g., use of curriculum texts) that mediate this association.

Similarly driven by the desire to understand the significance of teachers' mathematical knowledge for teaching and learning, the Professional Competence of Teachers, Cognitively Activating Instruction, and Development of Students' Mathematical Literacy (COACTIV) project in Germany developed open response measures of SMK (which they called content knowledge – CK) and PCK at the secondary level and used these together with data from varied records of practice of a large sample of grade 10 teachers and their students' progress (Baumert et al. 2010). COACTIV studies succeeded in distinguishing CK from PCK in secondary mathematics, conceptually and empirically. They validated their measures by testing them on groups of teachers that differed in their mathematical and teacher training, and through this confirmed the dependence of growth in teachers' PCK on the levels of CK, and the determining effects of CK acquired in initial training. At the same time, Baumert et al. (2010, p. 133) showed a substantial positive effect of teachers' PCK on students' learning gains, mediated by their pedagogic practice. In other words, while PCK was inconceivable without sufficient CK, CK cannot substitute for PCK.

In COACTIV, content knowledge (CK) is defined as "deep" or "profound" understanding of the mathematics taught in the secondary school and associated with the Klein's (1933) idea of "elementary mathematics from a higher viewpoint," on the one hand, and to Ma's (1999) notion of Profound Understanding of Mathematics on the other. CK is also distinguished from other notions of "content knowledge," including the everyday mathematical knowledge of adults, and described as lying between school land university-level mathematics. PCK in COACTIV is defined to include knowledge of tasks (including multiple representations of mathematical concepts), of students (e.g., typical errors), and of instruction. For Ball et al., knowledge of representing mathematics would align more with specialized content knowledge. Hence, difficulties with the boundary between SMK and PCK as distinct elements of MKT, noted by many others in the field, become evident.

Together with Shulman (1986, 1987), these two major studies have nevertheless evidenced a form of mathematical knowledge that is unique to teaching, that sits outside of, or between, school mathematics and university mathematics, with significant implications for the mathematical content preparation of teachers in teacher education.

## SMK, Pedagogy, and Teacher Education

Numerous other studies, particularly at the elementary level, based on examination of teachers' knowledge in use in practice, have resulted in different categorizations and descriptions. Turner and Rowland, in Rowland and Ruthven (2011), developed the Knowledge Quartet (KQ), comprising foundation, transformation, connection, and contingency knowledge. Foundation knowledge aligns with SMK, and the other three with PCK: the KQ serves as a useful tool for reflection with and by teachers on the content of their teaching. Through her study of US and Chinese teachers' responses to mathematical tasks situated in the context of teaching, Ma (1999) described the flexibility, depth, and coherence of the knowledge displayed by the Chinese teachers. She identified four key components of this "profound understanding of mathematics," connectedness, multiple perspectives, basic ideas, and longitudinal coherence, that together constitute a "package" of knowledge that was "deep, broad and thorough" (p. 122–123). Adler and Ball (2009), in their overview of a range of studies concerned with MKT, note the widening lexicon developing in the field. To this end, Petrou and Goulding, and Ruthven, both in Rowland and Ruthven (2011), provide useful overviews of texts on MKT.

All studies referred to above have episodes of mathematics teaching, or of a teacher engaged with a mathematics task of teaching as their unit of analysis. A different gaze from the perspective of the social production of knowledge moved the empirical site into mathematics teacher education and built descriptions of MKT from what is produced as mathematics in teacher education practice. For example, Adler and Davis, in Rowland and Ruthven (2011), illustrate how despite similar goals for deepening teachers' MKT opportunities for learning MKT in teacher upgrading programs in South Africa vary across contexts and practice, shaped in particular by different perspectives on knowledge and pedagogy.

Research on MKT with a focus on subject matter knowledge has thus evolved across empirical and cultural contexts and across levels of schooling and continues. A common conclusion can be drawn: if, as is now widely accepted, there is specialized knowledge that matters for practice, and initial or preservice education is paramount, then the inclusion of these forms of mathematical knowledge should not be left to chance and the context of practice, but become part of the content of professional training/education. Thus, not only is further research required with respect to precision and definition of MKT and its constitutive elements but also around the boundary around what constitutes mathematics in teacher preparation and professional development. We note with interest the explorations of MKT in teacher education, particularly in recent papers in the Journal of Mathematics Teacher Education.

## Cross-References

▶ Pedagogical Content Knowledge in Mathematics Education

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# Mathematical Language

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## Keywords

Algebraic notation; Communication; Genre; Language; Mathematical vocabulary; Multimodality; Objectification; Register; Representations; Semiotic systems

## Introduction: What Is Mathematical Language?

Specialized domains of activity generally have their own specialized vocabularies and ways of speaking and writing; consider, for example, the language used in the practices of law or computer science, fishing, or football. The specialized language enables participants to communicate efficiently about the objects peculiar to their practice and to get things done, though it may simultaneously serve to exclude other people who are not specialists in the domain. This is certainly the case for the specialized activity of mathematics: While some aspects of mathematical language, such as its high degree of abstraction, may be an obstacle to participation for some people, doing mathematics is highly dependent on using its specialized forms of language, not only to communicate with others but even to generate new mathematics. In making this claim, we need to be clearer about what mathematical language is.

For some, the language of mathematics is identified with its systems of formal notation. Certainly, like other languages, these systems include a "vocabulary" of symbols and grammatical rules governing the construction and manipulation of well-formed statements. A significant part of mathematical activity and communication can be achieved by forming and transforming sequences of such formal statements. In recent years, however, it has been widely recognized that not only other semiotic systems, including what is sometimes called "natural" language, but also specialized visual forms such as Cartesian graphs or geometric diagrams play an equally essential role in the doing and communicating of mathematics. This recognition has been strongly influenced by the work of the linguist Halliday and his notion of specialized languages or *registers* (Halliday 1974), by research applying and developing theories of semiotics in mathematics and mathematics education, and by more recent developments in multimodal semiotics that address the roles of multiple modes of communication (including gestures and the dynamic visual interactions afforded by new technologies). In this entry, it is not possible to provide a full characterization of all these aspects of mathematical language; in what follows, some of the most significant characteristics will be discussed.

## Characteristics of Mathematical Language

The most easily recognized aspect of the "natural" or verbal language component of mathematical language is the special vocabulary used to name mathematical objects and processes. This vocabulary was the focus of much of the early research conducted into language in mathematics education (see Austin and Howson 1979 for an overview of this research). This vocabulary includes not only some uniquely mathematical words (such as *hypotenuse*, *trigonometry*, and *parallelogram*) but, in addition, many words that are also used in everyday language, often with subtly different meanings. In English, words such as *prime*, *similar*, *multiply*, and *differentiate* originated in non-mathematical contexts and, in being adopted for mathematical use, have acquired new, more restrictive or precise definitions. The difficulties that learners may have in using such words in appropriately mathematical ways have been a focus of research; David Pimm's seminal book "Speaking Mathematically: Communication in Mathematics Classrooms" (Pimm 1987) provides a useful discussion of issues arising from this aspect of mathematical vocabulary. In national languages other than English, the specific relationships between mathematical and everyday vocabularies may vary, but similar issues for learners remain.

Another characteristic of mathematical vocabulary is the development of dense groups of words such as *lowest common denominator* or *topological vector space* or *integrate with respect to x*. Such expressions need to be understood as single units; understanding each word individually may not be sufficient. The formation of such lengthy locutions serves to pack large quantities of information into manageable units that may then be combined into statements with relatively simple grammatical structure. To consider a relatively simple example: if we wished to avoid using the complex locution *lowest common denominator*, the simple statement

*The lowest common denominator of these three fractions is 12.* would need to be unpacked into a grammatically more complex statement such as

*If we find fractions with different denominators equivalent to each of these three fractions, the lowest number that can be a denominator for all three of them is 12.*

The condensation of information achieved by complex locutions makes it possible to handle complex concepts in relatively simple ways. This is not unique to mathematics but is also a feature of the language of other scientific domains (Halliday and Martin 1993).

A further characteristic with a significant function in mathematics is the transformation of processes into objects; linguistically this is achieved by forming a noun (such as *rotation* or *equation*) out of a verb (*rotate* or *equate*). Like many of the special characteristics of mathematical language, this serves at least two functions that we may think of as relating to the nature of mathematical activity and to the ways in which human beings may relate to mathematics. In this case, by forming objects out of processes, the actors in the processes are obscured, contributing to an apparent absence of human agency in mathematical discourse. At the same time, however, changing processes (verbs) into objects (nouns) contributes to the construction of new mathematical objects that encapsulate the processes; the ability to think about ideas such as *function* both as a process and as an object that can itself be subject to other processes (e.g., addition or differentiation) is an important aspect of thinking mathematically. Sfard (2008) refers to these characteristics of mathematical language as *objectification* and *reification*, arguing that they both contribute to alienation – the distancing of human beings from mathematics. It is possible that alienation contributes to learners' difficulties in seeing themselves as potential active participants in mathematics. However, it is important to remember that many of the characteristics of mathematical language that seem to cause difficulties for learners are not arbitrary complexities but have important roles in enabling mathematical activity. Indeed, in Sfard's communicative theory of mathematical thinking, she makes no distinction between communicating and thinking: Thinking and doing mathematics are identified with participating in mathematical discourse, that is, communicating mathematically with others or with oneself.

## Variations in Language and Thinking Mathematically

Considering the relationship between language and thinking mathematically or doing mathematics also raises questions about the possible effects of using different national languages, especially those that do not share the structures and assumptions of the European languages that have dominated the development of modern academic mathematics. Even relatively simple linguistic differences, such as the ways in which number words are structured, have been argued to make a difference to children's learning of mathematics. Barton (2008) suggests that more substantial linguistic differences such as those found in some indigenous American or Australasian languages are related to different ways of thinking about the world that have the potential to lead to new forms of mathematics.

In focusing on features of verbal language, it is important not to forget the roles played by other semiotic systems in the doing and development of mathematics. A prime example to consider is the way in which Descartes' algebraization of geometry has transformed the development of the field. A powerful characteristic of algebraic notation is that it can be manipulated according to formal rules in order to form new statements that provide new insights and knowledge. In contrast, graphical forms tend not to allow this kind of manipulation, though they may instead enable a more holistic or dynamic comprehension of the objects represented. The different affordances for communication of verbal, algebraic, and graphical modes, analyzed in detail by O'Halloran (2005), mean that, even when dealing with the "same" mathematical object, different modes of communication will enable different kinds of messages. Consider, for example, which aspects you focus on and what actions you may perform when presented with a function expressed in verbal, algebraic, tabular, or graphical form.

Duval (2006) has argued that the differences between the affordances of different modes (which he calls registers) have an important consequence for learning: Converting from one mode to another (e.g., drawing the graph of a function given in algebraic form or determining the algebraic equation for a given graph) entails understanding and coordinating the mathematical structures of both modes and is hence an important activity for cognitive development. The design of environments involving making connections between different forms of representation has been a focus of researchers working with new technologies in mathematics education.

By speaking of mathematical language, as we have so far in this entry, it might seem that there is only one variety of mathematical language that has identical characteristics in all circumstances. This is clearly not the case; young children studying mathematics in the early years of schooling encounter and use specialized language in forms that are obviously different from the language of academic mathematicians. Even among academic mathematicians writing research papers, Burton and Morgan (2000) identified variation in the linguistic characteristics of publications, possibly relating to such variables as the status of the writers as well as to the specific field of mathematics. Researchers using discourse analytic approaches have studied the language used in a number of specific mathematical and mathematics education contexts. One way of thinking about the variation found across contexts is suggested by Mousley and Marks (1991): Different kinds of purpose in communicating mathematically demand the use of different forms of language or *genres*. Thus, for example, recounting what has been done in order to solve a problem will use language with different characteristics from that required in order to present a rigorous proof of a theorem. It may be that mathematical language should be thought of in terms of a cluster of forms of language with a family resemblance, differing in the extent to which they use the characteristics identified in this entry but sharing enough specialized features to enable us to recognize them all as mathematical. An important implication of recognizing the contextual variation in mathematical language is that research into the role of language in teaching and learning mathematics needs to be sensitive to the specificity of the practice being studied and cautious in its generalizations.

## Cross-References

- ▶ Bilingual/Multilingual Issues in Learning Mathematics
- ▶ Discourse Analytic Approaches in Mathematics Education
- ▶ Language Background in Mathematics Education
- ▶ Mathematical Language
- ▶ Semiotics in Mathematics Education

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# Mathematical Literacy

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## Keywords

Numeracy; Quantitative literacy; Critical mathematical literacy; Mathemacy; Matheracy; Statistical literacy

## Definition and Development

One of the first written occurrences of the term *mathematical literacy* was in 1944 in the USA, when a Commission of the National Council of Teachers of Mathematics (NCTM) on Post-War Plans (NCTM (1970/2002), p. 244) required that the school should ensure mathematical literacy for all who can possibly achieve it. Shortly after (in 1950), the term was used again in the Canadian Hope Report (NCTM (1970/2002), p. 401). In more recent times, the NCTM 1989 Standards (NCTM 1989, p. 5) spoke about mathematical literacy and mathematically literate students. Apparently, no definition of the term was offered in any of these texts. The 1989 Standards did, however, put forward five general goals serving the pursuit of mathematical literacy for all students: "(1) That they learn to value mathematics, (2) that they become confident with their ability to do mathematics, (3) that they become mathematical problem solvers, (4) that they learn to communicate mathematically, and (5) that they learn to reason mathematically" (op. cit., p.5). The IEA's Third International Mathematics and Science Study (TIMSS), first conducted in 1995, administered a *mathematics and science literacy* test to students in their final year of secondary school in 21 countries that aimed "to provide information about how prepared the overall population of school leavers in each country is to apply knowledge in mathematics and science to meet the challenges of life beyond school". The first attempt at an explicit definition appears to be found in the initial OECD framework for PISA (Programme for International Student Assessment) in 1999 (OECD 1999). The definition has been slightly altered a number of times for subsequent PISA cycles. The version for PISA 2012 reads (OECD 2010):

*Mathematical literacy* is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain, and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make well-founded judgments and decisions needed by constructive, engaged and reflective citizens.

In the mathematics education literature, one finds an array of related notions, such as *numeracy*, *quantitative literacy*, *critical mathematical literacy*, *mathemacy*, *matheracy*, as well as *statistical literacy*. While some of these concepts more clearly differ in extension and intension, some authors use "numeracy," "quantitative literacy," and "mathematical literacy" synonymously, whereas others distinguish also between these. While the term "mathematical literacy" seems to be of American descent, the term "numeracy" was coined in the UK. According to Brown et al. (1998, p. 363), it appeared for the first time in the so-called Crowther Report in 1959, meaning scientific literacy in a broad sense, and later obtained wide dissemination through the well-known Cockcroft Report (DES/WO 1982), which stated that its meaning had considerably narrowed by then. There have been further shifts in interpretation since then. A recent, rather wide, definition of "numeracy" can be found in OECD's PIAAC (Programme for the International Assessment of Adult Competencies) "numeracy" framework: "Numeracy is the knowledge and skills required to effectively manage and respond to the mathematical demands of diverse situations" (PIAAC Numeracy Expert Group 2009, p. 20). The term "quantitative literacy" is yet another term of American descent, going back to the work of Steen (see, e.g., Madison and Steen 2003).

Even though the notions above are interpreted differently by different authors (which suggests a need to pay serious attention to clear terminology), they do have in common that they stress awareness of the usefulness of and the ability to use mathematics in a range of different areas as an important goal of mathematics education. Furthermore, mathematical literacy and related notions are associated with education for the general public rather than with specialized academic training while at the same time stressing the connection between mathematical literacy and democratic participation. As in other combined phrases, such as "statistical literacy" or "computer literacy," the addition of "literacy" may suggest some level of critical understanding. In South Africa, the pursuit of mathematical literacy has motivated the introduction of a new stand-alone school mathematics subject area available for learners in grades 10–12, which aims at allowing "individuals to make sense of, participate in and contribute to the twenty-first century world – a world characterized by numbers, numerically based arguments and data represented and misrepresented in a number of different ways. Such competencies include the ability to reason, make decisions, solve problems, manage resources, interpret information, schedule events and use and apply technology" (DoBE 2011, p. 8). One motivation for introducing this mathematical subject was to increase student engagement with mathematics.

While "mathematical literacy," "quantitative literacy," and "numeracy" focus on mathematics as a tool for solving nonmathematical problems, the notions of *mathematical competence* (and *competencies*) and *mathematical proficiency* focus on what it means to master mathematics at large, including the capacity to solve mathematical as well as nonmathematical problems. The notion of "mathematical proficiency" (Kilpatrick et al. 2001) is meant to capture what successful mathematics learning means for everyone and is defined indirectly through five strands (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition). Furthermore, by referring to individuals' mental capacities, dispositions, and attitudes, the last two of these strands go beyond mastery of mathematics and include personal characteristics. The notion of "mathematical competence" has been developed, explored, and utilized in the Danish KOM Project (KOM is an abbreviation for "competencies and mathematics learning" in Danish) and elsewhere since the late 1990s (Niss and Højgaard 2011). Mathematical competence is an individual's capability and readiness to act appropriately, and in a knowledge-based manner, in situations and contexts in which mathematics actually plays or potentially could play a role. While mathematical competence is the overarching concept, its constituent components are, perhaps, the most important features. There are eight such constituents, called mathematical competencies: mathematical thinking, problem posing and solving, mathematical modeling, mathematical reasoning, handling mathematical representations, dealing with symbolism and formalism, communicating mathematically, and handling mathematical aids and tools. Mathematical competencies do not specifically focus on the learners of mathematics nor on mathematics teaching. Also, no personal characteristics such as capacities, dispositions, and attitudes are implicated in these notions.

## Motivations for Introducing Mathematical Literacy

There have always been endeavors amongst mathematics educators to go against the idea that the learning of basic or fundamental mathematics could be characterized solely in terms of facts and rules that have to be known (by rote) and procedures that have to be mastered (by rote). Mathematics educators have found this view reductionist, since it overlooks the importance of understanding when, and under what conditions, it is feasible to activate the knowledge and skills acquired, as well as the importance of flexibility in putting mathematics to use in novel intra- or extra-mathematical contexts and situations. For example, in the first IEA study on mathematics, which later became known as the First International Mathematics Study (FIMS), published in 1967, we read that in addition to testing factual and procedural knowledge and skills related to a set of mathematical topics, it was important to also look into five "cognitive behaviors": (1) *knowledge and information* (recall of definitions, notations, concepts), (2) *techniques and skills* (solutions), (3) *translation of data into symbols or schema* and vice versa, (4) *comprehension* (capacity to analyze problems and to follow reasoning), and (5) inventiveness (reasoning creatively in mathematics (our italics)). Another example is found in the NCTM document *An Agenda for Action: Recommendations for School Mathematics of the 1980s* (NCTM 1980). The document is partly written in reaction to the so-called "back-to-basics" movement in the USA in the 1970s, which in turn was a reaction to the "new mathematics" movement in the 1950s and 1960s. The document states:

> We recognize as valid and genuine the concern expressed by many segments of society that basic skills be part of the education of every child. However, the full scope of what is basic must include those things that are essential to meaningful and productive citizenship, both immediate and future (p. 5).

The document lists six recommendations, including:

> 2.1. The full scope of what is basic should contain at least the ten basic skill areas [. . .]. These areas are problem solving; applying mathematics in everyday situations; alertness to the reasonableness of results; estimation and approximation; appropriate computational skills; geometry; measurement; reading, interpreting, and constructing tables, charts, and graphs; using mathematics to predict; and computer literacy. (p. 6–7)
> 2.6 The higher-order mental processes of logical reasoning, information processing, and decision making should be considered basic to the application of mathematics. Mathematics curricula and teachers should set as objectives the development of logical processes, concepts, and language [. . .]. (p. 8)

These examples show that mathematics educators have been concerned with capturing "something more" (in addition to knowledge and skills regarding mathematical concepts, terms, conventions, rules, procedures, methods, theories, and results), which resembles what is indicated by the notion of mathematical literacy as it is, for example, used in the PISA. On the one hand, the arguments for broadening the scope of school mathematics have been utility oriented, based on the observation of students' lack of ability to use their mathematical knowledge for solving problems that are contextualized in extra-mathematical contexts, in school as well as out of school, an observation corroborated by a huge body of research. On the other hand, the constitution of mathematics as a school discipline in terms of "products" – concepts (definitions and terminology), results (theorems, methods, and algorithms), and techniques (for solving sets of similar tasks) – became challenged. Product-oriented curricula were complemented by, or contrasted with, a conception of mathematics that includes mathematical processes, such as heuristics for mathematical problem solving, mathematical argumentation, constructive and critical mathematical reasoning, and communicating mathematical matters.

There are different views about the amount of mathematical knowledge and basic skills needed for engagement in everyday practices and nonmathematically specialized professions, although it has been stressed that a certain level of proficiency in mathematics is necessary for developing mathematical literacy. The role of general mathematical competencies that transcend school mathematical subareas also has been stressed in the newer versions of conceptualizing mathematical literacy, most prominently in the versions promoted by the OECD-PISA (see above).

## Critique and Further Research

Even though the notion of mathematical literacy has gained momentum and is now widely invoked and used in various contexts, it has also encountered different sorts of conceptual and politico-educational criticism.

Some reservations against using the very term "mathematical literacy" concern the fact that it lacks counterparts in several languages. No suitable translation exists, for example, into German and Scandinavian languages, where there are only words for "illiteracy," which stands for the fundamental inability to read or write *any* text. Indeed, the term "literacy" (both mathematical and quantitative literacy) has been interpreted by some to connote the most basic and elementary aspects of arithmetic and mathematics, in the same way as linguistic literacy is often taken to mean the very ability to read and write, an ability that is seen to transcend the social contexts and associated values, in which reading and writing occurs. However, the demands for reading and writing substantially vary across a spectrum of texts and contexts, as do the social positions of the speakers or readers. The same is true for a range of contexts and situations in which mathematics is used. People's private, professional, social, occupational, political, and economic lives represent a multitude of different mathematical demands. So, today, for most mathematics educators, the term mathematical literacy signifies a competency far beyond a set of basic skills.

Another critique, going against attempts at capturing mathematical literacy in terms of transferable general competencies or process skills, consists in the observation that such a conception tends to ignore the interests and values involved in posing and solving particular problems by means of mathematics. Jablonka (2003) sees mathematical literacy as a socially and culturally embedded practice and argues that conceptions of mathematical literacy vary with respect to the culture and values of the stakeholders who promote it. Also, de Lange (2003) acknowledges the need to take into account cultural differences in conceptualizing mathematical literacy. There is no general agreement amongst mathematics educators as to the type of contexts with which a mathematically literate citizen will or should engage and to what ends. However, there is agreement that mathematical literate citizens include nonexperts and that mathematical literacy is based on knowledge that is/should be accessible to all.

In the same vein, mathematics educators have empirically and theoretically identified a variety of intentions for pursuing mathematical literacy. For example, Venkat and Graven (2007) investigated pedagogic practice and learners' experiences in the contexts of South African classrooms, in which the subject mathematical literacy is taught. They identified four different pedagogic agendas (related to different pedagogic challenges) that teachers pursued in teaching the subject. Jablonka (2003), through a review of literature, identifies five agendas on which conceptions of mathematical literacy are based. These are as follows: developing human capital (exemplified by the conception used in the OECD-PISA), maintaining cultural identity, pursuing social change, creating environmental awareness, and evaluating mathematical applications. Some terms have been introduced as alternatives to "mathematical literacy" in order to make the agenda visible. Frankenstein (e.g., 2010) uses *critical mathematical* numeracy, D'Ambrosio (2003) writes about *matheracy*, and Skovsmose (2002) refers to *mathemacy*. Relations of mathematical literacy to scientific and technological literacy have also been discussed (e.g., Keitel et al. 1993; Gellert and Jablonka 2007).

As to the role of mathematical literacy in assessment, discrepancies between actual assessment modes and the intentions of mathematical literacy have been pointed out by researchers in different contexts (Jahnke and Meyerhöfer 2007; North 2010). In the assessment literature, the contexts in which mathematically literate individuals are meant to engage are often referred to in vague or general terms, such as the "real-world," "everyday life," "personal life," "society," and attempts to categorize contexts often lack a theoretical foundation. Identifying the demands and knowledge bases for mathematically literate behavior in different contexts remains a major research agenda.

As far as the teaching of mathematical literacy is concerned, the transition between unspecialized context-based considerations and problem solutions that employ specialized mathematical knowledge is a continuing concern. Studies of curricula associated with teaching mathematics through and for exploring everyday practices, for example, have usefully drawn on theories of knowledge *recontextualization*.

These observations suggest that the meanings and usages associated with the notion of mathematical literacy and its relatives have not yet reached a stage of universally accepted conceptual clarification nor of general agreement about their place and role. Future theoretical and empirical research and development are needed for that to happen.

## Cross-References

▶ Competency Frameworks in Mathematics Education
▶ International Comparative Studies in Mathematics: An Overview
▶ Mathematical Modelling and Applications in Education
▶ Realistic Mathematics Education
▶ Recontextualization in Mathematics Education
▶ Word Problems in Mathematics Education

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# Mathematical Modelling and Applications in Education

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## Keywords

Modelling examples; Modelling cycle; Modelling competencies; Metacognition; Scaffolding

## Characteristics

The relevance of promoting applications and mathematical modelling in schools is currently consensus all over the world. The promotion of modelling competencies, i.e., the competencies to solve real-world problems using mathematics, is accepted as central goal for mathematics education worldwide, especially if mathematics education aims to promote responsible citizenship. In many national curricula, modelling competencies play a decisive role pointing out that the importance of mathematical modelling is accepted at a broad international level. However, beyond this consensus on the relevance of modelling, it is still disputed how to integrate mathematical modelling into the teaching and learning processes; various approaches are discussed and there is still a lack of strong empirical evidence on the effects of the integration of modelling examples into school practice.

### Theoretical Debate on Mathematical Modelling: Historical Development and Current State

Applications and modelling play an important role in the teaching and learning of mathematics; already in the nineteenth century, famous mathematics educator made a strong plea for the inclusion of contextual problems in mathematics education, mainly in elementary schools for the broad majority. At the turn to the twentieth century, Felix Klein – the first president of ICMI – laid out in the so-called syllabus from Meran the necessity to include applications in modelling in mathematics education for higher achieving children in grammar schools; however, he requested a strong balance between applications and pure mathematics. During and after the Second World War, applications lost significantly importance in many parts of the world. The claim to teach mathematics in application-oriented way has been put forth another time with the famous symposium "Why to teach mathematics so as to be useful" (Freudenthal 1968; Pollak 1968) which has been carried out in 1968. Why and how to include applications and modelling in mathematics education has been the focus of many research studies since then. This high amount of studies has not led to a unique picture on the relevance of applications and modelling in mathematics education; in contrast the arguments developed since then remained quite diverse. In addition the discussion, how to teach mathematics so as to be useful did not lead to a consistent argumentation. There have been several attempts to analyze the various theoretical approaches to teach mathematical modelling and applications and to clarify possible commonalities and differences; a few are described below.

Nearly twenty years ago, Kaiser-Meßmer (1986, p. 83) discriminated in her analysis of the applications and modelling discussion of that time various perspectives, namely, the following two main streams:

- A **pragmatic perspective**, focusing on utilitarian or pragmatic goals, i.e., the ability of learners to apply mathematics for the solution of practical problems. Henry Pollak (see, e.g., 1968) can be regarded as a prototypical researcher of this perspective.
- A **scientific-humanistic perspective**, which is oriented more towards mathematics as a science and humanistic ideals of education focusing on the ability of learners to create relations between mathematics and reality. The "early" Hans Freudenthal (see, e.g., 1973) might be viewed as a prototypical researcher of this approach.

The various perspectives of the discussion vary strongly due to their aims concerning application and modelling; for example, the following goals can be discriminated (Blum 1996; Kaiser-Meßmer 1986):

- **Pedagogical goals**: imparting abilities that enable students to understand central aspects of our world in a better way
- **Psychological goals**: fostering and enhancement of the motivation and attitude of learners towards mathematics and mathematics teaching
- **Subject-related goals**: structuring of learning processes, introduction of new mathematical concepts and methods including their illustration
- **Science-related goals**: imparting a realistic image of mathematics as science, giving insight into the overlapping of mathematical and extra-mathematical considerations of the historical development of mathematics

In their extensive survey on the state of the art, Blum and Niss (1991) focus a few years later on the arguments and goals for the inclusion of applications and modelling and discriminate five layers of arguments such as the formative argument related to the promotion of general competencies, critical competence argument, utility argument, picture of mathematics argument, and the promotion of mathematics learning argument. They make a strong plea for the promotion of three goals, namely, that students should be able to perform modelling processes, to acquire knowledge of existing models, and to critically analyze given examples of modelling processes.

Based on this position, they analyze the various approaches on how to consider applications and modelling in mathematics instruction and distinguish six different types of including applications and modelling in mathematics instruction, e.g., the *separation approach*, separating mathematics, and modelling in different courses or the *two-compartment approach* with a pure part and an applied part. A continuation of integrating applications and modelling into mathematics instruction is the *islands approach*, where small applied islands can be found within the pure course; the *mixing approach* is even stronger in fostering the integration of applications and modelling, i.e., newly developed mathematical concepts and methods are activated towards applications and modelling; whenever possible, however, in contrast to the next approach, the mathematics used is more or less given from the outset. In the *mathematics curriculum-integrated approach*, the problems come first and mathematics to deal with them is sought and developed subsequently. The most advanced approach, the *interdisciplinary-integrated approach*, operates with a full integration between mathematics and extra-mathematical activities where mathematics is not organized as separate subject.

At the beginning of the twenty-first century, Kaiser and Sriraman (2006) pointed out in their classification of the historical and more recent debate on mathematical modelling in school that several perspectives on mathematical modelling have been developed within the international discussion on mathematics education, partly new and different from the historical ones. Despite several commonalities, these strands of the discussion framed modelling and its pedagogical potential in different ways. In order to enhance the understanding of these different perspectives on modelling, Kaiser and Sriraman (2006) proposed a framework for the description of the various approaches, which classifies these conceptions according to the aims pursued with mathematical modelling, their epistemological background, and their relation to the initial perspectives.

The following perspectives were described, which continue positions already emphasized at the beginning of the modelling debate:

- *Realistic or applied modelling* fostering pragmatic-utilitarian goals and continuing traditions of the early pragmatically oriented approaches
- *Epistemological or theoretical modelling* placing theory-oriented goals into the foreground and being in the tradition of the scientific-humanistic approach
- *Educational modelling* emphasizing pedagogical and subject-related goals, which are integrating aspects of the realistic/applied and the epistemological/theoretical approaches taking up aspects of a so-called integrated approach being developed at the beginning of the nineties of the last century mainly within the German discussion

In addition the following new approaches have been developed:

- *Model eliciting and contextual approaches*, which emphasize problem-solving and psychological goals
- *Socio-critical and sociocultural modelling* fostering the goal of critical understanding of the surrounding world connected with the recognition of the sociol-cultural dependency of the modelling activities

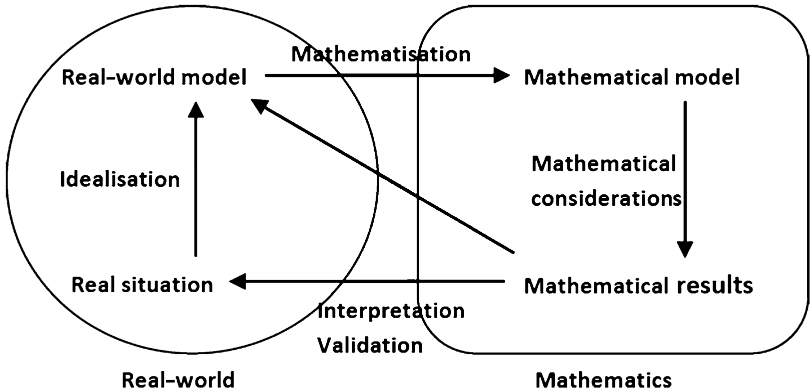
As kind of a meta-perspective, the following perspective is distinguished, which has been developed in the last decade reflecting demands on more detailed analysis of the students' modelling process and their cognitive and affective barriers.

- *Cognitive modelling* putting the analysis of students' modelling process and the promotion of mathematical thinking processes in the foreground

This classification points on the one hand to a continuity of the tradition on the teaching and learning of mathematical modelling; there still exist many commonalities between the historical approach already developed amongst others by Felix Klein and the new approaches. On the other hand, it becomes clear that new perspectives on modelling have been developed over the last decades emphasizing new aspects such as metacognition, the inclusion of socio-critical or socio-cultural issues, a more process-oriented view on modelling, and the modelling cycle.

## The Modelling Process as Key Feature of Modelling Activities

A key characteristic of these various perspectives is the way how the mathematical modelling process is understood, how the relation between mathematics and the "rest of the world" (Pollak 1968) is described. Analyses show that the modelling processes are differently used by the various




**Mathematical Modelling and Applications in Education, Fig. 1** Modelling process from Kaiser-Meßmer (1986) and Blum (1996)

perspectives and streams within the modelling debate, already since the beginning of the discussion. The perspectives described above developed different notions of the modelling process either emphasizing the solution of the original problem, as it is done by the realistic or applied modelling perspective, or the development of mathematical theory as it is done by the epistemological or theoretical approach. So, corresponding to the different perspectives on mathematical modelling, there exist various modelling cycles with specific emphasis, for example, designed primarily for mathematical purposes, research activities, or usage in classrooms (for an overview, see Borromeo Ferri 2006).

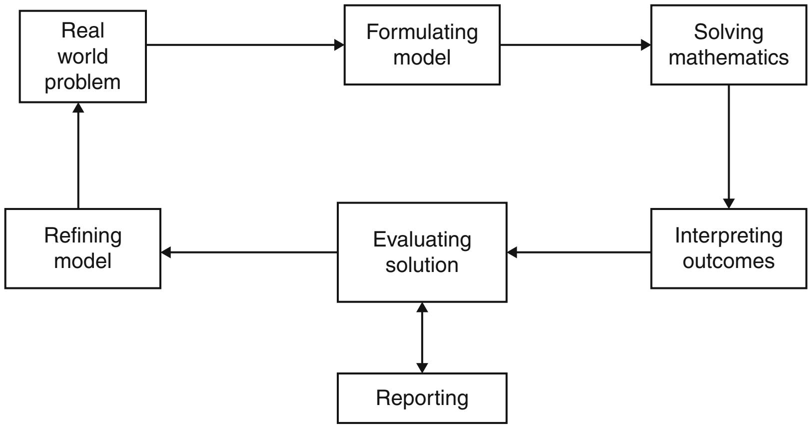
Although at the beginning of the modelling debate, a description of the modelling process as linear succession of the modelling activities was common or the differentiation between mathematics and the real world was seen more statically (e.g., by Burkhardt 1981), nowadays, despite some discrepancies, one common and widespread understanding of modelling processes has been developed. In nearly all approaches, the idealized process of mathematical modelling is described as a cyclic process to solve real problems by using mathematics, illustrated as a cycle comprising different steps or phases.

The modelling cycle developed by Blum (1996) and Kaiser-Meßmer (1986) is based amongst others on work by Pollak (1968, 1969) and serves as exemplary visualization of many similar approaches. This description contains the characteristics, which nowadays can be found in various modelling cycles: The given real-world problem is simplified in order to build a real model of the situation, amongst other many assumptions have to be made, and central influencing factors have to be detected. To create a mathematical model, the real-world model has to be translated into mathematics. However, the distinction between a real-world and a mathematical model is not always well defined, because the process of developing a real-world model and a mathematical model is interwoven, amongst others because the developed real-world model is related to the mathematical knowledge of the modeller. Inside the mathematical model, mathematical results are worked out by using mathematics. After interpreting the mathematical results, the real results have to be validated as well as the whole modelling process itself. There may be single parts or the whole process to go through again (Fig. 1).

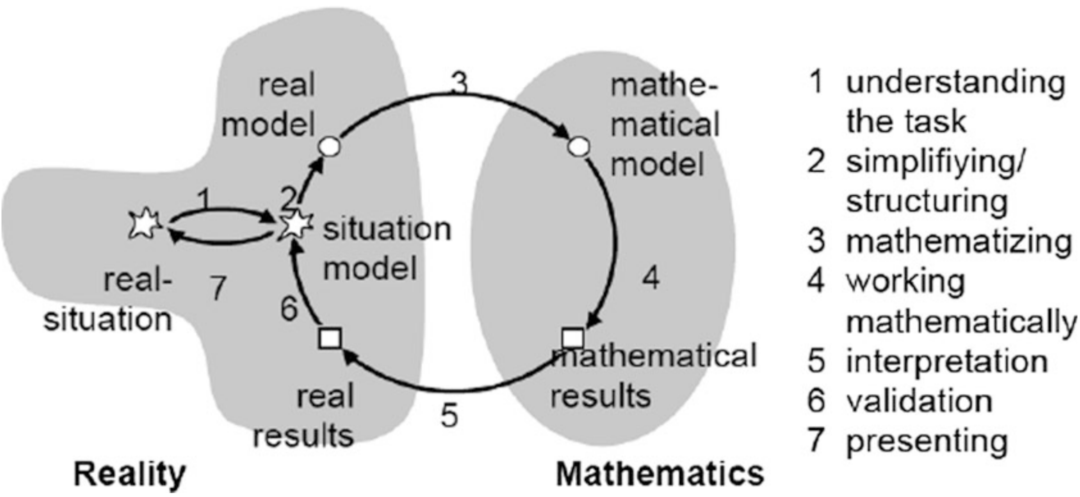
The shown cycle idealizes the modelling process. In reality, several mini-modelling cycles occur that are worked out either in linear sequential steps like the cycle or in a less ordered way. Most modelling processes include frequent switching between the different steps of the modelling cycles.

Other descriptions of the modelling cycle coming from applied mathematics, such as the one by Haines et al. (2000), emphasize the necessity to report the results of the process and include more explicitly the refinement of the model (Fig. 2).

Perspectives putting cognitive analyses in the foreground include an additional stage within




**Mathematical Modelling and Applications in Education, Fig. 2** Modelling process from Haines et al. (2000)




**Mathematical Modelling and Applications in Education, Fig. 3** Modelling process by Blum (2011)

the modelling process, the understanding of the situation by the students. The students develop a situation model, which is then translated into the real model; Blum in more recent work (e.g., 2011) and others (e.g., Leiß, Borromeo Ferri) have described modelling activities in such a way (Fig. 3).

## Detailed Description of One Modelling Cycle Based on the Lighthouse Example

The problem how far a ship is away from a lighthouse, when the crew sees the fire of the lighthouse the first time, is a well-known sea navigation problem with high relevance in former times, before most ships were equipped with GPS. This problem is proposed by protagonists of the educational modelling perspective for the teaching of mathematical modelling in school – especially Blum and Leiß – due to its mathematical richness and its easy accessibility and is adapted in the following to a local situation, namely, a lighthouse at the Northsea in Germany.

### Westerhever Lighthouse

The Westerhever lighthouse was built in 1906 at the German coast of the Northsea and is 41 m high. The lighthouse should in former times inform ships, which were approaching the coast, about their position against the coastline. How far off the coast is a ship when the crew is able to see

**Mathematical Modelling and Applications in Education, Fig. 4** Task on Westerhever lighthouse (photo by Thomas Raupach)

the light fire for the very first time over the horizon? (Round off whole kilometers) (Fig. 4).

Development of a Real-World Model
The students have to develop a real-world model based on different assumptions, i.e., they have to simplify the situation and idealize and structure it, taking into account the curvature of the earth as key influential factor.

Development of a Mathematical Model
The first step can comprise the translation of the real-world model into a two-dimensional mathematical model describing the earth as a circle and then using the Pythagorean Theorem to calculate the required distance from the ship to the lighthouse. Another attempt refers to the definition of the cosine, which can be used instead of the Theorem of Pythagoras.

An extension of this simple model takes into account that the observer who sees the lighthouse at first is not at the height of the waterline, but a few meters higher, e.g., in a look-out. A possible approach uses the Pythagorean Theorem twice, firstly with the right-angled triangle from the geocenter to the top of the lighthouse to the boundary point, where the line of sight meets the sea surface.

Interpretation and Validation
Afterwards the results need to be interpreted and validated using knowledge from other sources. The results need to be transferred back to reality and need to be questioned.

Further Explorations and Extensions
The example of the lighthouse allows many interesting explorations, for example, the reflection on the reverse question, how far away is the horizon?

That well-known problem is similar to the problem of the lighthouse, and its solution is mathematically equivalent to the first elementary model. However, from a cognitive point of view, the real-world model is much more difficult to develop, because the curvature and its central role are psychologically difficult to grasp.

The example above is a typical modelling example showing that there exists a rich variety of modelling examples ranging from small textbook examples to complex, authentic modelling activities. Many extracurricular materials have been developed in the last decades amongst others by COMAP or the Istron Group; many examples are nowadays included in textbooks for school teaching.

## Modelling Competencies and Their Promotion

A central goal of mathematical modelling is the promotion of modelling competencies, i.e., the ability and the volition to work out real-world problems with mathematical means (cf. Maaß 2006). The definition of modelling competencies corresponds with the different perspectives of mathematical modelling and is influenced by the taken perspective. A distinction is made between global modelling competencies and sub-competencies of mathematical modelling. Global modelling competencies refer to necessary abilities to perform the whole modelling process and to reflect on it. The sub-competencies of mathematical modelling refer to the modelling cycle; they include the different competencies that are essential for performing the single steps of the modelling cycle (Kaiser 2007). Based on the comprehensive studies by Maaß (2006) and Kaiser (2007), extensive work by Haines et al. (2000), and further studies and by referring to the various types of the modelling cycle as described above, the following sub-competencies of modelling competency can be distinguished (Kaiser 2007, p. 111):

- Competency to solve at least partly a real world problem through a mathematical description (that is, a model) developed by oneself;
- Competency to reflect about the modelling process by activating meta-knowledge about modelling processes;
- Insight into the connections between mathematics and reality;
- Insight into the perception of mathematics as process and not merely as product;
- Insight into the subjectivity of mathematical modelling, that is, the dependence of modelling processes on the aims and the available mathematical tools and students competencies;
- Social competencies such as the ability to work in groups and to communicate about and via mathematics.

This list is far from being complete since more extensive empirical studies are needed to receive well-founded knowledge about modelling competencies.

Obviously the sub-competencies are an essential part of the modelling competencies. In addition metacognitive competencies play a significant role within the modelling process (Maaß 2006; Stillman 2011). Missing metacognitive competencies may lead to problems during the modelling process, for example, at the transitions between the single steps of the modelling cycle or in situations where cognitive barriers appear (cf. Stillman 2011).

In the discussion on the teaching and learning of mathematical modelling, two different approaches of fostering mathematical modelling competencies can be distinguished: the holistic and the atomistic approach (Blomhøj and Jensen 2003). The holistic approach assumes that the development of modelling competencies should be fostered by performing complete processes of mathematical modelling, whereby the complexity and difficulty of the problems should be matched to the competencies of the learners. The atomistic approach, however, assumes that the implementation of complete modelling problems, especially at the beginning, would be too time-consuming and not sufficiently effective at fostering the individual modelling competencies. It is nowadays consensus that both approaches need to be integrated, although no secure empirical evaluation on the efficiency of both approaches or an integrated one has been carried out so far.

Obviously these two different approaches necessitate different ways of organizing the inclusion of modelling examples in schools: The atomistic approach seems to be more suitable for a "mixing approach," i.e., "in the teaching of mathematics, elements of applications and modelling are invoked to assist the introduction of mathematical concepts etc. Conversely, newly developed mathematical concepts, methods and results are activated towards applicational and modelling situations whenever possible" (Blum and Niss 1991, p. 61). The holistic approach can either be realized in a "separation approach," i.e., instead "of including modelling and applications work in the ordinary mathematics courses, such activities are cultivated in separate courses specially devoted to them" (Blum and Niss 1991, p. 60). Of course variations of these approaches. like the "two-compartment approach" or the "islands approach" described by Blum and Niss (1991) seem to be possible as well.

### Results of Empirical Studies on the Implementation of Mathematical Modelling in School

Several empirical studies have shown that each step in the modelling process is a potential cognitive barrier for students (see, e.g., Blum 2011, as overview). Stillman et al. (2010) describe in their studies these potential "blockages" or "red flag situations," in which there is either no progress made by the students, errors occur and are handled, or anomalous results occur. Stillman (2011) in her overview on the cognitively oriented debate on modelling emphasizes the importance of reflective metacognitive activity during mathematical modelling activities especially within transitions between phases in the modelling process. She identifies productive metacognitive acts promoting students' metacognitive competences at various levels and distinguishes routine metacognition responding to blockages or red flag situations from meta-metacognition being brought in by teachers trying to promote students' development of independent modelling competencies leading to reflective metacognition.

So far the role of the teacher within modelling activities has not been researched sufficiently: Until now not enough secure empirical evidence exists, how teachers can support students in independent modelling activities, how can they support them in overcoming cognitive blockages, and how can they foster metacognitive competencies. It is consensus that modelling activities need to be carried out in a permanent balance between minimal teacher guidance and maximal students' independence, following well-known pedagogical principles such as the principal of minimal help. Research calls for individual, adaptive, independence-preserving teacher interventions within modelling activities (Blum 2011), which relates modelling activities to the approach of scaffolding. Scaffolding can be according to well-known definitions described as a metaphor for tailored and temporary support that teachers offer students to help them solve a task that they would otherwise not be able to perform. Although scaffolding has been studied extensively in the last decades, it was found to be rare in classroom practice. Especially for modelling processes, which comprise complex cognitive activities, scaffolding seems to be especially necessary and appropriate. But scaffolding has to be based on a diagnosis of students' understanding of the learning content, which most teachers did not ascertain; in contrast most teachers provided immediate support or even favoured their own solution.

In the future, learning environments for modelling need to be established, which support independent modelling activities, for example, by sense-making using meaningful tasks, model-eliciting activities based on challenging tasks, or the usage of authentic tasks.

## Cross-References

- ▶ Interdisciplinary Approaches in Mathematics Education
- ▶ Mathematization as Social Process
- ▶ Realistic Mathematics Education
- ▶ Word Problems in Mathematics Education

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# Mathematical Proof, Argumentation, and Reasoning

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## Keywords

Argumentation; Logic; Proof; Reasoning; Visualization

## Definition

Argumentation, reasoning, and proof are concepts with ill-defined boundaries. More precisely, they are words that different people use in different ways. What one can perhaps say is that reasoning is the concept with the widest compass. Logic is usually taken to mean a more structured form of reasoning, with its own subset, formal logic, which is logic in its most rigidly structured form. Though people most closely associate logic with mathematics, all forms of reasoning have had, and continue to have, valuable roles in mathematical practice. For that reason and, perhaps even more important, because of their usefulness in teaching, the many forms of reasoning have also found their place in the mathematics curriculum.

## Characteristics

This entry will explore in more detail the concepts of argumentation, reasoning, and proof as understood by mathematicians and educators and present some of their implications for mathematics education. It will go on to describe some more recent thinking in mathematics education and in the field of mathematics itself.

### Mathematical Proof

Mathematics curricula worldwide aim at teaching students to understand and produce proofs, both to reflect proof's central position in mathematics and to reap its many educational benefits. Most documents addressed to teachers, such as that by the National Council of Teachers of Mathematics (NCTM 2000), give the following reasons for teaching proof: (1) to establish certainty; (2) to gain understanding; (3) to communicate ideas; (4) to meet an intellectual challenge; (5) to create something elegant, surprising, or insightful; and (6) to construct a larger mathematical theory.

This list encompasses not only justification but also considerations of understanding, insight, and aesthetics and in so doing further reflects mathematics itself. These additional considerations are important not only in the classroom but in mathematical practice as well: for mathematicians, too, a proof is much more than a sequence of logical steps that justifies an assertion.

Proof also plays other significant roles in mathematical practice. Proof can serve to present new methods and demonstrate their value, to inspire new hypotheses, and to show connections between different parts of mathematics. For practicing mathematicians, these too are valuable aspects of proof; yet the mathematics curricula, by and large, have failed to explore their educational potential.

Proof pervades all mathematical work. Unless it is considered an axiom, a mathematical assertion without a proof must remain a conjecture. To justify an assertion is the role of a proof. In the purest sense, a mathematical proof is a logical derivation of a given statement from axioms through an explicit chain of inferences obeying accepted rules of deduction. A "formal proof" will employ formal notation, syntax, and rules of inference ("axiomatic method"). Thus, strictly formal derivations will consist of unambiguous strings of symbols and conform to a mechanical procedure that will permit the correctness of the proof to be checked. Such proofs are considered highly reliable.

However, proofs in mathematical journals rarely conform to this pattern. As Rav (1999) pointed out, mathematicians express "ordinary" proofs in a mixture of natural and formal language, employing passages of explicit formal deductions only where appropriate. They bridge between these passages of formal deduction using passages of informal language in which they provide only the direction of the proof, by making reference to accepted chains of deduction. Consequently, most mathematicians would characterize ordinary proofs as informal arguments or "proof sketches."

Nevertheless, these ordinary informal proofs do provide a very high level of reliability, because the bridges are "derivation indicators" that are easily recognized by other mathematicians and provide enough detail to allow easy detection and repair of errors (Azzouni 2004). In this way, the social process by which such proofs are scrutinized and ultimately accepted improves their validity. In fact, most accepted mathematical proofs consist of valid arguments that may not lend themselves to easy formalization (Hanna 2000; Manin 1998; Thurston 1994).

To reflect mathematical practice, then, a mathematics curriculum has to present both formal and informal modes of proof. If they wish to teach students how to follow and evaluate a mathematical argument, make and test a conjecture, and develop and justify their own mathematical arguments and proofs, educators have to provide the students with the entire gamut of mathematical tools, including both the formal and informal ones. Without this important double approach, students will lack the body of mathematical knowledge that enables practicing mathematicians to communicate effectively by using "derivation indicators" and other mathematical shorthand (cf. Hanna and de Villiers 2012).

## Reasoning and Proof

Most mathematics curricula recognize that reasoning and proof are fundamental aspects of mathematics. In fact, much of the literature on mathematics teaching refers to them as one entity called "reasoning and proof."

We may take reasoning, in the broadest sense, to mean the common human ability to make inferences, deductive or otherwise. As Fischbein (1999) noted, everyday reasoning may differ from explicit mathematical reasoning in both process and result. In everyday reasoning, for example, we may even accept a statement without any type of proof at all, because we judge it to be self-evident or intuitively plausible, or at least more plausible than its contradiction. However, in many realms, including mathematics, such everyday reasoning provides little help (e.g., it is not intuitively clear that the sum of the angles in any triangle is always 180°). In all such cases we would need defined rules of reasoning in order to reach a valid conclusion. We would need to construct a correct chain of inference – that is, to construct a proof.

Thus, all mathematics educators aim to teach students the rules of reasoning. In the Western tradition, the rules of reasoning are derived from classical mathematics and philosophy and include, for example, the syllogism and such elementary rules as *modus ponens*, *modus tollens*, and *reductio ad absurdum*. Students typically first encounter these basic concepts of logic in the axiomatic proofs of Euclidean geometry.

Here the teacher's role is crucial. In addition to concepts specific to the mathematical topic, the teacher must make the students familiar with rules of reasoning, patterns of argumentation, and appropriate terms (e.g., assumption, conjecture, example, refutation, theorem, and axiom). How students actually learn these concepts is unfortunately a question of cognition that educators have yet to resolve, though researchers investigating this issue have proposed a number of promising models of cognition. One such model, the "cognitive development of proof," combines three worlds of mathematics: the conceptual/embodied, the proceptual/symbolic, and the axiomatic/formal (cf. Tall et al., chapter 2 in Hanna and de Villiers 2012). Another, based on extensive observations of college-level students learning mathematics, uses a psychological framework of "proof schemes" (Harel and Sowder 1998). Yet another (Balacheff 2010) aims at analyzing the learning of proof by considering how three "dimensions" – the subject, the milieu, and the problem – can be used to build a bridge between knowing and proving. Duval's (2009) model stresses that the cognitive processes needed to understand and devise a proof depend on students' learning "how proof really works" (learning its syntactic and deductive elements) and "how to be convinced by proof." Stylianides (2008) proposes that the processes of reasoning and proving encompass three "components" – mathematical, psychological, and pedagogical – while Reid and Knipping (2010) discuss still other variations.

## Argumentation and Proof

Many researchers in mathematics education have chosen to use the term "argumentation," which encompasses the various approaches to logical disputation, such as heuristics, plausible, and diagrammatic reasoning, and other arguments of widely differing degrees of formality (e.g., inductive, probabilistic, visual, intuitive, and empirical). Essentially, argumentation includes any technique that aims at persuading others that one's reasoning is right. As used by its proponents, the concept also implies exchange and cooperation in forming and criticizing arguments so as to arrive at the best conclusion despite imperfect knowledge. Evidently, the broad concept of argumentation encompasses mathematical proof as a special case.

In recent years, however, mathematics educators have been accustomed to use "argumentation" to mean "not yet proof" and "proof" to mean "mathematical proof." Consequently, opinion remains divided on the usefulness of encouraging students to engage in "argumentation" as a step in learning proof. Boero (in La lettre de la preuve 1999) and others see a great benefit in having students engage in conjecturing and argumentation as they develop an understanding of mathematical proof. Others take a quite different view, claiming that argumentation, because it aims only to establish plausibility, can never be more than a distraction from the task of teaching proof (e.g., Balacheff 1999; Duval – in La lettre de la preuve 1999). Despite these differences of opinion, however, the practice of teaching students the techniques of argumentation has recently been gaining ground in the classroom.

Durand-Guerrier et al. (Chapter 15 in Hanna and de Villiers 2012) reported on over 100 recent studies on argumentation in mathematics education that discuss the complex relationships between argumentation and proof from various mathematical and educational perspectives. Most of these studies reported that students can benefit from argumentation's openness of exploration and flexible validation rules as a prelude to the stricter uses of rules and symbols essential in constructing a mathematical proof. They also showed that appropriate learning environments can facilitate both argumentation and proof in mathematics classes.

Furthermore, some studies provided evidence that students who initially embarked upon heuristic argumentation in the classroom were nevertheless capable of going on to construct a valid mathematical proof. By way of explanation, Garuti et al. (1996) introduced the notion of "cognitive unity," referring to the potential continuity between producing a conjecture through argumentation and constructing its proof. Several other researchers have provided support for this idea and for other benefits or limitations of argumentation, particularly argumentation based on Toulmin's (1958) model of argument.

Toulmin's model, the one now most commonly used in mathematics education, proposes that an argument is best seen as comprising six elements: the Claim (C), which is the statement to be proved as a theorem or the conclusion of the argument; the Data (D), the premises; the Warrant (W) or justification, which is the connection between the Claim and the Data; the Backing (B), which gives authority to the Warrant; the Qualifier (Q), which indicates the strength of the Warrant by terms such as "necessarily," "presumably," "most," "usually," "always," and so on; and the Rebuttal (R), which specifies conditions that preclude the Claim (e.g., if the Warrant is not convincing).

Clearly, Toulmin's model reflects practical and plausible reasoning. It includes several types of inferences, admits of both inductive and deductive reasoning, and makes explicit both the premises and the conclusion, as well as the support that led from premises to conclusion. It is particularly relevant to mathematical proof in that it can include formal derivations of theorems by logical inference.

### Practical Classroom Approaches

In addition to argumentation, a number of other approaches have been investigated for their value in teaching mathematical reasoning. Educators have debated, for example, whether the study of symbolic logic, more particularly the propositional calculus, would help students understand and produce proofs. Durand-Guerrier et al. (Chapter 16 in Hanna and de Villiers 2012) have examined this question and provide some evidence for the value of integrating techniques of symbolic logic into the teaching of proof.

Visualization, and diagrammatic reasoning in particular, is another technique whose value in teaching mathematics, and especially proof, has been discussed extensively in the literature and in conferences, albeit inconclusively. After examining numerous research findings, Dreyfus et al. (Chapter 8 in Hanna and de Villiers 2012) concluded that the issue required further research; in fact, both philosophers of mathematics and mathematics educators are still debating the contribution of visualization to the production of proofs. Current computing technologies have offered mathematicians an array of powerful tools for experiments, explorations, and visual displays that can enhance mathematical reasoning and limit mathematical error. These techniques have classroom potential as well. Borwein (Chapter 4 in Hanna and de Villiers 2012) sees several roles for computer-assisted exploration, many of them related to proof: graphing to expose mathematical facts, rigorously testing (and especially falsifying) conjectures, exploring a possible result to see whether it merits formal proof, and suggesting approaches to formal proof. Considerable research has demonstrated that the judicious use of dynamic geometry software can foster an understanding of proof at the school level (de Villiers 2003; Jones et al. 2000).

Physical artifacts (such as abaci, rulers, and other ancient and modern tools) provide another technique for facilitating the teaching of proof. Arzarello et al. (Chapter 5 in Hanna and de Villiers 2012) demonstrate how using such material aids can help students make the transition from exploring to proving. In particular, they show that students who use the artifacts improve their ability to understand mathematical concepts, engage in productive explorations, make conjectures, and come up with successful proofs.

### Trends in Proof

In mathematical practice, as we have seen, ordinary informal proofs are considered appropriate and suitable for publication. Still, mathematicians would like to have access to a higher level of certainty than those informal proofs afford. For this reason, contemporary mathematical practice is trending toward the production of proofs much more rigorous and formal than those of a century ago (Wiedijk 2008). In practice, however, one cannot write out in full any formal proof that is not trivial, because it encompasses far too many logical inferences and calculations.

The last 20 years have seen the advent of several computer programs known as "automatic proof checkers" or "proof assistants." Because computers are better than humans at checking conformance to formal rules and making massive calculations, these new programs can check the correctness of a proof to a level no human can match. According to Wiedijk (2008), such programs have been successful in confirming the validity of several well-known theorems, such as the Fundamental Theorem of Algebra (2000) and the Prime Number Theorem (2008).

Mathematics educators and students have already benefitted greatly from educational software packages in areas other than proof, such as Dynamic Geometric Software (DGS) and Computer Algebra Systems (CAS), and researchers are working on advanced proof software specifically for mathematics education. For example, there is now a fully functional version of Theorem-Prover System (TPS) appropriate for the school and undergraduate levels, named eduTPS (Maric and Neuper 2011). The role of Artificial Intelligence in mathematics education, and in particular that of automated proof assistants, has already been the subject of several doctoral dissertations. Unfortunately, mathematics educators have not yet tested the proof software or tried it in the classroom, so its usefulness for teaching mathematics has not yet been firmly established.

## Cross-References


▶ Argumentation in Mathematics
▶ Argumentation in Mathematics Education
▶ Deductive Reasoning in Mathematics Education
▶ Logic in Mathematics Education


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## Mathematical Representations

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### Keywords

Cognitive configurations; Concrete embodiments; Diagrams; External representations; Graphs; Imagery; Inscriptions; Interpretation; Internal representations; Language; Manipulatives; Meanings; Models; Productions; Representational systems; Semiotics; Signification; Symbols; Symbolization; Visualization

### Definitions

As most commonly interpreted in education, *mathematical representations* are visible or tangible productions – such as diagrams, number lines, graphs, arrangements of concrete objects or manipulatives, physical models, mathematical expressions, formulas and equations, or depictions on the screen of a computer or calculator – that encode, stand for, or embody mathematical ideas or relationships. Such a production is sometimes called an *inscription* when the intent is to focus on a particular instance without referring, even tacitly, to any interpretation. To call something a *representation* thus includes reference to some meaning or signification it is taken to have. Such representations are called *external* – i.e., they are external to the individual who produced them, and accessible to others for observation, discussion, interpretation, and/or manipulation.

The term *representation* is also used very importantly to refer to a person's *mental* or *cognitive* constructs, concepts, or configurations. Then the mathematical representation is called *internal* to the individual. Examples include individuals' visual and/or spatial representation of geometrical objects or mathematical patterns, operations, or situations; their kinesthetic encoding of operations, shapes, and motions; their internal conceptual models of mathematical ideas; the language that they use internally to describe mathematical situations; their heuristic plans and strategies for problem solving; and their affective states in relation to mathematical problems and situations. The idea of external representation is expressible in German as *Darstellung* and that of internal representation as *Vorstellung*.

*Representation* also refers to the *act* or *process* of inventing or producing representations – so that "mathematical representation" is something that students and others *do*. Reference may be to the physical production of external representations as well as to the *cognitive* or *mental processes* involved in constructing internal or external representations. The term also describes the semiotic relation between external productions and the internal mathematical ideas they are said to represent. Finally, it may refer specifically to the mathematical encoding of *nonmathematical* patterns – i.e., using the ideas and notations of mathematics as a *language* to represent concepts in physics, chemistry, biology, and economics, to describe quantitatively the laws that govern phenomena, to make predictions, and to solve problems.



### Characteristics

Representations are considered to be mathematically *conventional*, or standard, when they are based on assumptions and conventions shared by the wider mathematical community. Examples of such conventional mathematical representations include base ten numerals, abaci, number lines, Cartesian graphs, and algebraic equations written using standard notation. In contrast, mathematical representations created on specific occasions by students are frequently *idiosyncratic*. Examples may include pictures, diagrams, illustrative gestures, physical movements, and original or nonstandard notations invented by the individual.

Even when they are unconventional, mathematical representations can be *shared* and not simply personal. That is, the forms and meanings of representations may be *negotiated* during class discussions or group problem solving.

Concrete structured manipulative materials such as geoboards, Cuisenaire rods, base ten blocks, pegboards, and attribute blocks, as well as calculators, graphing calculators, and a wide variety of computer environments, facilitate students' construction, discussion, interpretation, and sharing of many different kinds of external representations, both standard and idiosyncratic.

Likewise internal mathematical representations, depending on their degree of consistency with the internal representations of others, can be characterized as conventional or idiosyncratic, shared or personal.

In discussion one often refers to a mathematical representation "in the abstract." For instance, to talk about "examining the graph of the equation $y = 3x-2$" is to suggest (among other things) a kind of *idealized* or *generic* external representation in which a straight line has been drawn intersecting the horizontal $x$-axis at the point $x = 2/3$ and the vertical $y$-axis at the point $y = -2$. This is in contrast to discussing a particular *instance* of the graph as it might occur in a textbook illustration (with particular scales, ranges of values, and so forth), in a blackboard drawing (perhaps imperfect), or on a graphing calculator. Internal representations are frequently considered "in the abstract," as one refers, for example, to idealized mathematical ideas, concept images, or visualized symbol configurations.

An essential feature of mathematical representations is that not only do they have signification but they belong to or are situated within *structured systems of representation* within which other configurations have similar signifying relationships. This is analogous to the way words and sentences occur, not as discrete entities in isolation from each other, but within natural languages endowed with grammar, syntax, and networks of semantic relationships. Furthermore, representational systems (like languages) *evolve*. And previously developed systems of mathematical representation serve up to a point as "scaffolds" or "templates" for the development of new systems, making reference to the representing relationship between configurations in the new system and their meanings in the prior system.

For example, algebra as a representational system entails the interpretation of letters as variables that can assume numerical values. But it also involves algebraic expressions, operational symbols, and equality and inequality symbols, configured according to fairly precise syntactic rules, as well as processes for manipulating and transforming them. Up to a point, the prior arithmetic system of representation serves as a kind of template for the development of algebraic notation. The system evolved historically, and it evolves within learners in interaction with external environments. As mathematics is learned, the structured nature of the mathematical representations creates a certain tension between a student's interpretation of meanings, acquisition of procedures, and eventual apprehension of underlying structures (e.g., Gravemeijer et al. 2010).

Characteristics of conventional structured mathematical representational systems can often be described in considerable detail. A particular written or printed numeral may represent a natural number, but it does so within our base ten Hindu-Arabic system of notation, a representational system of numeration involving the conventional signs {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}, rules for writing multidigit numerals, and conventions for interpreting "place value." A particular Cartesian graph of an equation in two variables occurs within the wider conventional system of graphical representation based on two orthogonal coordinate axes in the plane, a method of locating points in the plane corresponding to ordered pairs of coordinates, the use of certain letters to signify variables that can take on numerical values, and conventions involving positive and negative directions.

The precision of such characterizations is, of course, a prized feature of mathematics. Furthermore, an important aspect of the power of abstract mathematics is that mathematical constructs map to other constructs (i.e., can be represented) in ways that *respect or preserve the mathematical structure*. When the structure thus respected is algebraic, such maps are called homomorphisms or isomorphisms. For example, in mathematics a *group representation* is a precisely defined notion: a homomorphism from a given, abstract mathematical group to a group of linear operators acting on a vector space.

However the mathematical representations that occur in educational contexts, even when conventional, are extremely varied. They are most often incomplete and almost always highly ambiguous. Indeed, *ambiguity* and *context-dependence* are characteristic features of the interpretation of mathematical representations and systems of representation. Resolution of ambiguity in the process of interpreting a representation often entails making use of contextual and/or tacit information that is outside the representational system within which the ambiguity has occurred.

Mathematical representations and systems of representation are frequently characterized according to the nature of the representing configurations – e.g., internal or external; enactive, iconic, or symbolic; verbal, visual, spatial, auditory, or kinesthetic; concrete or abstract/symbolic; and static or dynamic. Mathematical metaphors are representations that typically involve words or phrases, visual imagery, and some enactive or kinesthetic encoding of mathematical ideas. Different representational systems may be *linked*; and (with today's computer technology) external, dynamic systems of representation may be multiply linked for purposes of mathematics teaching.

## Research

Contrasting philosophical views that have greatly influenced mathematics education sometimes exclude or limit the study of representations as such within their respective paradigms. Behaviorism is based on the idea that mental states of any kind are inadmissible as explanations of observable learning or problem solving. This permits external productions or configurations and their manipulation to be discussed, but not to be regarded as representing internal mathematical conceptualizations or as being represented by them. External configurations might only have some possible observable correspondence with other external ones. Radical constructivism is based on the tenet that each of us has access only to his or her own world of experience and none to the "real world." The exclusive emphasis on "experiential reality" permits internal configurations only to "re-present" other internal mathematical experience in different ways. Still other viewpoints are based on the idea that the external-internal distinction itself entails a Cartesian mind-body dualism that is not tenable.

Nevertheless research on representations and systems of representation in mathematics education has been ongoing for well over half a century and continues apace. Jerome Bruner, whose thinking contributed to some of the visionary ideas proposed by advocates of the "new mathematics" during the 1960s, characterized and discussed three kinds of representation by learners – enactive, iconic, and symbolic – seen as predominant during successive stages of a child's learning a concept (Bruner 1966). Semiotic and cognitive science approaches to mathematics education incorporated mathematical representation in its various interpretations (e.g., Palmer 1978; Skemp 1982; Davis 1984).

During the 1980s and 1990s, continuing research on representation by many (e.g., Janvier 1987; Goldin and Kaput 1996; Goldin and Janvier 1998) helped lay the groundwork for the inclusion by the National Council of Teachers of Mathematics (NCTM) in the United States of "Representations" as one of the major strands in its *Principles and Standards for School Mathematics* (NCTM 2000). The NCTM also devoted its 2001 Yearbook to the subject (Cuoco and Curcio 2001).

The NCTM's standards included many of the different meanings of mathematical representation described here:

> The term representation refers both to process and to product – in other words, to the act of capturing a mathematical concept or relationships in some form and to the form itself. . . . Moreover, the term applies to processes and products that are observable externally as well as those that occur 'internally,' in the minds of people doing mathematics. (NCTM 2000, p. 67)

Continuing research on mathematical representation in education has included work on cognition and affect, on the affordances for mathematics learning offered by technology-based dynamic representation and linked representations, on sociocultural contexts and their influences, and on the role of representations in particular conceptual domains of mathematics (e.g., Goldin 1998, 2008; Hitt 2002; Kaput et al. 2002; Lesh and Doerr 2003; Duval 2006; Moreno-Armella et al. 2008; Anderson et al. 2009; Roth 2009; Gravemeijer et al. 2010).

Teachers and researchers try to infer features of students' internal representations from the external representations they produce or with which they are presented. The representing relationship is usually understood in research to be in principle two-way, "bridging" the external and the internal. In addition, distinct external representations can represent each other (e.g., equations, graphs, and tables of values) in a student's thinking, and distinct internal representations can do likewise (e.g., as the student visualizes or imagines a function of a real variable as a formula, a graph, a machine generating outputs from inputs, or a set of ordered pairs satisfying some conditions). However in any particular situation, one cannot simply assume a close or one-to-one correspondence between external and internal representations or between distinct external or internal ones. Different researchers have offered different perspectives on what it is that representations actually represent and the nature of the representing relationship.

Much research on mathematical representation in education is devoted to the study of particular conceptual domains, such as number, fractions or rational numbers, integers (positive and negative), algebra, geometry and spatial concepts, and functions and graphs. The goal is frequently to study, in a specific domain, how students generate representations, interact and move within various representations, translate between representations, or interpret one representation using another. Researchers seek to characterize students' understandings in terms of multiple representations, to infer students' thinking from the representations they produce and manipulate, to identify the affordances and obstacles associated with particular kinds of representation, and to develop new representational teaching methods using new media.

When representations are embodied in different media, different features of a conceptual domain of mathematics may become the most salient. Thus the mathematical meanings may be regarded as *distributed* across various representational media in which they are encoded. With the advent of increasingly diverse and sophisticated technological environments, *dynamic* and *linked* mathematical representations are becoming increasingly important. These are built to respond to learners' actions, touches, or gestures according to preestablished structures and may eventually lead not only to novel teaching methods but to quite new interpretations of what it means to understand mathematics.

When a mathematical representation is first introduced, it is typically assigned a particular meaning or signification. For instance, a specific number-word may correspond to the result of counting fingers or objects; a positive whole number exponent may be defined as a way to abbreviate repeated multiplication; or the letter $x$ may stand for an unknown number in a problem. Sometimes the initial signification is taken to be so fundamental that it poses a cognitive or epistemological obstacle to reinterpretation or later generalization. Certain misconceptions or alternative conceptions can be understood in this way. But particular representations do not exist in isolation from each other; and as relationships develop their meanings evolve, transfer to new contexts, and eventually may change profoundly. Such processes occur across the history of mathematics, within particular cultures, and within individual learners (e.g., Moreno-Armella et al. 2008; Anderson et al. 2009; Moreno-Armella and Sriraman 2010).

## Cross-References


▶ Affect in Mathematics Education
▶ Algebra Teaching and Learning
▶ Constructivism in Mathematics Education
▶ History of Mathematics and Education
▶ Functions Learning and Teaching
▶ Manipulatives in Mathematics Education

- ▶ Mathematical Language
- ▶ Metaphors in Mathematics Education
- ▶ Misconceptions and Alternative Conceptions in Mathematics Education
- ▶ Number Lines in Mathematics Education
- ▶ Number Teaching and Learning
- ▶ Problem Solving in Mathematics Education
- ▶ Semiotics in Mathematics Education
- ▶ Teaching Practices in Digital Environments
- ▶ Visualization and Learning in Mathematics Education

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# Mathematics Classroom Assessment


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## Keywords

Formative assessment; Assessment tasks; Questioning; Assessment rubrics; Feedback; Self-assessment


## Definition

*Classroom assessment* refers to the activities undertaken by teachers in eliciting and interpreting evidence of student learning and using this evidence to inform subsequent action.

*Classroom assessment* can be distinguished from *external assessment*, which often involves standardized tests carried out on a large scale. The most important difference between classroom assessment and external assessment arises from their different purposes. Wiliam (2007) summarizes the main purposes of assessment as:

1. Certifying the achievement or level of performance of individual students (summative)
2. Supporting students' learning and informing teachers' instructional decisions (formative)
3. Evaluating the quality of educational programs or institutions (evaluative)

Although teachers may design *classroom assessments* for both summative and formative purposes, it is more common to use this term to refer to assessment that is intended to support learning and teaching, in other words, formative assessment (Van den Heuvel-Panhuizen and Becker 2003; Wilson and Kenney 2003). On the other hand, *external assessment* is most often used for summative or evaluative purposes.

## Background

Throughout the twentieth century, educational assessment was increasingly associated with externally administered tests that measure the performance of students, as well as teachers, schools, and whole school systems. This measurement paradigm continues to influence classroom assessment practices, despite the emergence of new theories of learning and curriculum that require new approaches to assessment. Shepard (2000) argues that classroom assessment should be epistemologically consistent with instruction, and indeed this was the case for much of the twentieth century when social efficiency models of curriculum and associationist and behaviorist theories of learning informed educational thinking and practice. These psychological theories assumed that learning is most efficient when knowledge and skills are broken into small steps and accumulated sequentially. Closely aligned with such theories is the idea of scientific measurement of skill mastery, which led to development of the "objective" test as the dominant method of assessing student achievement.

Time-restricted objective tests that require only recall of previously learned facts and rehearsed procedures are still a common form of mathematics classroom assessment in many countries. However, this traditional approach to assessment is out of alignment with the broadly social-constructivist conceptual frameworks that shape current understandings of learning and curriculum. Learning mathematics is now viewed as a process of constructing knowledge within a social and cultural context, and deep understanding, problem solving, and mathematical reasoning have become valued curricular goals. As the goals of mathematics education change, along with understanding of how students learn mathematics, new approaches to classroom assessment are called for that make students' thinking visible while enhancing teachers' assessment abilities (Van den Heuvel-Panhuizen and Becker 2003).

## A Social-Constructivist Approach to Classroom Assessment

Work on developing a social-constructivist approach to mathematics classroom assessment is less advanced than research on mathematics learning, but key principles informing a new approach to assessment are well established and have been promulgated via research reports (Shepard 2000; Wiliam 2007; Wilson and Kenney 2003), curriculum documents (National Council of Teachers of Mathematics 1995, 2000), and professional development resources (Clarke 1997). Three overarching principles that correspond to each of the elements of the definition of assessment provided above are shown in Table 1, with particular reference to classroom assessment in mathematics.

### Eliciting Evidence of Student Learning

The principle of modeling good mathematical practice in classroom assessment is consistent with curriculum goals that value sophisticated mathematical thinking (abstraction, contextualization, making connections between concepts and representations) and appropriate use of mathematical language and tools.

**Mathematics Classroom Assessment, Table 1** Classroom assessment principles

| Definition of classroom assessment | Assessment principle | Assessment examples |
|---|---|---|
| Classroom assessment involves teachers in . . . eliciting evidence of student learning | Assessment should model good mathematical practice | Tasks |
| | | Classroom discussion and questioning |
| Classroom assessment involves teachers in . . . interpreting evidence of student learning | Assessment should promote valid judgments of the quality of student learning | Alignment |
| | | Multiple forms of evidence |
| | | Explicit criteria and standards |
| Classroom assessment involves teachers in . . . acting on evidence of student learning | Assessment should enhance mathematics learning | Feedback |
| | | Self-assessment |

Classroom assessment can provide insights into students' mathematical thinking through tasks that have more than one correct answer or more than one solution pathway, require application of knowledge in familiar and unfamiliar contexts, and invite multiple modes of communication and representation for demonstrating understanding. Time-restricted tests are usually unsuitable for revealing students' thinking in these ways. While investigative projects and mathematical modeling tasks provide rich opportunities for students to demonstrate understanding of significant mathematics, so too do more modest tasks such as "good" questions (Sullivan and Clarke 1991). Good questions are open-ended, elicit a range of responses, and can reveal what a student knows before and after studying a topic. These questions can easily be adapted from more conventional tasks that have only one correct answer, as demonstrated in Table 2.

Assessment is something that teachers are doing all the time, not only through tasks designed for assessment purposes but also in classroom discussion. In mathematics education, social-constructivist research carried out by Cobb, Forman, Lampert, O'Connor, and Wood has investigated the teacher's role in initiating students into mathematical discourse and practices (Lampert and Cobb 2003; Forman 2003). From an assessment perspective, a teacher purposefully orchestrating classroom discussion is collecting evidence of students' understanding that can inform subsequent instruction.

**Mathematics Classroom Assessment, Table 2** Converting conventional questions to "good" questions

| Conventional question | Open-ended "good" question |
|---|---|
| Find the mean of these three numbers: 12, 16, 26 | The mean age of three people is 18. What might their ages be? |
| Find the area of a rectangle with length 3 units and width 4 units | Draw a triangle with an area of 6 square units |
| Find the equation of the line passing through the points (2, 1) and (−1, 3) | Write the equations of at least five lines passing through the point (2, 1) |



## Interpreting Evidence of Student Learning

Teachers do not have direct access to students' thinking, and so assessment relies on interpretation of observable performance to enable judgments to be made about the quality of students' learning. Shepard (2000) notes that teachers are often reluctant to trust qualitative judgments because they believe that assessment needs to be "objective", requiring formula-based methods that rely on numerical marks or scores. This is a reductionist approach more consistent with the scientific measurement paradigm of assessment than the social-constructivist paradigm, where the goal of assessment is to provide a valid portrayal of students' learning (Clarke 1997).

The validity of teachers' assessment judgments can be strengthened by ensuring that assessment practices are aligned with curriculum goals and instruction. This means that the form and content of mathematics classroom assessments should reflect the ideas about good mathematical practice envisioned in curriculum documents and (ideally) enacted in classrooms. Assessment promotes valid judgments when it draws on multiple forms of evidence, as no single assessment tool can reveal the full range of student learning.

Validity is also enhanced when teachers explicitly communicate to students the criteria and standards that will be used to judge the quality of their performance (Wiliam 2007). Sadler's (1989) work on ways of specifying achievement standards has been influential in stimulating the development of assessment rubrics that use verbal descriptors to communicate the characteristics of task performance that will be assessed (criteria) and the benchmarks for describing the quality of performance (standards). A well-constructed rubric can make explicit the mathematical practices that teachers value, but students will not necessarily understand the verbal descriptors in the same way as the teacher. There is an opportunity here for teachers to engage students in discussion about the meaning of the criteria and what counts as good quality performance. Some researchers suggest that teachers can involve students in the development of rubrics in the process of looking at samples of their own or other students' work (Clarke 1997; Wiliam 2007). In this way, students can become familiar with notions of quality and develop the metacognitive ability to judge the quality of their own mathematical performances.

### Acting on Evidence of Student Learning

One of the most important ways in which assessment can enhance mathematics learning is through the provision of feedback that can be used by students to close the gap between actual and desired performance. The notion of feedback had its origins in engineering and cybernetics, but finds extensive application in education. Ramaprasad's (1983) definition of feedback makes it clear that feedback is only formative if the information provided to the student is used in some way to improve performance. Reviews of research on feedback have identified characteristics of effective formative feedback in relation to quantity, timeliness, and strategies for engaging students in task-related activities that focus on improvement (Bangert-Drowns et al. 1991). However, Shepard (2000), arguing from a social-constructivist perspective, points out that these studies are mostly of little value because they are informed by behaviorist assumptions about learning and assessment. Drawing on Vygotsky's idea of the zone of proximal development, she calls for more research on dynamic assessment where the teacher uses scaffolded feedback to guide students through the solution process for a problem.

Involving students in self-assessment can enhance metacognitive self-regulation and help students become familiar with the criteria and standards that will be used to judge their performance. Controlled experiments have shown that structured self-assessment improves students' mathematics performance, but classroom self-assessment can also be used informally to gain insights into how students experience mathematics lessons. The IMPACT (Interactive Monitoring Program for Accessing Children's Thinking) procedure described by Clarke (1997) is one such approach. It invites students to write about important things they have learned in mathematics in the past month, problems they have found difficult, what they would like more help with, and how they feel in mathematics classes at the moment. This is a self-assessment tool that makes assessment a more open process and recognizes the important role of student affect in mathematics learning.

## Issues in Classroom Assessment

A social-constructivist approach to classroom assessment places significant demands on mathematics teachers' knowledge and expertise. This includes knowing:

- How to design tasks and orchestrate classroom discussions that elicit students' mathematical thinking
- How to formulate assessment criteria and standards that reflect valued mathematical activity
- How to make balanced judgments about the quality of student performance across a range of different tasks

- How to provide contingent, “real-time” feedback that moves students forward in their learning
- How to encourage students to share ownership of the assessment process

Teachers’ beliefs about what counts as “fair” or “objective” assessment also need to be taken into consideration, since the scientific measurement paradigm still exerts a strong influence on teachers’ assessment practices. Although there are many research studies investigating social-constructivist mathematics teaching, the possibilities for introducing new approaches to mathematics classroom assessment require further research focusing in particular on supporting teacher development and change.

## Cross-References

▶ Competency Frameworks in Mathematics Education
▶ External Assessment in Mathematics Education
▶ Mathematical Modelling and Applications in Education
▶ Metacognition
▶ Questioning in Mathematics Education
▶ Scaffolding in Mathematics Education
▶ Zone of Proximal Development in Mathematics Education

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# Mathematics Curriculum Evaluation

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## Keywords

Mathematics curricula; Curriculum evaluation; Assessment; Curriculum coherence; Alignment; Math standards; Mathematical competencies

## Definition

Mathematics curriculum evaluation is the process of collecting and analyzing data with the purpose of making decisions about whether to keep, modify, or completely change a mathematics curriculum or some of its components.

## Notions and Meanings

Though the definition above provides a sense of what *mathematics curriculum evaluation* means, the fact is that because of evasive meanings of the terms involved, it is difficult to adopt one agreed-upon definition. Defining *mathematics curriculum evaluation* draws on the more general concepts of *curriculum* and *curriculum evaluation*, taking into consideration the specific characteristics of mathematics as a discipline.

### Curriculum

Historically, the term *curriculum* has been used in different meanings, including one or more of the following: goals and objectives determining the expectations of learning that are set by policy makers, textbooks used to guide teaching, instructional methods, plan of experiences, and/or actual experiences that learners go through in order to reach the specified learning goals. Larger meanings of *curriculum* include, in addition, the pedagogical framework or philosophy underlying the teaching practices and materials, training programs for supporting teachers, and/or guidelines for assessing students' learning. There is, however, a wide agreement that a curriculum may not be limited to a syllabus or list of topics set for teaching and learning.

The different processes involved at any point in the design, development, and implementation of a curriculum affect the ways the intentions of the curriculum are conceptualized, actualized, and implemented (Stein et al. 2007). As a result, educators distinguish different manifestations of a curriculum. Bauersfeld (1979) introduced the distinction between three entities, *the matter meant, the matter taught*, and *the matter learnt*, the first referring to the expectations set for learning mathematics, usually reflected in official documents such as a curriculum plan, standards, and/or textbooks; the second referring to the curriculum as taught and actualized by teachers through their classroom practices; and the third referring to what is actually learned by students. This distinction has later been used under different names and sometimes with added curriculum manifestations. The International Association for the Evaluation of Educational Achievement (IEA) used the names *intended, implemented*, and *attained curricula*, which have subsequently been widely used in mathematics education (e.g., Akker 2003; Cai 2010). The *assessed* curriculum came to be added to the threesome, to refer to the contents and mathematical processes that are addressed in assessments such as achievement tests.

Akker (2003) identifies two more specific aspects for the *intended* curriculum, which are the *ideal* curriculum (philosophical foundations) and the *formal/written* curriculum (intentions as specified in curriculum documents); two for the *implemented* curriculum, the *perceived* curriculum (interpretations by users, e.g., teachers) and the *operational* curriculum (as enacted in the classroom); and two for the *attained* curriculum, the *experiential* curriculum (learning experiences as pupils perceive them) and the *learned* curriculum (achieved learner outcomes).

### Curriculum Evaluation

This complexity and the manifold nature of the notion of *curriculum* make it even more difficult to capture the notion of *curriculum evaluation.* It is frequently found in implicit or informal forms, inherent to making decisions about daily teaching practices, interpretations of students' results on tests, and actions of developing or supplementing teaching materials. Such actions may be taken by individuals (e.g., teacher, school principal) or groups (e.g., teachers in a math department, parents, employers). More explicit and formal aspects of evaluation are adopted when decisions need to be made about more general curriculum components at the institutional or national level (e.g., school board, educational committees, Ministry of Education). With such actions, "there is a need to convince the community, educators, teachers, parents, etc." (Howson et al. 1981), hence the need for explicit and evidence-based curriculum evaluation.

Curriculum evaluation always has, to various extents, dimensions of institutional, social, cultural, and political nature. Designing, developing, implementing, and evaluating a curriculum involve different actors and are affected by social, economical, and political forces as well as by different cultural groups in the community. This is, for instance, made clear in Artigue and Bednarz (2012) where the authors compare the results of several case studies of math curriculum design, development, and follow-up in some French-speaking countries or regions, namely, Belgium, Burkina Faso, France, Québec, Romand Switzerland, and Tunisia, using as a filter the notion of *social contract* due to Rousseau. The social contract considered here is determining, explicitly but also partly implicitly, the relationships between school and nation (or region), by fixing the authorities and obligations of the different institutions involved in the educational endeavor, the rights and duties of the different actors, as well as the respective expectations.

Though the terms *evaluation* and *assessment* are sometimes used interchangeably, their meanings came gradually to be more precisely defined and distinguished. Niss (1993) refers to the Discussion Document of the 1990 ICMI study on *Assessment in mathematics education and its effects* to highlight this distinction: "Assessment in mathematics education is taken to concern the judging of the mathematical capacity, performance and achievement – all three notions to be taken in their broadest sense – of students whether as individuals or in groups (...). Evaluation in mathematics education is taken to be the judging of educational or instructional systems, in its entirety or in parts, as far as mathematics teaching is concerned." (p. 3).

Evaluation is often perceived as an integral phase of the curriculum development process seen as a cycle. Sowell (2005) identifies four phases: (1) planning, that is, determining curriculum aims and objectives, naming the key issues and trends as global content areas, and considering the needs; (2) developing curriculum content or subject matter according to specific criteria or standards; (3) implementing, through teaching strategies that convert the written curriculum into instruction; and (4) evaluating, based on criteria that help in identifying the curriculum's strengths and weaknesses.

When a curriculum evaluation action is to be taken, the complexity of the curriculum, its numerous components and actors involved, leads to raising many questions as to the aspects to be evaluated, for example, the quality of textbooks, students' learning, teaching practices, and consistency between specific components. For evaluating these different aspects, different techniques, tools, and instruments are needed. Other questions would be about the criteria on which to base the evaluation. Talmage (1985) identified five types of "value questions" to be considered for the evaluation of a curriculum: (a) the question of intrinsic value, related to the appropriateness and worth of the curriculum; (b) the question of *instrumental* value, related to whether the curriculum is achieving what it is supposed to achieve, and concerned with the consistency of the program components with its goals and objectives and with its philosophical or psychological orientation; (c) the question of *comparative* value, asked when comparing a new program to the old one or comparing different curricula; (d) the question of *idealization* value posed throughout the delivery of the new program and concerned with finding ways to make the program the best possible; and (e) the question of *decision* value asking about whether to retain, modify, or eliminate the curriculum.

Particularly, the concept of *curriculum alignment* is used in many sources and evaluation studies (e.g., Romberg et al. 1991; Schmidt et al. 2005; Osta 2007). According to Schmidt et al. (2005), alignment is the degree to which various "policy instruments," such as standards, textbooks, and assessments, accord with each other and with school practice. Curriculum alignment may also be defined as the consistency between the various manifestations of a curriculum: the intended, the implemented (also called enacted), the assessed, and the attained curriculum. Porter (2004) defines curriculum assessment as "measuring the academic content of the intended, enacted, and assessed curricula as well as the content similarities and differences among them. (...) To the extent content is the same, they are said to be aligned" (p. 12).

Alignment is also referred to as *curriculum coherence*. The term coherence received more attention with the studies motivated by TIMSS results, especially in the USA. Schmidt and Prawat (2006) claim that the term *curriculum coherence* was defined as *alignment* in most of the studies that were conducted before the release of TIMSS results in 1997. In their study on "curriculum coherence and national control of education," several types of alignment were measured: "Alignment between content standards and textbooks, alignment between textbooks and teacher coverage, and alignment between content standards and teacher coverage" (p. 4). Globally, a curriculum is said to be coherent if its components are aligned with one another.

Evaluation may be *formative* or *summative*. Formative evaluation takes place during the process of development of the curriculum. It includes pilot studies of teaching units, interviews with teachers, and/or tests to assess students' learning from those units. Its aim is to adjust the process of development based on the results. Procedures used for formative evaluation are usually informal, unsystematic, and sometimes implicit. Summative evaluation is conducted to determine the worth or quality of a curriculum that is completely developed and implemented. Its main purpose is to make decisions about the continuation, alteration, or replacement of the curriculum or some of its components.

## Models of Mathematics Curriculum Evaluation

Many types of activities conducted throughout the years, in formal and/or informal ways, in different regions of the world, have aimed at the evaluation of mathematics curricula. Such activities contributed to shaping the meaning of math curriculum evaluation as used today and to the development and refinement of techniques and instruments used. As this process evolved in different places of the world and in different societies and communities, different models emerged that may be distinguished by their level of formality, the level of rigidity of the tools or instruments they use, and the scope of factors and actors they involve in the analysis. The following examples may provide a sense of these differences:

Since the first large-scale projects of curricular reform and evaluation in the USA and other Anglo-Saxon societies, the experiences in mathematics curriculum evaluation tended toward more and more systematization and control by sets of criteria and detailed guidelines. Guides for curriculum evaluation are abundant. In their guide for reviewing school mathematics programs, for example, Blume and Nicely (1991) provided a list of criteria that characterize "an exemplary mathematics program" (p. 7), which should systematically develop mathematical concepts and skills; be sequential, articulated, and integrated; help students develop problem-solving skills and higher-order thinking; encourage students to develop their full potential in mathematics; promote a belief in the utility and value of mathematics; relate mathematics to students' world; use technology to enhance instruction; and be taught by knowledgeable, proficient, and active professionals. The guide then provides rubrics that help in determining the extent to which each one of those criteria is met by the mathematics curriculum under evaluation. Similarly, Bright et al. (1993) insist, in their "guide to evaluation," on the importance of examining the quality of curricula in a systematic and an ongoing way, based on selected criteria. For specific aspects of mathematics – problem solving, transition from arithmetic to algebra, materials for teaching statistics, and manipulative resources for mathematics instruction – the guide provides ways to focus the evaluation, pose evaluation questions, collect and analyze data, and report results.

Other models of math curriculum evaluation use more flexible approaches that take into consideration the rapport that the different actors (teachers, principals, educational authorities, etc.) have with the curriculum. For instance, the curricular reform in Québec, started in 1995 and presented by Bednarz et al. (2012), is qualified by these authors as a *hybrid* model, characterized by its long-term span, the involvement of actors with different perspectives, creating multiple interactions among them, and the involvement of teachers and school personnel. The evaluation model presented is formative and rather informal, regulated by the roles assigned to the actors, and perceiving the curriculum as being in continuous development, according to the experiences lived by different groups of practice. Concurrently, programs for raising teachers' awareness of the major directions and principles are created, aiming at teachers' appropriation of, and adherence to, a curriculum that is "alive" and open to debate.

The examples above show the richness and complexity of tasks of curriculum evaluation. They also show that these tasks cannot be separated from the culture and the characteristics of societies in which they emerge and develop.

## Mathematics Curriculum Evaluation and Large-Scale Reforms

The notion of *mathematics curriculum evaluation* has been, since its first-known instances in the history of mathematics education, associated with major reforms in mathematics contents, teaching materials, and methods. When stakeholders, decision makers, governmental or nongovernmental agencies, educators, or mathematicians start questioning mathematics teaching practices and materials currently in effect, actions are usually undertaken for evaluating their worth and developing alternative programs, which in turn call for evaluation.

Following are briefly some of the major landmark reforms and evaluation initiatives that had a considerable international impact.

The 1960s witnessed the wave of *New Mathematics* curricula, based on the Bourbakist view of mathematics. *New Math* programs were worldwide taught in schools in most countries. They resulted in a proliferation of textbooks to support instruction. They were also paralleled with large projects for piloting those textbooks as they were developed, especially in the USA (e.g., SMSG, School Mathematics Study Group) and in the UK (e.g., SMP, School Mathematics Project). Those projects resulted in a considerable body of research, widely disseminating a culture of evidence-based evaluation of mathematics curriculum materials. But serious problems of credibility and validity were raised, since many of the evaluative studies were conducted by the same groups which participated in the development of the curriculum materials. SMSG, for example, undertook a large enterprise of curriculum development and conducted a large-scale evaluation in the context of the National Longitudinal Study of Mathematical Abilities (NLSMA). The NLSMA study adopted a model that was based on two dimensions of analysis. The first is by *categories of mathematical content* (number systems, geometry, and algebra), and the second is by *levels of behavior* (namely, computation, comprehension, application, and analysis). Such two-entry model will later, with different extents of modification, guide many of the mathematics curriculum evaluation studies around the world.

According to Begle and Wilson (1970), the major research design adopted for the pilot studies was the experimental design, by which student achievement in experimental classes, where the tested materials were used, was compared to achievement in control classes that used "traditional" materials. Two types of tests were used and administered to both groups, standardized tests and tests to evaluate mathematical knowledge according to the new math content. Major concerns about the validity of those comparisons were raised, especially because they use, with both groups, tests developed to assess the learning of the new content, which privileged the experimental group. The use of standardized tests was also contested, as these only provide scores which don't uncover the real learning problems, and which focus on recalling information and computation skills rather than mathematical thinking.

During and after their implementation, New Math curricula motivated debates and evaluation actions, formal as well as informal, in various parts of the world, because of their elitism and extreme mathematical formalism and because of the difficulties faced by teachers who were not prepared to cope with them. Most of those evaluation actions were motivated by the two opposing positions that arose in the mathematics education community. While one position advocated the *New Math* curricula as improving student learning, the other maintained that they were causing a drastic loss of students' basic mathematical skills.

Other landmarks that motivated many studies for evaluating mathematics curricula worldwide were the NCTM's Standards (NCTM 1989, 1995). These documents were influential, not only in the USA but in the conception of mathematics curricula in many other countries. Many research studies were conducted that tried to evaluate the alignment of mathematics curriculum materials and textbooks with the Standards.

The beginnings of the twenty-first century witnessed a new wave of calls for reform, characterized by increase of state control, core requirements, and systematic evidence-based evaluation of mathematics curricula, because of the international assessments and studies. An extensive body of worldwide research for evaluating mathematics curricula was motivated by the Third International Mathematics and Science Study (TIMSS), later known as Trends in International Mathematics and Science Study, conducted since 1995 on a regular 4-year cycle, and the Program for International Student Assessment (PISA), conducted since 2000 on a regular 3-year cycle. Many of those studies used the rich cross-national data to compare and evaluate participating countries' curricula. Schmidt et al. (2005) advocated that "the presence of content standards is not sufficient to guarantee curricula that lead to high-quality instruction and achievement" (p. 525). The lack of coherence between the *intended* and the *enacted* curricula was found to be one of the main reasons for relatively low scores in international comparative tests. Houang and Schmidt (2008) present the 1995 TIMSS ICA (International Curriculum Analysis) cross-national study which "captures" the curriculum from the participating countries, using the tripartite model of curriculum: the intended, implemented, and attained curricula. The study established methodological procedures and instruments to encode curriculum documents and textbooks (Houang and Schmidt 2008).

As a reaction to the results of international assessments in mathematics and science (TIMSS and PISA), we see many countries tending to more standardization and centralization in their math curricular procedures and practices. Central governments are taking more and more control in countries where more freedom and authority used to be left to states, districts, cantons, or even smaller communities. The concern of accountability of educational systems and the pressure of international assessments are prevalent. A compulsory common core is imposing itself as a solution in countries where no central curriculum was adopted before. For example, many USA states have already started implementing the Common Core State Standards (CCSS 2010).

The international assessments, especially PISA, motivated, on the other hand, an increasing trend in many countries toward designing mathematics curricula, according to a set of mathematical competencies, to be used for student learning assessment. This influence is made clear in the study by Artigue and Bednarz (2012) already mentioned. In Denmark as well, the eight mathematical competencies set by the KOM project (Niss 2003) aiming at an "in-depth reform of mathematics education" are very close, almost identical for some, to the PISA framework's cognitive competencies.

The increase of governmental control and the rise of calls for evidence-based judgments of educational systems' performance, added to the increasing pressure of the international assessments, are expected to motivate new waves of curriculum monitoring and evaluative procedures. Crucial questions and new problems will be awaiting investigation. Particularly the rise of the "evaluation by competencies" trend for assessing students' learning will lead to changes in the ways the evaluation of mathematics curricula is approached. These changes will raise new types of research questions and create a need for rethinking the different techniques, categories, and criteria used for mathematics curriculum evaluation.

## Cross-References

▶ Competency Frameworks in Mathematics Education
▶ Curriculum Resources and Textbooks in Mathematics Education
▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes
▶ Mathematics Teachers and Curricula
▶ Technology and Curricula in Mathematics Education

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## Mathematics Teacher as Learner

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### Keywords

Teacher as learner; Design-based learning; Sources and strategies; Reflection and enaction; Individual learning versus institutional learning

### Rationale

Analogous to mathematics power as goal for student learning, mathematics teachers learn to increase their pedagogical power of identifying challenges in a specific classroom environment and properly applying strategies to solve those challenges. Nurturing the power requires a complex and lifelong learning process through which teachers gradually go beyond themselves as they dig into the essences of mathematics learning and have ability to structure lessons for students to experience the learning accordingly.

Research on teacher learning can be generally categorized into two trends. One is tied to person, inheriting the research in psychology. Fuller (1969) conceptualized teacher concern into three major phases: nonconcern, concern with self, and concern with pupils. Clarke and Hollingsworth (2002) elaborated teacher growth as a nonlinear and interconnected learning process involving personal attributes, teaching experimentation, perception of professional communities, and the observation of salient outcomes.

Another trend originates from Vygotsky's work, focusing on interpersonal relationships and identities in teaching and learning interactions as well as the modes of thinking linked to forms of social practices. Learning inherently is viewed as increasing participation in socially organized practices (Lave and Wenger 1991). The conception, Zone of Proximal Development (ZPD), is also adopted to describe teacher learning in relation to the social setting and the goals and actions of tiers of participants (e.g., Goos and Geiger 2010). Additionally, Putnam and Borko (2000) combined both psychology and sociocultural perspectives, stating that teacher learning involves a process of enculturation and *construction*, which can be investigated by lines of research with roots in various disciplines (e.g., anthropology).

Reflection and enaction have been treated as crucial and inseparable mechanisms for teacher growth. Reflective thinking instead of routine thinking can effectively help teachers to overcome challenges (Dewey 1933). The distinction between reflection-in-action and reflection-on-action further presents how both mechanisms interact and lead to the learning (Schön 1983). Specifically, the power of institutional learning where school teachers work together as a term for their growth should be highlighted because school-based environments entail the norms and rationality for teachers to frequently implement new ideas into teaching practices and have ample opportunities to learn from each other in their daily-life teaching.

### Sources and Strategies

A variety of sources and strategies have been proposed to facilitate teacher learning. Narrative cases offer teachers opportunities to situate their teaching for detecting and challenging the pedagogical problems. Analyzing mathematics tasks allows teachers to evaluate cognitive complexity of the tasks, converse the tasks into lesson structures, and properly enact them with students in class. Research findings can be materials as well to facilitate teachers' understanding of students' cognitive behaviors and improve the teaching quality. Strategies such as peer coaching or lesson study also make possible the learning of teachers by observing and analyzing peers' teaching experiences.

Of importance are the design-based professional development programs in which teachers can learn from educators, peer teachers, and students. Design-based approach has the capacity of encompassing all strengths for the facilitation of teacher learning listed above. By participating in designing tasks, teachers actively challenge the pedagogical problems that they concern. Designing tasks and enacting them with students also develops teachers' competence in coordinating experiences from different learning environments into the refinement of the tasks and the teaching. Particularly, as any of the existing instructional materials (e.g., test items) can be the sources to initiate new designs for promoting students' active thinking, this strategy is powerful to engender the ongoing learning journeys of teachers.

## Teacher Learning Theory

Theories of student learning have been used to construct models and frameworks to facilitate teacher learning. Nevertheless, fundamental theories for teacher learning have not been well established yet. In light with the perspective viewing mathematics as the core for the learning of educators, teachers, and students (Mason 2008), it is particularly important to develop teachers' mathematical pedagogical thinking, the notion created by making analogy to mathematical thinking, and use the pedagogical thinking as principles to solve teachers' teaching problems (e.g., the use of specializing and generalizing thinking for probing students' error patterns across different mathematics topics).

## Cross-References


- ▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes
- ▶ Models of In-Service Mathematics Teacher Education Professional Development
- ▶ Professional Learning Communities in Mathematics Education
- ▶ Teacher as Researcher in Mathematics Education


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# Mathematics Teacher Education Organization, Curriculum, and Outcomes


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## Keywords

Preservice teacher education; In-service teacher education; Professional teacher's competences; Mathematics content and mathematics pedagogy content knowledge; Concurrent and consecutive study programs; Teaching practicum; Mathematics and teacher educators


## Characteristics

Tatto et al. (2010) stated: "We know little about the organization of the opportunities to learn mathematics and mathematics pedagogy offered to prospective and practicing teachers across the world and their relative effectiveness." (p. 313). The quote comes from a paper based on reports from 20 participating countries collected as part of the 2005 Conference of the International Commission on Mathematics Instruction (ICMI-15) (see Tatto et al. 2009). Since then the Teacher Education and Development Study in Mathematics or TEDS-M* (see Tatto et al. 2012) was implemented in 2008 to begin to answer such questions.

In the 7 years between the ICMI-15 and the TEDS-M studies, the education of teachers has become an important policy issue. While we know more about the structure and characteristics of teacher education, the image that emerges is one of increased complexity. On the one hand, there are efforts by supranational institutions (e.g., European Union) to unify the system of teacher education, while on the other hand, countries and regions under the influence of globalization forces struggle to implement fast-paced reforms that may threaten or end up reaffirming more traditional teacher preparation systems. The fast development of information technologies, growth of multiculturalism, economic development, and globalization – all these place a great deal of pressure on education systems and also on teacher education. Educators, politicians, sociologists, as well as the general public all over the world ask the same questions: what skills, knowledge but also attitudes, and values should be passed on to the new generation? How can children, young people, and their teachers be prepared for what they can expect in their future everyday life and career? More specifically regarding teachers, what are the characteristics of teacher education programs that can prepare their graduates effectively for what is now needed? How can the outcomes of teacher education programs for teachers of mathematics be measured in ways that are reliable and valid? What kinds of policies are effective in recruiting qualified teachers of mathematics from diverse backgrounds?

In contrast to the above-quoted studies, this text is an encyclopedia entry which only outlines the main ideas but can never be exhaustive. The reader is advised to consult the sources we cite here and other relevant sources to obtain more exhaustive information on a whole range of questions concerning mathematics teacher education.

## Institutions

The range of institutions preparing future teachers is large and includes secondary as well as tertiary schools (universities, national teacher colleges, both public and private). In some countries it is also possible to read a course in mathematics and, only after having graduated and having made the decision to teach, to take a course in pedagogy and pedagogical content designed for in-service teachers who lack pedagogical education.

In many countries teachers can also achieve credentials in practice (such as the notable *Teach for America* program in the US and its variants now making inroads in many other countries). In some countries it is possible to begin to teach without a proper teacher credential, but the situation is changing rapidly.

In some countries preservice and in-service teachers can also attend distance courses (increasingly offered on-line), usually organized by universities. They may be attended either by in-service unqualified teachers or by in-service teachers who make the decision to extend their qualification by another subject. They may also be selected by anybody else who is working elsewhere but wants to prepare for the teaching profession.

In-service training is necessary also for practicing teachers who have already achieved credentials but want professional development and support. In many countries these development programs are supported by the government and authorities as it is understood that in the teachers' professional lifetime, they cannot be expected to teach the same contents using the same methods (see Schwille and Dembélé 2007; Tatto 2008). Just as doctors are expected to follow the newest trends and technologies, teachers must be expected to keep up with the latest developments, both in content and pedagogical content. That is why some countries financially motivate their teachers to develop, offering better salaries to those who are willing to learn by engaging in further study. It is also true that many in-service teachers welcome the possibility for further training as it gives them confidence, self-esteem, motivation, and a feeling of belonging to a professional community.

### Study Programs

In general, there are two possibilities of organization of teacher education. Programs may be concurrent which means that the preservice teacher takes at the same time mathematics, didactics, and general pedagogy seminars and lectures of. This system is sometimes criticized because it may fail to provide future teachers with in-depth content knowledge, considered as a prerequisite to mastering teaching methods, and by an overly formal pedagogical training. The other possible model is consecutive, which means that the preservice teachers first study the content and only subsequently methodology, psychology, and pedagogy. This may work well if it does not result in neglect of pedagogy and pedagogical content knowledge, which is sometimes the case especially among preservice teachers for secondary schools. This also depends on who teaches the future teachers, which will be discussed later.

The advantage of some consecutive programs is that it enables the structuring of university studies to include a bachelor's and master's degree, where the preservice teachers spend their time in the bachelor's studies focusing only on mathematics, and the master's course focusing on the study of pedagogical content knowledge. This organization may be a way of preventing recent reform efforts emerging in some countries to shorten the study time of preservice preparation (e.g., to 3 years), claiming that a bachelor's degree is sufficient to become a teacher.

The preparation of primary school teachers, on the other hand, tends to be concurrent as the general belief is that teachers for this stage should be real experts in pedagogical disciplines. The scope of subjects future primary school teachers study often results on superficial knowledge across all the disciplines.

Teacher education programs typically include teaching practice or practicum which may take various forms. It may be one semester spent on an affiliated school supervised by an accredited practicing teacher. It may include a couple of hours a week for a longer period of time. Or it may be few years following graduation, the so-called induction, when the fresh teacher is supervised and supported until he/she gets more experience of classroom and school practice (see Britton et al. 2003). This part of teacher education is considered very important under the assumption that only hands-on experience and advice of an experienced practitioner would enable mastering the necessary skills and that theoretical knowledge, albeit of pedagogical content and pedagogy, will never make a complete teacher.

### Who Teaches Future Teachers?

For the most part, future teachers of mathematics are taught by mathematicians, mathematics educators (usually with a degree in mathematics and pedagogy), and teacher educators. In practical experience, future teachers are often supervised by experienced practitioners. Comprehensive teacher education requires the combination of all these aspects.

Countries that offer in-service teacher professional development sometimes organize them outside university walls in various kinds of pedagogical centers. They hire trainers (from pedagogical centers, experience practitioners, etc.) to deliver different seminars and courses. One must stress that even these trainers must be trained too. The value of trainer training through formal programs of professional development and support has emerged as an area of concern. It may seem strange, even unnecessary, to suggest that the training of trainers ("trainer education" or "formal professional development" for trainers) needs to be justified. But while the value of the professional development opportunities for teacher educators is significant, it is rarely done or documented. If in-service teachers report the need of growing self-esteem, the team spirit, it would follow that the same must apply to teacher trainers and educators. While the academic world of universities and many international conferences and projects offer university teacher educators the chance to grow, develop, exchange information, and cooperate at the international level, teacher trainers still need other more formal avenues of professional development.

### Who Enters the Profession?

The study programs offered by universities and national teacher preparation institutions may be selective or nonselective. This means that some institutions require from their participants to pass entrance exams or to have passed certain school leaving exams at the secondary school level. We have no knowledge that the candidates would be asked to pass any aptitude tests to show their *predisposition* for the profession in any countries although they are asked to demonstrate academic proficiency in the disciplines. It is a question whether or not it would help the education systems if only candidates of certain skills and talents were accepted to study education programs. It would definitely not be easy to specify which predispositions are essential for success in future work with pupils.

In case of nonselective admittance to universities, personal choice is what matters, but even if admittance is restrictive, only people with talent for the subject are likely to enrol. The problem in many countries is that teaching is not the most glamorous career, the job is poorly paid and the reputation of teachers is low. The unfortunate consequence then is that education programs are entered only by those candidates who failed in other entrance exams to more demanding and desirable fields of study.

The TEDS-M study found that different countries' policies designed to shape teachers' career trajectories have a very important influence on who enters teacher education and eventually who becomes a teacher. These policies can be characterized as of two major types (with a number of variations in between): career-based systems where teachers are recruited at a relatively young age and remain in the public or civil service system throughout their working lives and position-based systems where teachers are not hired into the civil or teacher service but rather are hired into specific teaching positions within an unpredictable career-long progression of assignments. In a career-based system there is more investment in initial teacher preparation, knowing that the education system will likely realize the return on this investment throughout the teacher's working life. While career-based systems have been the norm in many countries, increasingly the tendency is toward position-based systems. In general, position-based systems, with teachers hired on fixed, limited-term contracts, are less expensive for governments to maintain. At the same time, one long-term policy evident in all TEDS-M countries is that of requiring teachers to have university degrees, thus securing a teaching force where all its members have higher education degrees. These policy changes have increased the individual costs of becoming a teacher while also increasing the level of uncertainty of teaching as a career.

### Professional Teacher's Competences

What skills, abilities, knowledge, and attitudes should graduate of teacher preparation programs master? For a long time, designers of teacher preparation programs have struggled to balance the theoretical with the practical knowledge and skills (Ball and Bass 2000). However, there is no consensus on the proportion of the different teacher preparation "ingredients."

It is clear that a good teacher of mathematics must understand more than the mathematical discipline. They must master other skills in order to be able to plan and manage their lessons, to transmit knowledge, and especially to facilitate their pupils' learning. They must get introduced to various types of classroom management (whole class, group work, pair work, individual work) and understand the advantages and disadvantages in different activities; they must learn how to work with pupils with specific learning needs and problems and how to work with mixed-ability classes to answer the needs of the talented as well as below average students. They must learn to pose motivating and challenging questions, learn how to facilitate pupils' work, must be aware of the difference in pupils' learning styles, and must be experts in efficient communication and appropriate language use.

They must be able to work with mistakes. They must also know the demands in the output, what the pupils will be expected to master, and in what form they will be expected to show their knowledge and skills. They must be able to mediate the increasing demands for excelling in examinations and developing deep and relevant learning. They should be able to manage the development and the administration of summative or formative assessments to inform and plan their teaching; they should be able to understand the advantages of each of these types of assessments (Even and Ball 2009). These of course cannot be acquired in purely mathematical courses and preservice teachers must undergo more extensive preparation.

According to Shulman (1987) the knowledge that teachers must master consists of content knowledge (in this case mathematics), pedagogical content knowledge (didactics and methodology of the studied subject, the ability to act adequately directly in the course of lessons) and pedagogy (philosophy of education, history of education, educational psychology, sociology of education), knowledge of pupils, and knowledge of context. In several studies, knowledge, beliefs, and attitudes toward mathematics and practical skills are highlighted (see, e.g., Nieto 1996).

Whatever classification or division we choose, the fact remains that it is at this point impossible to give one answer to the question of how much time and attention should be paid to each of the components. The problem is that it is impossible to state objectively which part of this knowledge makes a really good teacher. In general terms it can be said that usually future primary school teachers get much more training in pedagogy and psychology, while future secondary school teachers get more training in the mathematics itself. The problem of the first situation is the lack of the teacher's knowledge of mathematics which often results in lack of self-confidence. Unaware of the underlying mathematical structures, the teacher may be hardly expected to identify the sources of pupils' mistakes and misconceptions, let alone correct them. Primary school teachers report that this lack of self-confidence in the discipline prevents them from adequate reactions to their pupils' questions and problems. If it is true that mathematics that has already been discovered is "dead" mathematics and is brought to life by teachers (Sarrazy and Novotná to be published), the teacher must know it and be able to assist in this rediscovery.

In contrast, if teachers are not trained adequately in pedagogy and pedagogical content knowledge, they may fail to pinpoint the sources of their pupils' problems as they may be related to their cognitive abilities, age, and methods used in lessons, among others.

The problem with mathematical content knowledge is that there is wide disagreement regarding the extent and depth of the mathematical content pupils should be taught to make use of in their future life. If there is disagreement regarding what pupils need to know, there is also disagreement on the mathematics their teachers need. The current trend emphasizing transversal-horizontal skills (learning to learn, social competences, cross-curricular topics) seems to put more emphasis on everything but the mathematical content. However, there is no doubt that pupils must learn also mathematics as they will be using it in many everyday situations in their future. Calculators and computers will never really substitute human mathematical thinking.

This problem of lack of agreement of what mathematics to teach and how much of it to teach well known to those involved in mathematics education at all levels. One of the strategies recently introduced to solve this problem is the development of content standards currently implemented in a considerable number of countries. They might differ in form, in the degree of obligation, and in the level of details included, but they certainly share one characteristic: they define the framework for the volume of mathematics that teachers will have to teach and consequently the bases for the mathematical content to be included in the teacher education curriculum.

The TEDS-M study shows that there are topics and areas that can be found in the curricula of teacher education programs in a considerable number of countries and may therefore be regarded as the cornerstones of mathematics education. These topics are numbers; measurement; geometry; functions, relations, and equations; data representation, probability, and statistics; calculus; and validation, structuring, and abstracting. The opportunity to learn these topics varies according to the grade levels future teachers are prepared to teach with primary teachers predominantly studying topics such as numbers, measurement, and geometry. As programs prepare teachers for higher grades, the proportion of areas reported as having been studied increases. Importantly TEDS-M found that the Asian countries and other countries whose future teachers did well on the TEDS-M assessments did offer algebra and calculus as part of future primary and lower secondary teacher education (see Tatto et al. 2012).

## Cross-References


▶ Communities of Inquiry in Mathematics Teacher Education
▶ Communities of Practice in Mathematics Teacher Education
▶ Education of Mathematics Teacher Educators
▶ Mathematics Teacher as Learner
▶ Mathematics Teachers and Curricula
▶ Models of In-Service Mathematics Teacher Education Professional Development
▶ Teacher Education Development Study-Mathematics (TEDS-M)


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## Further Readings


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## Mathematics Teacher Educator as Learner

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### Keywords

Action research; Intervention research; Lifelong learning; Professional development; Reflection; Role model; Teacher education

### Mathematics Teacher Educators: Definition

Mathematics teacher educators in postsecondary institutions are academics educating prospective or practicing teachers; in many cases they do both. Thus, teacher educators initiate, guide, and support teacher learning across the teacher's professional lifespan (see also the entry "► Education of Mathematics Teacher Educators" and, Even and Ball 2009). Most teacher educators have the task not only to teach (and to evaluate their teaching) but also to do research (including systematic and self-critical evaluation) and to do organizational administrative work. The quality of teaching, research, and organization is based on teacher educators' attitude towards and competence in continuous learning. The more complex teacher education activities are (e.g., running a challenging master's program or leading a professional development program for a couple of schools), the more the components of teaching, research, and organization are interwoven and influence each other.

Since teachers also have the task to teach, to critically reflect on their work (and maybe to do or be involved in research), and to do administrative work, observing teacher educators' actions may serve as a learning opportunity for teachers. Thus, teacher educators can be seen as role models for teachers. This makes teacher education a complex endeavour (see Krainer and Llinares 2010) since a serious teacher educator needs to live the goals he or she is claiming to his or her participants: it would be inconsistent and an obstacle for the learning process if, for example, a teacher educator stresses students' active learning but mainly designs his or her courses in a way where passive learning (listening to lectures) is dominating. This affords teacher educators to reflect the (explicit or implicit) "learning theory" underlying their teaching and – in best case – to make it transparent and discussable in the teacher education process. One of the challenges of teacher educators is to create genuine learning situations for teachers, often through carefully designed tasks, in which teachers experience as learners the kind of learning that the mathematics teacher educator wishes to convey (Zaslavsky 2007).

#### Mathematics Teacher Educators' Learning Through Research

Research on teacher educators' learning as practitioners is sparse, however increasing (see, e.g., in general: Russell and Korthagen 1995; Cochran-Smith 2003; Swennen and van der Klink 2009; directly related to mathematics teacher education: Zaslavsky and Leikin 2004; Even 2005; Jaworski and Wood 2008) with growing interest in the mathematics education community evidenced by discussion groups in recent international mathematics education conferences (e.g., PME 35 proceedings and ICME 12 preconference proceedings). Most opportunities for teacher educators to learn are not offered as formal courses. Such formats are discussed in the entry "▶ Education of Mathematics Teacher Educators." The emphasis here is on teacher educators' autonomous efforts to learn, in particular, through reflection and research on their practice.

Teachers' ability to critically reflect on their work is a crucial competence (see, e.g., Llinares and Krainer 2006). Teacher educators need to evoke this inquiry stance (link to entry "▶ Inquiry-Based Mathematics Education") of teachers as a basis of their learning. From this perspective, teacher educators learn from their practice through ongoing reflection on their thinking and actions as an inherent aspect of their work with teacher (i.e., as reflective practitioners – Schön 1983) and/or through systematic, intentional inquiry of their teaching in order to create something new or different in terms of their knowledge, "practical theories" (see Altrichter et al. 2008, pp. 64–72), and teaching. However, this dual role of researcher and instructor when educators inquire into their own practice puts a special focus on the question of how teacher education and research are interwoven.

A survey of recent research in mathematics teacher education published in international journals, handbooks, and mathematics education conference proceedings (see Adler et al. 2005) claims that most teacher education research is conducted by teacher educators studying the teachers with whom they are working. Such studies could involve studying characteristics of their students or the instructional approaches in which they engage their students. This presents a challenging situation for educator-researchers who need to reflect on their dual role to guard against unintentional biases that could influence the outcome of the research and their learning. For example, "success stories" that dominate the research literature may suggest that teacher educators' learning generally involves situations that improve teachers' learning and knowledge. However, this could be explained at least by two reasons: such published research of teacher education projects might be planned more carefully than others, and the readability to publish successful projects is higher than to publish less successful ones.

In spite of this challenge, there are good reasons for teacher educators to study teachers' learning through their own courses and programs. In system theory it is taken for granted that we only have a chance to understand a system (e.g., teachers in a mathematics teacher education course) if we try to bring about change in this system. This means that trying to understand is important to achieve improvement, and trying to improve is important to increase understanding. However, the researcher needs to reflect carefully on the strengths and weaknesses of distance and nearness to the practical field being investigated. For example, telling a "rich story", taking into account systematic self-reflection on one's own role as a teacher educator and researcher in the process, being based on a viable research question and building on evidence and critical data-analysis, is an important means to gather relevant results in teacher education research.

## Mathematics Teacher Educators' Learning Through Action Research and Intervention Research

Action research and intervention research are two of the common methods mathematics teacher educators might engage in when conducting research as a basis of their learning. These methods allow them to investigate their own practice in order to improve it. This investigation process might be supported by other persons, but it is the teacher educators who decide which problem is chosen, which data are gathered, which interpretations and decisions are taken, etc. Action research challenges the assumption that knowledge is separate from and superior to practice. Thus, through it, teacher educators' production of "local knowledge" is seen as equally important as general knowledge, and "particularization" (e.g., understanding a specific student's mathematical thinking) is seen as equally important as "generalization" (e.g., working out a classification of typical errors).

"Intervention research" (see, e.g., Krainer 2003) done by teacher educators to investigate teachers' learning can take place in their classrooms influenced by interventions of their colleagues or often – as research shows – by their own interventions (e.g., see Chapman 2008) or in the field where it does not only apply knowledge that has been generated within the university, but much more, it generates "local knowledge" that could not be generated outside the practice. Thus, this kind of research is mostly process oriented and context bounded, generated through continuous interaction and communication with practice. Intervention research tries to overcome the institutionalized division of labor between science and practice. It aims both at balancing the interests in developing and understanding and at balancing the wish to particularize and generalize. Action research as intervention research done by practitioners themselves (first-order action research) can also provide a basis for teacher educators to investigate their own intervention practice (second-order action research, see, e.g., Elliott 1991).

Worldwide, there is an increasing number of initiatives in mathematics education based on action research or intervention research. However, most of them are related to teachers' action research (see, e.g., Chapman 2011; Crawford and Adler 1996; papers in JMTE 6(2) and 9(3); Benke et al. 2008; Kieran et al. 2013). In some cases, even the traditional role names (teachers vs. researchers) are changed in order to express that both, individual learning and knowledge production for the field, are a two-way street. For example, in the Norwegian Learning Communities in Mathematics (LCM) project (Jaworski et al. 2007), the team decided to replace "researchers and practitioners" with "teachers and educators" ("both of whom are also researchers"). There are a lot of projects in which teachers document their (evidence-based) experiences in reflecting papers. In Austria, for example, nearly 1,000 papers – written by teachers for teachers – have been gathered since the 1980s within the context of programs like PFL (see, e.g., Krainer 1998) and IMST (Pegg and Krainer 2008; Krainer and Zehetmeier 2013) and can be searched by key word in an Internet database (http://imst.ac.at). The most extensive and nationally widespread version of action research by teachers is practiced in Japan within the framework of "lesson study" (see, e.g., Hart et al. 2011). In general, teacher educators who participate directly or indirectly in such cases of teachers' action research are afforded opportunities to learn in and from these experiences.

## Cross-References

▶ Education of Mathematics Teacher Educators
▶ Inquiry-Based Mathematics Education

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# Mathematics Teacher Identity

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## Keywords

Teacher identity; Specialisation; Pedagogic identity

## Definition

Mathematics teacher identity (MTI) is commonly "defined" or conceptualized in recent publications of the mathematics education research community as ways of being, becoming, and belonging, as developing trajectories, and in narrative and discursive terms.

## Characteristics

### A Brief History

The concept of identity can be traced to Mead (1934) and Erikson (1968), the former seeing identity as developed in interaction with the environment, and thus multiple, though it appears more unified to the individual (Lerman 2012). The latter saw identity as something that develops throughout one's life and is seen as more unified. The study of teacher identity is more recent. Perspectives focus on images of self (Nias 1989) as determining how teachers develop or on roles (Goodson and Cole 1994). One can argue that societal expectations and perceptions and at the same time the teacher's own sense of what matters to them play key roles in teachers' professional identity. Beijaard et al. (2004) argue, in their review, that 1988 saw the emergence of teacher identity as a research field. Perhaps the key impetus for this focus on identity in mathematics education can be attributed to Jean Lave and Etienne Wenger who wrote, "We have argued that, from the perspective we have developed here, learning and a sense of identity are inseparable: They are aspects of the same phenomenon." (Lave and Wenger 1991, p. 115).

Research on teacher identity has gained prominence over the past decade. Special issues of teacher education journals focusing on teacher identity attest to this (e.g., *Teaching and Teacher Education* 21, 2005; *Teacher Education Quarterly*, June 2008). Related issues revealed across the literature include "the problem of defining the concept; the place of the self, and related issues of agency, emotion, narrative and discourse; the role of reflection; and the influence of contextual factors" (Beauchamp and Thomas 2009, p. 175). As Grootenboer and Zevenbergen (2008, p. 243) argue, in relation to understanding the learning of mathematics, identity is "a unifying and connective concept that brings together elements such as life histories, affective qualities and cognitive dimensions."

Despite increasing engagement with mathematical learning and identity, many have argued that researchers working with the notion of mathematics teacher identity have not clearly defined and operationalized the notion (e.g., Sfard and Prusak 2005). MTI is increasingly accepted as a dynamic rather than a fixed construct even while debates continue as to whether an individual has one identity with multiple aspects or multiple identities (see Grootenboer and Ballantyne 2010). These post-structuralist interpretations of identity are more "contingent and fragile than previously thought and thus open to reconstruction" (Zembylas 2005, p. 936). The agency of the teacher to reconstruct or reauthor her story in relation to participation within classroom, pre- and in-service practices in which teachers participate, is a central opportunity for much of the literature focused on identity in relation to mathematics teacher support (e.g., Hodgen and Askew 2007; Lerman 2012). In contexts where mathematics teacher morale is low and teacher identities are portrayed as mathematically deficient (supported by poor results on international studies such as TIMMS), this framework opens the space for teacher education to focus explicitly on the reauthoring of negative and damaging narratives (e.g., Graven 2012).

Clusters of issues illuminated in recent published mathematics teacher education research include:

- Discipline specialization is widely considered to be highly significant in teacher identity both generally and in mathematics teacher research specifically (Hodgen and Askew 2007). Relating teacher identity and teacher emotion is argued by some to be particularly important in relation to mathematics teacher identity where many teachers teach the subject without disciplinary specialization in their teacher training and with histories of negative experiences of learning mathematics within their own schooling (Graven 2004; Hodgen and Askew 2007; Grootenboer and Zevenbergen 2008; Lerman 2012). For other literature outside of mathematics education that connects teacher identity with researching teacher emotions, see the special issue no. 21 of *Teaching and Teacher Education* which focused on this relationship (e.g., Zembylas 2005).
- Research into mathematics teacher identities often deals separately with primary nonspecialist teachers, who teach across subjects, and with secondary teachers, who teach only or predominantly mathematics and who may or may not have specialized in mathematics in their preservice education. The nature of the way in which the discipline specificity of mathematics influences evolving teacher identities differs in relation to whether one is identified as a generalist teacher or a mathematics teacher. While it can be internationally accepted that many more secondary mathematics teachers have discipline specific training in their preservice studies, the extent to which this is the case differs across

countries. As Grootenboer and Zevenbergen (2008) point out, depending on the extent of the shortage of qualified mathematics teachers, secondary school mathematics classes are often taught by nonspecialist teachers. Shortage of qualified mathematics teachers can be particularly high for developing countries (for example in South Africa). In this respect research into supporting such teachers to strengthen their mathematics teacher identities becomes important. Graven (2004) engages with how teachers in a longitudinal mathematics education in-service program transformed their identities from accidental "teachers of mathematics" (they specialized in other subjects in their training) to "professional mathematics teachers" with long term trajectories of further learning and increasing specialization in the subject. Research also tends to deal separately with either preservice or in-service teachers as the way in which identities emerge for these groups of teachers differs in relation to the different practices in which they participate.

- Mathematics teacher identity is also increasingly being considered in relation to studies researching mathematics teacher retention. The ICME-12 Discussion Group (DG11) on teacher retention included as a key theme the notion of identity and mathematics teacher retention and several of the papers presented in this DG highlighted the role of strengthened professional identities, increasing sense of belonging and developing leadership identities and trajectories as enabling factors contributing to teacher retention. Presenters in this discussion group were from the USA, South Africa, Israel, New Zealand, Norway, and India. Research on mathematics teacher identity seems to be of particular interest in these countries as well as in the UK and Australia (see reference list). Similarly research into the relationship between teacher identity and sustaining commitment to teaching (more generally than only for mathematics teaching) has been argued across US and Australian contexts (e.g., Day et al. 2005).
- A growing area of research in MTI explores the relationship between mathematics "teacher change" or teacher learning and radical curriculum change (e.g., Schifter 1996; Van Zoest and Bohl 2005). Similarly research is beginning to investigate the relationship between teacher identity/positions and the increasing use of national standardized assessments across various contexts (Morgan et al. 2002). As in Lasky's (2005) research across subject teachers this research often points to disjuncture between mathematics teacher identities and expectations (and contradictory messages) of reform mandates and to the ways in which these constrain teacher identities. As Wenger (1998) notes, national education departments can design roles but they cannot design (local) identities of teachers.

In this respect the work of Bernstein becomes useful in providing a more macro perspective on teacher identity and the way policy, curriculum, and assessment practices shape this. His work complements localized case study analyses of identity within teacher communities with a broader concept of identity connected to macro structures of power and control. Bernstein's model (1996) shows:

> how the distribution of power and principles of control translate into pedagogic codes and their modalities. I have also shown how these codes are acquired and so shape consciousness. In this way, a connection has been made between macro structures of power and control and the micro process of the formation of pedagogic consciousness. (p. 37).

Bernstein first introduced the concept of identity in 1971 (Bernstein and Solomon 1999). This analysis did not focus on identity in terms of regulation and realization in practice but rather on identity in terms of the "construction of identity modalities and their change within an institutional level" (p. 271). Thus Bernstein approaches identity from a broader systemic level, which of course impacts on enabling and constraining the emergence of localized individual teacher identities. Bernstein's notion of "Projected Pedagogic Identities" (Bernstein and Solomon 1999) provides a way of analyzing macro promoted identities within contemporary curriculum change which is the context within which teacher roles are elaborated in curriculum documents. South African and British Mathematics Educators have particularly drawn on the work of Bernstein to analyze positions available to teachers within often contradictory and shifting "official" discourses (see, e.g., Parker and Adler 2005; Graven 2002; Morgan et al. 2002).

## Conclusion

The references seem to suggest that "identity" as an alternative way of identifying teacher learning is not necessarily a global perspective. Research on international interpretations of the relevance of the notion is needed. At the same time the notion is ubiquitous in the social sciences, and mathematics education researchers working with "identity" need to specify how they are using the term, what the sources are for their perspectives, and the relevance for the teaching and learning of mathematics.

## Cross-References

- ► Communities of Practice in Mathematics Teacher Education
- ► Teacher Beliefs, Attitudes, and Self-Efficacy in Mathematics Education

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## Mathematics Teachers and Curricula

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### Keywords

Collaborative work; Curricula development; Mathematics teachers; Production of materials; Text books; Professional development of teachers

### Definition and Historical Background

The word curriculum has had several meanings over time and has been interpreted broadly not only as a project about *what* should be learned by students but, in the context of teachers and curriculum, as all the experiences which occur within a classroom. These different meanings are grounded in different assumptions about teaching and the nature of interactions of the teacher with ideas that support curriculum guidelines (Clandinin and Connelly 1992). These different meanings have defined several roles of teachers in mathematics curriculum development. Regarding these roles, the relationship between teachers and curricula can be described as the history of a shift from teachers as curriculum users to teachers with roles as curriculum interpreters and/or curriculum makers. Whereas the former view assumes curricula to be "teacher-proof," the latter includes teachers' activities like reflecting, negotiating issues of curricula, and disseminating to their peers. This shift mirrors acknowledgement of the centrality of the teacher in curricula issues in particular (Clarke et al. 1996; Hershkowitz et al. 2002; Lappan et al. 2012) and viewing teachers as key stakeholders of educational change in general (Krainer 2011). These meanings are located along a continuum from a view of curricula as fixed, embodying discernible and complete images of practice, to a view of curricula guidelines as possible influencing forces in the construction of practice.

In the 1970s, Stenhouse (1975) defined curriculum as "an attempt to communicate the essential principles and features of an educational proposal in such a form that it is open to critical scrutiny and capable of effective translation into practice" (p. 4). The teacher is central to this translation into practice. A model that is commonly used for analysis in mathematics education sees curricula as located at three levels: the *intended curriculum* (at the system level, the proposal), the *implemented curriculum* (at the class level, the teacher's role), and the *attained curriculum* (at the student level, the learning that takes place) (Clarke et al. 1996).

Focusing on the implemented curriculum, Stenhouse began the "teachers as researchers" movement. He believed that the "development of teaching strategies can never be a priori. New strategies [principled actions] must be worked out by groups of teachers collaborating within a research and development framework [. . .] grounded in the study of classroom practice" (p. 25). The development of this idea in the mathematics education field illustrated the complexity of teaching and the key roles played by teachers in students' learning, underlining the importance of teachers' processes of interpretation of curricula materials (Zack et al. 1997).

## Different Cultures Shaping Different Forms of Interaction Between Teachers and Curricula

The relation between teacher and curricula depends on internal and external influences. For example, teachers frame their approach to curricula differently dependent on their conceptions of different components of curricula (Lloyd 1999) and/or through the different structures of professional development initiatives. Locally, teachers' knowledge and pedagogical beliefs are influences as they engage with curricula materials. Furthermore, the content and form of curricula materials influence the ways in which teachers interpret, evaluate, and adapt these materials considering their students' responses and needs in a specific institutional context. Globally, countries have different curricular traditions shaping different conditions for teachers' roles in curriculum development. Thus, the diversity of cultures and features of each country's system generates different modes of interaction between teachers and curricula, as well as different needs and trends in teacher professional development related to curriculum reform (Clarke et al. 1996). The main elements which have been proved to affect the relation between teachers and curricula are, for instance, the distance that usually exists between the intended curriculum and the implemented curriculum; whatever the level of detail and prescription of the curriculum description, for years after curriculum reform, the implemented curriculum remains a subtle composition of the old and the new, differing between one teacher and another. In this sense, curricula are related with teacher practice, and curricula change is linked to how teachers continuously further develop or change their current practice, in particular with regard to teaching and assessment and professional development initiatives (Krainer and Llinares 2010).

## Teachers and Curricula Within a Collaborative Perspective

From this view of interaction between teacher and curriculum, curricula development initiatives are a context for teacher professional development reconstructing wisdom through inquiry. There is a long tradition of teachers developing curricula materials in collaborative groups.

In the United Kingdom in the late 1970s and early 1980s, Philip Waterhouse's research (2001, updated by Chris Dickinson), supported by the Nuffield Foundation, led to the founding of a number of curricula development organizations called Resources for Learning Development Units. In these units, the mathematics editor (one of a cross-curricular team of editors) worked with groups of not more than ten teachers, facilitating their work on either developing materials related to government initiatives or from perceived needs of teachers themselves. The explicit focus for the teachers was on the development and then production of materials that had been tried out in their classrooms. However, the implicit focus of the editor was on the professional development of those teachers in the groups. It was still important for the materials to be designed, printed, and distributed.

Also, in France, since the 1970s, the IREM network has functioned on the basis of mixed groups of academics, mathematicians, and teachers inquiring, experimenting in classrooms, producing innovative curriculum material, and organizing teacher professional development sessions relying on their experience (cf. www.univ-irem.fr/). In recent views of how teachers interact with, draw on, refer to, and are influenced by curricula resources, teachers are challenged to express their professional knowledge keeping a balance between the needs of their specific classrooms and their conceptions. In many countries, as mathematics education research has matured, there is increasingly development of curricula materials by teachers themselves working collaboratively in groups, possibly in association with researchers, and the organization of teacher professional development around such collaborative actions based on a more developed idea of the teacher.

Barbara Jaworski, working in Norway from 2003 to 2010, has led research projects in partnership with teachers to investigate "Learning Communities in Mathematics" and "Teaching Better Mathematics" (see, e.g., Kieran et al. 2013). In Canada, led by Michael Fullan, there is a large-scale project supporting professional development of teachers through curriculum reform in literacy and numeracy based on in-school collaborative groupings of teachers attending a central "fair" to present their inquiry work once a year. This project, *Reach Every Student, energizing Ontario Education*, works on the attained curriculum through the implemented one and has led to Fullan's (2008) book *Six Secrets of Change*.

More recently, with the spread of ideas through international conferences, meetings, and research collaborations, ideas such as the Japanese "lesson study" have spread widely (Alston 2011). Lesson study is a professional development process that teachers engage in to systematically examine their practice. It is considered to be a means of supporting the dissemination of documents like standards, benchmarks, and nationally validated curricula. Other collaborative groups take many forms, frequently facilitated by a university academic or sometimes with university mathematicians and mathematics teacher educators forming part of the group. These multiple views define distinctive professional development pathways through curricula reforms. These pathways influence teachers' professional identities and work practices.

Social perspectives on the role of teachers in curricula reforms are being reported by Kieran and others (2013), where the major focus is on the role and nature of teachers' interactions within a group of teachers. From this perspective, teachers are motivated by collaborative inquiry activities (teams, communities, and networks) aiming at interpreting and implementing curricula materials, "participation with." How do teachers actively engage or collaborate with curricular resources (Remillard 2005)? How do teachers collaborate with other groups of participants (Pegg and Krainer 2008)? Both engagements must be understood in light of their particular local and global contexts.

Teachers' learning through collaborative inquiry activities, contextualized in curriculum development initiatives, has allowed the contextual conditions in which curriculum is implemented in different traditions to be made explicit. Pegg and Krainer (2008) reported examples of large-scale projects involving national reform initiatives in mathematics where the focus was initiating purposeful pedagogical change through involving teachers in rich professional learning experiences. The motivation for these initiatives was a perceived deficiency in students' knowledge of mathematics (and science) understood as the attained curriculum. In all of these programs, collaboration, communication, and partnerships played a major role among teachers and university staff members of the program and within these groups. In these programs, the teachers were seen not only as participants but crucial change agents who were regarded as collaborators and experts (Pegg and Krainer 2008). This view of teachers as change agents emerged from the close collaboration among groups of stakeholders and the different forms of communications that developed.

## Open Questions

The relationship between teacher and curricula defines a set of open questions in different realms. These questions are linked to the fact that the relationship between teachers and curricula is moving, due to a diversity of factors: the increasing autonomy and power given to teachers regarding curriculum design and implementation in some countries at least, the development of collaborative practices and networks in teachers' communities, the evolution of relationships between researchers and teachers, the explosion of curriculum resources and their easier accessibility thanks to the Internet.... So, some open questions are:

1. What are the implications of the school-based partial transfer of power in curriculum decision-making to teachers based on teachers' practical, personal reflective experience and networks?
2. What role do collegial networks play in how ideas about curricula change are shared (e.g., using electronic communications, practical coaching)?
3. How are new kinds of practices and teaching objectives emerging as a consequence of new resources influencing the relation between teacher and curricula?

4. How can reform initiatives cope with the balance between national frameworks for curricula (e.g., educational standards as expressions of societal demands) and local views on curricula as negotiated between the teachers of one school?
5. What role students play in bringing in ideas related to curricula (e.g., starting topics based on students' interests, questions)?

## Cross-References


▶ Communities of Practice in Mathematics Education
▶ Concept Development in Mathematics Education
▶ Curriculum Resources and Textbooks in Mathematics Education
▶ Lesson Study in Mathematics Education
▶ Mathematics Teacher as Learner
▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes
▶ Models of In-Service Mathematics Teacher Education Professional Development


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# Mathematization as Social Process


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## Keywords

Mathematization; Demathematization; Mathematics in action; Technological imagination; Hypothetical reasoning; Justification; Legitimation; Realization; Elimination of responsibility; Critical mathematics education

## Definition

Mathematization refers to the formatting of production, decision-making, economic management, means of communication, schemes for surveilling and control, war power, medical techniques, etc. by means of mathematical insight and techniques.

Mathematization provides a particular challenge for mathematics education as it becomes important to develop a critical position to mathematical rationality as well as new approaches to the construction of meaning.

## Characteristics

### Mathematization and Dematheatization

The notions of mathematization and dematheatization, the claim that there is mathematics everywhere, and mathematics in action are addressed, before we get to the challenges that mathematics education is going to face.

It is easy to do shopping in a supermarket. One puts a lot of things into the trolley and pushes it to the checkout desk. Here an electronic device used by the cashier makes a pling-pling-pling melody, and the total to be paid is shown. One gets out the credit card, and after a few movements by the fingers, one has bought whatever. No mathematics in this operation.

However, if we look at the technologies that are configuring the practice of shopping, one finds an extremely large amount of advanced mathematics being brought in operation: The items are coded and the codes are read mechanically; the codes are connected to a database containing the prices of all items; the prices are added up; the credit card is read; the amount is subtracted from the bank account associated to the credit card; security matters are observed; schemes for coding and decoding are taking place.

We have to do with a mathematized daily practice, and we are immersed in such practices. We live in a mathematized society (see Keitel et al. 1993, for an initial discussion of such processes). Gellert and Jablonka (2009) characterize the mathematization of society in the following way: “Mathematics has penetrated many parts of our lives. It has capitalised on its abstract consideration of number, space, time, pattern, structure, and its deductive course of argument, thus gaining an enormous descriptive, predictive and prescriptive power” (p. 19).

However, most often the mathematics that is brought into action is operating beneath the surface of the practice. At the supermarket there is no mathematics in sight. In this sense, as also emphasized by Jablonka and Gellert (2007), a dematheatization is accompanying a mathematization.

### There Is Mathematics Everywhere

Mathematization and the accompanying dematheatization have a tremendous impact on all forms of practices. Mathematics-based technology is found everywhere.

One can see the modern computer as a materialized mathematical construct. Certainly the computer plays a defining part of a huge range of technologies. It is defining for the formation of databases and for the processing of information and knowledge.

Processes of production are continuously taking new forms due to new possibilities for automatization, which in turn can be considered a materialized mathematical algorithm. Any form of production – being of TV sets, mobile phones, kitchen utensils, cars, shoes, whatever – represents a certain composition of automatic processes and manual labor. However, this composition is always changing due to new technologies, new needs for controlling the production process, new conditions for outsourcing, and new salary demands. Crucial for such changes is not only the development of mathematics-based technologies of automatization but also of mathematics-based procedures for decision-making.

In general mathematical techniques have a huge impact on management and decision-making. As an indication, one can think of the magnitude of cost-benefit analyses. Such analyses are crucial, in order not only to identify new strategies for production and marketing but to decision-making in general.

Complex cost-benefit analyses depend on the calculation power that can be executed by the computer. The accompanying assumption is that a *pro et contra* argumentation can be turned into a straightforward calculation. This approach to decision-making often embraces an ideology of certainty claiming that mathematics represents objectivity and neutrality. Thus in decision-making we find an example not only of a broad application of mathematical techniques but also an impact of ideological assumptions associated with mathematics.

Mathematics-based technologies play crucial roles in different domains, and we can think of medicine as an example. Here we find mathematics-based technologies for making diagnoses, for defining normality, for conducting a treatment, and for completing a surgical operation. Furthermore, the validation of medical research is closely related to mathematics. Thus any new type of medical treatment needs to be carefully documented, and statistics is crucial for doing this.

Not only medicine but also modern warfare is mathematized. As an example one can consider the drone, the unmanned aircraft, which has been used by the USA, for instance, in the war in Afghanistan. The operation of the drone includes a range of mathematics brought in action. The identification of a target includes complex algorithms for pattern recognition. The operation of a drone can only take place through the most sophisticated channels of communication, which in turn must be protected by advanced cryptography. Channels of communication as well as cryptography are completely mathematized. The decision of whether to fire or not is based on cost-benefit analyses: Which target has been identified? How significant is the target? What is the probability that the target has been identified correctly? What is the probability that other people might be killed? What is the price of the missile? Mathematics is operating in the middle of this military logic.

### Mathematics in Action

The notion of mathematics in action – that can be seen as a further development of "formatting power of mathematics" (Skovsmose 1994) – can be used for interpreting processes of mathematization (see, for instance, Christensen et al. 2009; Skovsmose 2009, 2010, in print; Yasukawa et al. in print). Mathematics in action can be characterized in terms of the following issues:

*Technological imagination* refers to the conceptualization of technological possibilities. We can think of technology of all kinds: design and construction of machines, artifacts, tools, robots, automatic processes, networks, etc.; decision-making concerning management, advertising, investments, etc.; and organization with respect to production, surveillance, communication, money processing, etc. In all such domains mathematics-based technological imagination has been put into operation. A paradigmatic example is the conceptualization of the computer in terms of the Turing machine. Even certain limits of computational calculations were identified before any experimentation was completed. One can also think of the conceptualization of the Internet, of new schemes for surveilling and robotting (see, for instance, Skovsmose 2012), and of new approaches in cryptography (see, for instance, Skovsmose and Yasukawa 2009). In all such cases mathematics is essential for identifying new possibilities.

*Hypothetical reasoning* addresses consequences of not-yet-realized technological constructions and initiatives. Reasoning of the form "if p then q, although p is not the case" is essential to any kind of technological enterprise. Such hypothetical reasoning is most often model based: one tries to grasp implications of a new technological construct by investigating a mathematical representation (model) of the construct. Hypothetical reasoning makes part of decision-making about where to build an atomic power plant, what investment to make, what outsourcing to make, etc. In all such cases one tries to provide a forecasting and to investigate possible scenarios using mathematical models. Naturally a mathematical representation is principally different from the construct itself, and the real-life implication might turn out to be very different from calculated implications. Accompanied by (mischievous) mathematics-based hypothetical reasoning, we are entering the risk society.

*Legitimation* or *justification* refers to possible validations of technological actions. While the notion of justification includes an assumption that some degree of logical honesty has been exercised, the notion of legitimation does not include such an assumption. In fact, mathematics in action might blur any distinction between justification and legitimation. When brought into effect, a mathematical model can serve any kind of interests.

*Realization* refers to the phenomenon that mathematics itself comes to be part of reality, as was the case at the supermarket. A mathematical model becomes part of our environment. Our lifeworld is formed through techniques as well as through discourses emerging from mathematics. Real-life practices become formed through mathematics in action. It is this phenomenon that has been referred to as the formatting power of mathematics.

*Elimination of responsibility* might occur when ethical issues related to implemented action are removed from the general discourse about technological initiatives. Mathematics in action seems to be missing an acting subject. As a consequence, mathematics-based actions easily appear to be conducted in an ethical vacuum. They might appear to be determined by some "objective" authority as they represent a logical necessity provided by mathematics. However, the "objectivity" of mathematics is a myth that needs to be challenged.

Mathematics in action includes features of imagination, hypothetical reasoning, legitimation, justification, realization including a dematheматization of many practices, as well as an elimination of responsibility. Mathematics in action represents a tremendous knowledge-power dynamics.

### New Challenges

Mathematics in action brings about several challenges to mathematics education of which I want to mention some.

Over centuries mathematics has been celebrated as crucial for obtaining insight into nature, as being decisive for technological development, and as being a pure science. Consistent or not, these assumptions form a general celebration of mathematics. This celebration can be seen as almost a defining part of modernity. However, by acknowledging the complexity of mathematics in action such celebration cannot be sustained. Mathematics in action has to be addressed critically in all its different instantiations. Like any form of action, mathematics in action may have any kind of qualities, such as being productive, risky, dangerous, benevolent, expensive, dubious, promising, and brutal. It is crucial for any mathematics education to provide conditions for reflecting critically on any form of mathematics in action.

This is a challenge to mathematics education both as an educational practice and research. It becomes important to investigate mathematics in action as part of complex sociopolitical processes. Such investigations have been developed with reference to ethnomathematical studies, but many more issues are waiting for being addressed (see, for instance, D'Ambrosio's 2012 presentation of a broad concept of social justice).

Due to processes of mathematization and not least to the accompanying processes of dematheматization, one has to face new challenges in creating meaningful activities in the classroom. Experiences of meaning have to do with experiences of relationships. How can we construct classroom activities that, on the one hand, acknowledge the complex mathematization of social practices and, on the other hand, acknowledge the profound dematheматization of such practices? This general issue has to be interpreted with reference to particular groups of students in particular sociopolitical contexts (see, for instance, Gutstein 2012).

To break from any general celebration of mathematics, to search for new dimensions of meaningful mathematics education, and to open for critical reflections on any form of mathematics in action are general concerns of critical mathematics education (see also "▶ Critical Mathematics Education" in this Encyclopedia).

## Cross-References

- ▶ Critical Mathematics Education
- ▶ Critical Thinking in Mathematics Education
- ▶ Dialogic Teaching and Learning in Mathematics Education
- ▶ Mathematical Literacy

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# Metacognition


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## Keywords

Metamemory; Metacognitive knowledge; Metacognitive experiences; Metacognitive strategies


## Definition

Any knowledge or cognitive activity that takes as its object, or monitors, or regulates any aspect of cognitive activity; that is, knowledge about, and thinking about, one's own thinking.

## Characteristics

Although the construct, metacognition, is used quite widely and researched in various fields of psychology and education, its history is relatively short beginning with the early work of John Flavell on *metamemory* in the 1970s. Metamemory was a global concept encompassing a person's knowledge of "all possible aspects of information storage and retrieval" (Schneider and Artelt 2010). Flavell's (1979) model of metacognition and cognitive monitoring has underpinned much of the research on metacognition since he first articulated it. It was a revised version of his taxonomy of metamemory that he had developed with Wellman (Flavell and Wellman 1977). According to his model, a person's ability to control "a wide variety of cognitive enterprises occurs through the actions and interactions among four classes of phenomena: (a) metacognitive knowledge, (b) metacognitive experiences, (c) goals (or tasks), and (d) actions (or strategies)" (p. 906). *Metacognitive knowledge* incorporates three interacting categories of knowledge, namely, personal, task, and strategy knowledge. It involves one's (a) *sensitivity* to knowing how and when to apply selected forms and depths of cognitive processing appropriately to a given situation (similar to subsequent definitions of partly what is called *procedural metacognitive knowledge*), (b) intuitions about intra-individual and inter-individual differences in terms of beliefs, feelings, and ideas, (c) knowledge about task demands which govern the choice of processed information, and (d) a stored repertoire of the nature and utility of cognitive strategies for attaining cognitive goals. The first of these is mostly implicit knowledge, whereas the remaining three are explicit, conscious knowledge. *Metacognitive experiences* are any conscious cognitive or affective experiences which control or regulate cognitive activity. Achieving *metacognitive goals* are the objectives of any metacognitive activity. *Metacognitive strategies* are used to regulate and monitor cognitive processes and thus achieve metacognitive goals.

In the two decades that followed when Flavell and his colleagues had initiated research into metacognition (Flavell 1976, 1979, 1981), the use of the term became a buzzword resulting in an extensive array of constructs with subtle differences in meaning all referred to as metacognition (Weinert and Kluwe 1987). This work was primarily in the area of metacognitive research on reading; however, from the early 1980s, work in mathematics education had begun mainly related to problem solving (Lester and Garofalo 1982) particularly inspired by Schoenfeld (1983, 1985, 1987) and Garofalo and Lester (1985). Cognition and metacognition were often difficult to distinguish in practice, so Garofalo and Lester (1985) proposed an operational definition distinguishing cognition and metacognition which clearly demarcates the two, namely, cognition is "involved in doing," whereas metacognition is "involved in choosing and planning what to do and monitoring what is being done" (p. 164). This has been used subsequently by many researchers to be able to delineate the two.

Today, the majority of researchers in metacognitive research in mathematics education have returned to the roots of the term and share Flavell's early definition and elaborations (Desoete and Veenman 2006). The field has firmly established the foundations of the construct and by building on these foundations, several researchers have extended Flavell's work usefully and there is an expanding body of knowledge in the area. The elements of his model have been extended by others (e.g., elaborations of metacognitive experiences, see Efklides 2001, 2002) or are the subject of debate (e.g., motivational and emotional knowledge as a component of metacognitive knowledge, see Op 't Eynde et al. 2006). Subsequently, it has led to many theoretical elaborations, interventions, and ascertaining studies in mathematics education research (Schneider and Artelt 2010).

Flavell did not expect metacognition to be evident in students before Piaget's stage of formal operational thought, but more recent work by others has shown that preschool children already start to develop metacognitive awareness. Work in developmental and educational psychology as well as mathematics education has shown that metacognitive ability, that is, the ability to gainfully apply metacognitive knowledge and strategies, develops slowly over the years of schooling and there is room for improvement in both adolescence and adulthood. Furthermore, studying the developmental trajectory of metacognitive expertise in mathematics entails examining both frequency of use and the level of adequacy of utilization of metacognition. Higher frequency of use does not necessarily imply higher quality of application, with several researchers reporting such phenomena as *metacognitive vandalism*, *metacognitive mirage* and *metacognitive misdirection*. Metacognitive vandalism occurs when the response to a perceived metacognitive trigger ("red flag") involves taking drastic and destructive actions that not only fail to address the difficulty but also could change the nature of the task being undertaken. Metacognitive mirage results when unnecessary actions are engaged in, because a difficulty has been perceived, but in reality, it does not exist. Metacognitive misdirection is the relatively common situation where there is a potentially relevant but inappropriate response to a metacognitive trigger that is purely inadequacy on the part of the task solver not deliberate vandalism. Recent research shows that as metacognitive abilities in mathematics develop, not only is there increased usage but also the quality of that usage increases.

The popularity of the metacognition construct stems from the belief that it is a crucial part of everyday reasoning, social interaction as occurs in whole class and small group work and more complex cognitive tasks such as mathematical problem solving, problem finding and posing, mathematical modeling, investigation, and inquiry based learning.

## Cross-References

► Problem Solving in Mathematics Education

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# Metaphors in Mathematics Education

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## Keywords

Metaphor; Conceptual metaphor; Metaphoring; Reification; Embodied cognition; Gestures; Analogy; Representations

## Definition

Etymologically metaphor means "transfer," from the Greek meta (trans) + pherein (to carry). Metaphor is in fact "transfer of meaning."

## Introduction

Metaphors are very likely as old as humankind. Recall Indra's net, a 2,500-year-old Buddhist metaphor of dependent origination and interconnectedness (Cook 1977; Capra 1982), consisting of an infinite network of pearls, each one reflecting all others, in a never-ending process of reflections of reflections, highly appreciated by mathematicians (Mumford et al. 2002).

It was Aristotle, however, with his taxonomic genius, who first christened and characterized metaphors c. 350 BC in his Poetics: "Metaphor consists in giving the thing a name that belongs to something else; the transference being either from genus to species, or from species to genus, or from species to species, on the grounds of analogy" (Aristotle 1984, 21:1457b). Interestingly for education, Aristotle added:

> The greatest thing by far is to be a master of metaphor. It is the one thing that cannot be learned from others; it is also a sign of genius, since a good metaphor implies an eye for resemblance. (loc. cit. 21:1459a)

But time has not passed in vain since Aristotle. Widespread agreement has been reached (Richards 1936; Black 1962, 1979; Ortony 1993; Ricoeur 1977; Reddy 1993; Gibbs 2008, 2008; Indurkhya 1992, 2006; Johnson and Lakoff 2003; Lakoff and Núñez 2000; Wu 2001; Sfard 1994, 1997, 2009) that metaphor serves as the often unknowing foundation for human thought (Gibbs 2008) since our ordinary conceptual system, in terms of which we both think and act, is fundamentally metaphorical in nature (Johnson and Lakoff 2003).

## Characteristics

### Metaphors for Metaphor

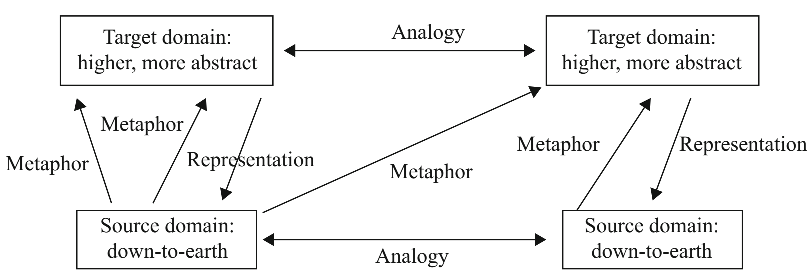
"There is no non metaphorical standpoint from which one could look upon metaphor" remarked Ricoeur (1977). To Bruner (1986) "Metaphors are crutches to help us to get up the abstract mountain," but "once up we throw them away (even hide them) . . . (p. 48). Empirical evidence suggests however that metaphor is a permanent resource rather than a temporary scaffold becoming later a "dead metaphor" (Chiu 2000). We find also *theory-constitutive metaphors* that do not "worn out" like literary metaphors and provide us with heuristics and guide our research (Boyd 1993; Lakoff and Núñez 1997). Recall the "tree of life" metaphor in Darwin's theory of evolution or the "encapsulation metaphor" in Dubinsky's APOS theory (Dubinsky and McDonald 2001).

In the field of mathematics education proper, it has been progressively recognized during the last decades (e.g., Chiu 2000, 2001; van Dormolen 1991; Edwards 2005; English 1997; Ferrara 2003; Gentner 1982, 1983; Lakoff and Núñez 2000; Parzysz et al. 2007; Pimm 1987; Presmeg 1997; Sfard 1994, 1997, 2009; Soto-Andrade 2006, 2007) that metaphors are powerful cognitive tools that help us in grasping or building new mathematical concepts, as well as in solving problems in an efficient and friendly way: "metaphors we calculate by" (Bills 2003).

According to Lakoff and Núñez (2000), (conceptual) metaphors appear as mappings from a *source* domain into a *target* domain, carrying the inferential structure of the first domain into the one of the second, enabling us to understand the latter, often more abstract and opaque, in terms of the former, more down-to-earth and transparent. In the classical example "A teacher is a gardener," the *source* is gardening, and the *target* is education.

Figure 1 maps metaphors, analogies, and representations and their relationships (Soto-Andrade 2007).

We thus see metaphor as bringing the target concept into being rather than just shedding a new light on an already existing notion, as




**Metaphors in Mathematics Education, Fig. 1** A topographic metaphor for metaphors, representations, and analogies representation usually does, whereas analogy states a similarity between two concepts already constructed (Sfard 1997). Since new concepts arise from a crossbreeding of several metaphors rather than from a single one, multiple metaphors, as well as the ability to transiting between them, may be necessary for the learner to make sense of a new concept (Sfard 2009). Teaching with multiple metaphors, as an antidote to unwanted entailments of one single metaphor, has been recommended (e.g., Low 2008; Sfard 2009; Chiu 2000, 2001).

### Metaphor and Reification

Sfard (1994) named *reification* the metaphorical creation of abstract entities, seen as the transition from an *operational* to a *structural* mode of thinking. Experientially, the sudden appearance of reification is an "aha!" moment, the birth of a metaphor that brings a mathematical concept into existence. Reification is however a double-edged sword: Its *poietic* (generating) edge brings abstract ideas into being, and its *constraining* edge bounds our imagination and understanding within the confines of our former experience and conceptions (Sfard 2009). This "metaphorical constraint" (Sfard 1997) explains why it is not quite true that anybody can invent anything, anywhere, anytime, and why metaphors are often "conceptual recycling." For instance, the construction of complex numbers was hindered for a long time by *overprojection* of the metaphor "number is quantity" until the new metaphor "imaginary numbers live in another dimension" installed them in the "complex plane." "To understand a new concept, I must create an appropriate metaphor…" says one of the mathematicians interviewed by Sfard (1994).

### Metaphor, Embodied Cognition, and Gestures

Contemporary evidence from cognitive neuroscience shows that our brains process literal and metaphorical versions of a concept in the same localization (Knops et al. 2009; Sapolsky 2010). Gibbs and Mattlock (2008) show that real and imagined body movements help people create embodied simulations of metaphorical meanings involving haptic-kinesthetic experiences. The underlying mechanism of cross-domain mappings may explain how abstract concepts can emerge in brains that evolved to steer the body through the physical, social, and cultural world (Coulson 2008). It has been proposed that acquiring metaphoric items might be facilitated by acting them out, as in total physical response learning (Low 2008).

The didactical chasm existing between the ubiquitous motion metaphors in the teaching of calculus and the static and timeless character of current formal definitions (Kaput 1979) is in fact bridged by the often unconscious gestures (Yoon et al. 2011) that lecturers enact in real time while speaking and thinking in an instructional context (Núñez 2008). So gestures inform mathematics education better than traditional disembodied mathematics (Núñez 2007).

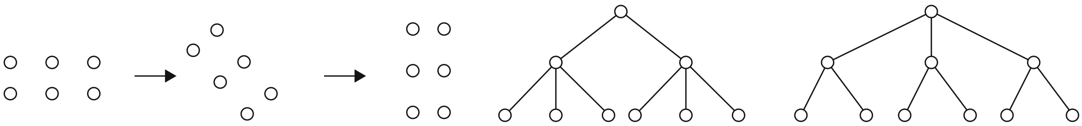
### Metaphors for Teaching and Learning

When confronted with the metaphor "teaching is transmitting knowledge," many teachers say: This is not a metaphor, teaching *is* transmitting knowledge! What else could it be? Unperceived here is the "Acquisition Metaphor," dominant in mathematics education, that sees learning as acquiring an accumulated commodity. The alternative, complementary, metaphor is the Participation Metaphor: learning as participation (Sfard 1998). Plutarch agreed when he said "A mind is a fire to be kindled, not a vessel to be filled" (Sfard, 2009).

### Educational Metaphors

*Grounding* and *linking* metaphors are used in forming mathematical ideas (Lakoff and Núñez 2000). The former "ground" our understanding of mathematics in familiar domains of experience, the latter link one branch of mathematics to another.

Lakoff and Núñez (1997) point out that often mathematics teachers attempt to concoct ad hoc extensions of grounding metaphors beyond their natural domain, like "helium balloons" or "anti-matter objects" for negative numbers. Although the grounding "motion metaphor" extends better to negative numbers: $-3$ steps means walking backwards 3 steps and multiplying by $-1$ is turning around, they consider this extension



**Metaphors in Mathematics Education, Fig. 2** Two metaphors for commutativity of multiplication

a forced "educational metaphor." For an explicit account of such educational metaphors, see Chiu (1996, 2000, 2001). Negative numbers arise more naturally, however, via flows in a graph: A "negative flow" of 3 units from agent A to agent B "is" a usual flow of 3 units from B to A.

## Metaphoring (Metaphorical Thinking) in Mathematics Education

Presmeg (2004) studied idiosyncratic metaphors spontaneously generated by students in problem-solving as well as their influence on their sense making. Students generating their own metaphors increase their critical thinking, questioning, and problem-solving skills (Low 2008). There are however potential pitfalls occasioned by invalid inferences and overgeneralization.

Building on their embodied prior knowledge, students can understand difficult concepts metaphorically (Lakoff and Núñez 1997). Explicit examples have been given by Chiu (2000, 2001), e.g., students using their knowledge of motion to make sense of static polygons through the "polygons are paths" metaphor, and so "seeing" that the sum of the exterior angles is a whole turn and that exterior angles are more "natural" than interior angles! "Polygons are enclosures between crossing sticks" elicits different approaches. Source understanding overcomes age to determine metaphoring capacity, since 13-month infants can already metaphorize (Chiu 2000). Also, a person's prior (nonmetaphorical) target understanding can curtail or block metaphoring (loc. cit.).

### Examples of Metaphors for Multiplication

Chiu (2000) indicates the following:

"Multiplication A × B is replacing the original A pieces by B replications of them."

"Multiplication A × B is cutting each of the current A objects into B pieces."

"Area metaphor" and "Branching metaphor" for multiplication (Soto-Andrade 2007) are illustrated in Fig. 2.

In the area metaphor, commutativity is perceived as invariance of area under rotation. We "see" that $2 \times 3 = 3 \times 2$, *without counting and knowing that it is 6*. In the branching metaphor, commutativity is less obvious unless this metaphor becomes a "met-before" (McGowen and Tall 2010) because you know trees very well. Our trees also suggest a "hydraulic metaphor," useful to grasp multiplication of fractions: A litre of water drains evenly from the tree apex, through the ducts. Then 1/6 appears as 1/3 *of* 1/2 in the left tree and also as 1/2 *of* 1/3 in the right tree. Our hydraulic metaphor enables us to see the "two sides of the multiplicative coin": $2 \times 3$ is bigger but $1/2 \times 1/3$ is smaller than both factors. It also opens up the way to a deeper metaphor for multiplication: "multiplication is concatenation", a generating metaphor for category theory in mathematics.

## On the Metaphorical Nature of Mathematics

Lakoff and Núñez's claim that mathematics consists entirely of conceptual metaphors has stirred controversy among mathematicians and mathematics educators. Dubinsky (1999) suggests that formalism can be more effective than metaphor for constructing meaning. Goldin (1998, 2001) warns that the extreme view that all thought is metaphorical will be no more helpful than earlier views that it was propositional and finds that Lakoff and Núñez's "ultrarelativistism" dismisses perennial values central to mathematics education like mathematical truth and processes of abstraction, reasoning, and proof among others (Goldin 2003).

However some distinguished mathematicians dissent. Manin (2007), referring to Metaphor and

Proof, complains about the imbalance between various basic values which is produced by the emphasis on proof (just one of the mathematical genres) that works against values like "activities", "beauty" and "understanding", essential in high school teaching and later, neglecting which a teacher or professor tragically fails. He also claims that controverted Thom's Catastrophe Theory "is one of the developed mathematical metaphors and should only be judged as such". Thom himself complains that "analogy, since positivism, has been considered as a remain of magical thinking, to be condemned absolutely, being nowadays hardly considered as more than a rhetorical figure (Thom, 1994). He sees catastrophe theory as a pioneering theory of analogy and points out that narrow minded scientists objecting the theory because it provides nothing more than analogies and metaphors, do not realize that they are stating its true purpose: to classify all possible types of analogical situations (Porte, 2013).

The preface to Mumford et al. (2002) reads: "Our dream is that this book will reveal to our readers that mathematics is not alien and remote but just a very human exploration of the patterns of the world, one which thrives on play and surprise and beauty."

McGowen and Tall (2010) argue that even more important than metaphor for mathematical thinking are the particular mental structures built from experience that an individual has "met-before." Then one can analyze the met-befores of mathematicians, mathematics educators, and developers of theories of learning to reveal implicit assumptions that support their thinking in some ways and hinder it in others. They criticize the top-down nature of Lakoff and Núñez "mathematical idea analysis" and their unawareness of their own embodied background and implicit met-befores that shape their theory.

### Open Ends and Questions

Further research is needed on methods and techniques of teaching metaphor.

Facts on how the neural substrate of perception and action is co-opted by higher-level processes suggest further research on comparing visual, auditory, and kinesthetic metaphors.

How can teachers facilitate the emergence of idiosyncratic metaphors in the students?

May idiosyncratic metaphors be voltaic arcs that spring when didactical tension is high enough in the classroom?

How and where do students learn relevant metaphors: from teachers, textbooks, or sources outside of the classroom?

How can we facilitate students' transiting between metaphors?

How can teaching trigger change in students' metaphors?

What roles should the teacher play in metaphor teaching?

What happens when there is a mismatch between teacher and student's metaphors?

Do experts continue using the same metaphors as novices? If yes, do they use them in the same way?

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▶ Mathematical Representations

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## Misconceptions and Alternative Conceptions in Mathematics Education

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### Keywords

Understanding; Learning; Constructivism; Cognitive models; Child's perspective; Cognitive conflict; Child's conceptions

### Definition

The term "misconception" implies incorrectness or error due to the prefix "mis." However its connotation never implies errors from a child's perspective. From a child's perspective, it is a reasonable and viable conception based on their experiences in different contexts or in their daily life activities. When children's conceptions are deemed to be in conflict with the accepted meanings in mathematics, the term misconceptions has tended to be used. Therefore some researchers or educators prefer to use the term "alternative conception" instead of "misconception." Other terms sometimes used for misconceptions or terms related to misconceptions include students' mental models, children's arithmetic, preconceptions, naïve theories, conceptual primitives, private concepts, alternative frameworks, and critical barriers.

Some researchers avoid using the term "misconceptions," as they consider them as misapprehensions and partial comprehensions that develop and change over the years of school. For example, Watson (2011), based on an extensive program of research, identifies developmental pathways that can be observed as middle school students move towards more sophisticated understandings of statistical concepts, culminating in a hierarchical model incorporating six levels of statistical literacy (p. 202).

## Characteristics

Research on misconceptions in mathematics and science commenced in the mid-1970s, with the science education community researching the area much more vigorously. This research carefully rejected the tabula rasa assumption that children enter school without preconceptions about a concept or topic that a teacher tries to teach in class. The first international seminar Misconceptions and Educational Strategies in Science and Mathematics was held at Cornell University, Ithaca, NY, in 1983, with researchers from all over the world gathering to present research papers in this area – although the majority of research papers were in the field of science education.

In mathematics education, according to Confrey (1987), research on misconceptions began with the work of researchers such as Erlwanger (1975), Davis (1976), and Ginsburg (1976), who pioneered work focusing on students' conceptions. In the proceedings of the second seminar: Misconceptions and Educational Strategies in Science and Mathematics, Confrey (1987) used constructivism as a framework for a deep analysis of research on misconceptions. Almost two decades later, Confrey and Kazak (2006) identified examples of misconceptions which have been extensively discussed by the mathematics education community – for example, "Multiplication makes bigger, division makes smaller," "The graph as a picture of the path of an object," "Adding equal amounts to numerators and denominators preserves proportionality," and "longer decimal number are bigger, so the 1.217 > 1.3" (pp. 306–307). Concerning decimals, a longitudinal study by Stacey (2005) showed that this misconception is persistent and pervasive across age and educational experience. In another extensive study, Ryan and Williams (2007) examined a variety of misconceptions among 4–15-year-old students in number, space and measurement, algebra, probability, and statistics, as well as preservice teachers' mathematics subject matter knowledge of these areas.

From the teacher's perspective, a misconception is not a trivial error that is easy to fix, but rather it is resilient or pervasive when one tries to get rid of it. The reason why misconceptions are stubborn is that they are viable, useful, workable, or functional in other domains or contexts. Therefore, it is important for teachers not only to treat misconceptions with equal importance to mathematical concepts but also to identify what exactly the misconception is in the learning context and to clarify the relationship between the misconception and the mathematical concept to be taught. In other words, the teacher needs to construct the task for the lesson taking the misconception into consideration in order to resolve the conflict between the misconception and the mathematical concept. By doing this the lesson may open up a new pathway to children's deeper and wider understanding of the mathematical concept to be taught.

So far many misconceptions have been identified at the elementary and secondary levels, however only a few of them are considered for inclusion in actual teaching situations. While very few of these are incorporated in mathematics textbooks, one exception is the misconception that figures with the same perimeter have the same area. For example, Takahashi (2006) describes an activity used in a fourth-grade Japanese textbook to introduce the formula for the area of a rectangle that asks students to compare the areas of carefully chosen figures that have the same perimeter – for example, 3 cm $\times$ 5 cm and 4 cm $\times$ 4 cm rectangles.

Further research is needed to develop how to incorporate misconceptions into textbook or teaching materials in order to not only resolve the misconception but also to deepen and expand children's understanding of mathematical concepts.

## Cross-References

▶ Concept Development in Mathematics Education
▶ Constructivism in Mathematics Education
▶ Constructivist Teaching Experiment
▶ Learning Environments in Mathematics Education
▶ Situated Cognition in Mathematics Education

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# Models of In-Service Mathematics Teacher Education Professional Development

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Essentially ongoing improvement in learning is connected to the knowledge of the teacher. This knowledge can be about the mathematics they will teach, communicating that mathematics, finding out what students know and what they find difficult to learn, and managing the classroom to maximize the learning of students. This entry is about approaches to in-service mathematics teacher education and highlighting where emphases are placed. The basic organizer is around teacher decision making since effective classroom teaching is essentially about planning experiences that engage students in activities that are mathematically rich, relevant, accessible, and the management of the learning that results. As Zaslavsky and Sullivan (2011) propose, educating practising teachers involves facilitating growth from "uncritical perspectives on teaching and learning to more knowledgeable, adaptable, judicious, insightful, resourceful, reflective and competent professionals ready to address the challenges of teaching" (p. 1).

The entry is structured around an adaptation of the Clark and Peterson (1986) schematic in which three background factors – specifically teacher knowledge; the constraints they anticipate they will experience; and their attitudes, beliefs, and self-goals – influence each other and together inform teachers' intentions to act and ultimately their classroom actions. Because the schematic essentially connects background considerations with practice, it is ideal for structuring the education of practising mathematics teachers.



The first of these background factors refers to teacher knowledge. A model informing the design of practising teacher education directed at improving their knowledge was proposed by Hill et al. (2008) in which there were two major categories: subject matter knowledge and pedagogical content knowledge. Hill et al. described *Subject Matter Knowledge* as consisting of common content knowledge, specialized content knowledge, and knowledge at the mathematical horizon. For each of these, the emphasis is on developing in teachers the capacity not only to learn any new mathematics they need but also to view the mathematics they know in new ways. Generally, both of these orientations are facilitated by connecting this learning to the further development of their pedagogical content knowledge. Hill et al. argued that *Pedagogical Content Knowledge* includes knowledge of content and teaching, knowledge of content and students, and knowledge of curriculum. In addressing knowledge of content and teaching, Zaslavsky and Sullivan (2011) proposed focusing teacher learning on experiences such as those involving comparing and contrasting between and across topics to identify patterns and make connections, designing and solving problems for use in their classrooms, fostering awareness of similarities and differences between tasks and concepts, and developing the capacity of teachers to adapt successful experiences to match new content. Knowledge of content and students is primarily about the effective use of data to inform planning and teaching. Essentially, the goal is to examine what students know, as distinct from what they do not. In terms of knowledge of curriculum, Sullivan et al. (2012) described processes where teachers evaluate resources, draw on the experience of colleagues, analyze assessment data to make judgments on what the students know, and interpret curriculum documents to identify important ideas (Charles 2005) as the first level of knowing the curriculum. The subsequent levels involve selecting, sequencing, and adapting experiences for the students, followed by planning the teaching. All of these can inform the design of practising teacher education.

The second background factor refers to the constraints that teachers anticipate they may confront. Such constraints can be exacerbated by the socioeconomic, cultural, or language background of the students, geographic factors, and gender. A further constraint is the diversity of readiness that teachers experience in all classes, even those grouped to maximize homogeneity. Sullivan et al. (2006) described a planning framework that includes accessible tasks, explicit pedagogies, and specific enabling prompts for students experiencing difficulty. Such prompts involve slightly lowering an aspect of the task demand, such as the form of representation, the size of the number, or the number of steps, so that a student experiencing difficulties can proceed at that new level; and then if successful can proceed with the original task. Teacher educators can encourage practising teachers to examine the existence and sources of constraints and strategies that can be effective in overcoming those constraints.

The third background factor includes teachers' beliefs about the nature of mathematics and the way it is learned. Particularly important is whether teachers believe that all students can learn mathematics or whether such learning is just for some (Hannula 2004). Also important is whether teachers see their own and students' achievement as incremental and amenable to improvement through effort (Dweck 2000). Teacher education can include experiences that address this by, for example, examining forms of affirmation, studying tasks that foster inclusion, and developing awareness of threats such as self-fulfilling prophecy effects (Brophy 1983).

Having formed intentions, teachers act in classrooms. Rather than compartmentalizing the elements of the background factors described above, it is preferable that the education of practising teachers incorporate all elements together, a suitable framework for which is the study of practice. The most famous example of teacher learning from the study of practice is *Japanese Lesson Study* which is widely reported in the Japanese context (e.g., Fernandez and Yoshida 2004; Inoue 2010) and has been adapted to Western contexts (e.g., Lewis et al. 2004). Other examples of learning through the study of practice include realistic simulations offered by videotaped study of exemplary lessons (Clarke and Hollingsworth 2000); interactive study of recorded exemplars (e.g., Merseth and Lacey 1993); case methods of teaching dilemmas that problematize aspects of teaching (e.g., Stein et al. 2000); and *Learning Study* which is similar to Japanese Lesson Study but which focuses on student learning (Runesson et al. 2011).

An associated factor is the need for effective school-based leadership of the mathematics teachers. If the focus is on sustainable, collaborative school-based approaches to improving teaching, this needs active and sensitive leadership. Such leaders can be assisted to study processes for leadership, as well as developing their confidence to lead the aspects of planning, teaching, and assessment described above.

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# Models of Preservice Mathematics Teacher Education

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## Keywords

Mathematics teacher education; Preservice teachers; Professional development; Prospective teachers

## Definition

Models of preservice teacher education are understood as structures of professional learning set up by intention for prospective mathematics teachers.

## Characteristics

Preservice teacher education is widely considered as necessary for preparing prospective mathematics teachers for mastering the challenges of the mathematics classroom. To this end, models of preservice teacher education have been developed and are subject to ongoing investigations. For the profession of teaching mathematics, specific professional knowledge is necessary. In particular, designing learning opportunities and exploring the students' understanding or adaptive strategies of fostering mathematical competency require not only mathematical knowledge and pedagogical knowledge but also pedagogical content knowledge (Shulman 1986; Ball et al. 2008; Bromme 1992). This knowledge encompasses declarative and procedural components (e.g., Baumert et al. 2010; Ball et al. 2008), as well as prescriptive views and epistemological orientations (e.g., Pajares 1992; McLeod 1989; Törner 2002); it ranges from rather global components (cf. Törner 2002) to content-specific or even classroom situation-specific components (Kuntze 2012; Lerman 1990).

The goal of developing such a multifaceted professional knowledge underpins the significance of specific and structured environments for initial professional learning. However, it is widely agreed that models of preservice teacher education have to be seen as subcomponents in the larger context of continued professional learning throughout the whole working period of teachers rather than being considered as an accomplished level of qualification. Even though these models of preservice teacher education are framed by various institutional contexts and influenced by different cultural environments (Leung et al. 2006; Bishop 1988), the following fundamental aspects which are faced by many such models of preservice teacher education may be considered:

- Theoretical pedagogical content knowledge is essential for designing opportunities of rich conceptual learning in the classroom. Hence, in models of preservice teacher education, theoretical knowledge such as knowledge about dealing with representations or knowledge about frequent misconceptions of learners (cf. Ball 1993) is being supported in particular methodological formats which may take the form, e.g., of lectures, seminars, or focused interventions accompanying a learning-on-the job phase (Lin and Cooney 2001).
- Linking theory to practice is a crucial challenge of models of preservice teacher education. The relevance of professional knowledge for acting and reacting in the classroom is asserted to be supported by an integration of theoretical knowledge with instructional practice. In models of preservice teacher education, this challenge is addressed by methodological approaches such as school internships, frequently with accompanying seminars and elements of coaching (cf. Joyce and Showers 1982; Staub 2001; Kuntze et al. 2009), and specific approaches such as lesson study (Takahashi and Yoshida 2004), video-based work (e.g., Sherin and Han 2003; Seago 2004; Dreher and Kuntze 2012; Kuntze 2006), or work with lesson transcripts. For several decades, approaches such as "microteaching" (e.g., Klinzing 2002) had emphasized forms of teacher training centered in practicing routines for specific instructional situations. Seen under today's perspective, the latter approach tends to underemphasize the goal of supporting reflective competencies of prospective teachers which tend to be transferable across contents and across specific classroom situations (Tillema 2000).
- Developing competencies of instruction- and content-related reflection is a major goal in preservice teacher education. Accordingly, learning opportunities such as the analysis and the design of mathematical tasks (e.g., Sullivan et al. 2009, cf. Biza et al. 2007), the exploration of overarching ideas linked to mathematical contents or content domains (Kuntze et al. 2011), or the analysis of videotaped classroom situations (Sherin and Han 2003; Reusser 2005; Kuntze et al. 2008) are integrated in models of preservice mathematics education, supporting preservice teachers to build up reflective competencies or to become "reflective practitioners" (e.g., Smith 2003; Atkinson 2012).

The scenarios mentioned above indicate that there are a wide variety of possible models of preservice teacher education, as it has also been observed in comparative studies of institutional frameworks (König et al. 2011; Tatto et al. 2008). In contrast, research on the effectiveness of different models of preservice teacher education is still relatively scarce. Studies like TEDS-M (Tatto et al. 2008) constitute a step into this direction and set the stage for follow-up research not only in processes of professional learning in the settings of specific models of preservice teacher education but also into effects of specific professional learning environments, as they can be explored in quasi-experimental studies. In addition to a variety of existing qualitative case studies, especially quantitative evidence about models of preservice teacher education is still needed (cf. Adler et al. 2005). Such evidence from future research should systematically identify characteristics of effective preservice teacher education. Moreover empirical research about models of preservice teacher education should give insight how characteristics of effective professional development for in-service mathematics teachers (Lipowsky 2004) may translate into the context of the work with preservice teachers, which differs from professional development of in-service teachers (da Ponte 2001).

## Cross-References

### Encyclopedia of Mathematics Education


▶ Communities of Practice in Mathematics Teacher Education
▶ Lesson Study in Mathematics Education
▶ Mathematical Knowledge for Teaching
▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes
▶ Models of In-Service Mathematics Teacher Education Professional Development
▶ Pedagogical Content Knowledge in Mathematics Education
▶ Reflective Practitioner in Mathematics Education
▶ Teacher Beliefs, Attitudes, and Self-Efficacy in Mathematics Education
▶ Teacher Education Development Study-Mathematics (TEDS-M)


### Encyclopedia of Science Education


▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes
▶ Models of In-Service Mathematics Teacher Education Professional Development


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# Motivation in Mathematics Learning

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## Keywords

Motivation; Affect; Self-efficacy; Math anxiety; Disposition

## Definition

The impetus for and maintenance of mathematical activity. Mathematics learning, as goal-directed behavior, involves the development of expectations, values, and habits that constitute the reasons why people choose to engage and persevere on the one hand or disengage and avoid on the other, in mathematics and mathematically related pursuits.

## Characteristics and Findings from Various Theoretical Perspectives

The history of motivation research applied to mathematics learning began with the study of biological drives and incentive in the first decades of the twentieth century (see Brownell 1939 for a good review of this perspective as applied to education). Following the tenets of classical and operant (instrumental) conditioning, it was found that if a reinforcer was provided for successfully completing a behavior, the probability of that behavior occurring in the future under similar circumstances would increase. Additionally, Thorndike found that the intensity of the behavior would increase as a function of the reinforcement value (1927). These general theories of the use of incentives to motivate student learning dominated educational theory roughly until the middle of the 1960s.

They are still valuable to educators today, particularly in the use of behavior modification techniques, which regulate the use of rewards and other reinforcers contingent upon the learner's successive approximation of the desired behavioral outcomes, which could be successful skill attainment or increase in positive self-statements to reduce math anxiety and so on (Bettinger 2008).

Since the mid-1960s, research on motivation in the psychology of learning has focused on six different, but not distinct, theoretical constructs: Attributions, Goal Theory, Intrinsic Motivation, Self-Regulated Learning, Social Motivation, and Affect. These factors grew out of a general cognitive tradition in psychology but recently have begun to explain the impact of social forces, particularly classroom communities and teacher-student relationships on student enjoyment and engagement in mathematical subject matter (see Middleton and Spanias 1999 for a review comparing these perspectives).

### Attribution Theory

Learners' beliefs about the causes of their successes and failures in mathematics determine motivation based on the locus of the cause (internal or external to the learner) and its stability (stable or unstable). Productive motivational attributions tend to focus on internal, stable causes (like ability and effort) for success as these lead to increased persistence, self-efficacy, satisfaction, and positive learning outcomes. Lower performing demographic populations tend to show more external and unstable attributional patterns. These appear to be caused by systematic educational biases (Kloosterman 1988; Pedro et al. 1981; Weiner 1980).

### Goal Theory

Goal theories focus on the stated and unstated reasons people have for engaging in mathematical tasks. Goals can focus on *Learning* (also called *Mastery*), *Ego* (also called *Performance*), or *Work Avoidance*. People with learning goals tend to define success as improvement of their performance or knowledge. Working towards these kinds of goals shows results in the valuation of challenge, better metacognitive awareness, and improved learning than people with ego goals. Work avoidance goals are debilitating, psychologically, as they result from learned helplessness and other negative attributional patterns (Wolters 2004; Covington 2000; Gentile and Monaco 1986).

### Intrinsic Motivation and Interest

The level of interest a student has in mathematics, the more effort he or she is willing to put out, the more he or she thinks the activity is enjoyable, and the more they are willing to persist in the face of difficulties (Middleton 1995; Middleton and Spanias 1999; Middleton and Toluk 1999). Intrinsic Motivation and Interest theories have shown that mathematical tasks can be designed to improve the probability that a person will exhibit task-specific interest and that this task-specific interest, over time, can be nurtured into long-term valuation of mathematics and its applications (Hidi and Renninger 2006; Köller et al. 2001; Cordova and Lepper 1996).

### Self-Regulated Learning

Taken together, these primary theoretical perspectives can be organized under a larger umbrella concept: Self-Regulated Learning (SRL). Internal, stable attributions are a natural outcome of Learning Goals, and Interest is a natural outcome of internal, stable, attributions.

Each of these perspectives contributes to the research on the others such that the field of motivation in general, and in mathematics education specifically, is now able to use these principles to design classroom environments, tasks, and interventions to improve mathematics motivation and performance (Zimmerman and Schunk 2011; Eccles and Wigfield 2002; Wolters and Pintrich 1998).

### Social Motivation

In addition to the aforementioned psychological theories, study of students in classrooms has recently yielded principles for understanding how social groups motivate themselves. In general these theories show that needs for affiliation and relatedness with peers, fear of disapproval, and the need to demonstrate competence interact in complex ways in the classroom (Urdan and Schoenfelder 2006). Intellectual goals and social needs therefore are integrally related. Additionally, the need for social concern is a critical motivator for student prosocial learning (Jansen 2006). Students who feel a concern for the struggles of others are able to provide support for the learning of others. This is a key component of effective group work and social discourse in mathematics classrooms.

### Affect

The outcomes of learning environments consist of cognitive as well as affective responses. People tend to enjoy mathematics more when they find it interesting and useful, and they tend to dislike or even fear engagement in mathematics when they believe they will not be successful (Hoffman 2010). Goldin et al. (2011) have shown that people build affective structures which allow them to predict the emotional content and probable outcomes of mathematical activity. Activity forms a physiological feedback loop between behavior and goals and therefore has both an informational role as well as a reinforcement role (Hannula 2012). These cognitive structures are integral to self-regulation and decision-making regarding when and how deeply to engage in mathematics tasks.

## Cross-References

▶ Affect in Mathematics Education
▶ Communities of Practice in Mathematics Education
▶ Creativity in Mathematics Education
▶ Equity and Access in Mathematics Education
▶ Gender in Mathematics Education
▶ Mathematics Teacher Identity

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# N

## Noticing of Mathematics Teachers

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### Keywords

Attention; Learning to notice; Noticing; Teacher professional development; Student conceptions; Teacher education

### Definition

*Noticing* is a term used in everyday language to indicate the act of observing or recognizing something, and people engage in this activity regularly while they navigate a perceptually complex world. At the same time, individual professions have strategic ways of noticing, and understanding and promoting productive noticing by mathematics teachers has become a growing area of inquiry among researchers (For a compilation, see Sherin et al. 2011). The field currently embraces a range of conceptualizations of noticing, but many researchers use as their foundation Goodwin's ideas about *professional vision* (1994), Mason's (2002) *discipline of noticing*, and research on expertise.

### Conceptualizations and Contributions of Mathematics Teacher Noticing

Teachers, in particular, have always been confronted with a "blooming, buzzing confusion of sensory data" (B. Sherin and Star in M.G. Sherin et al. 2011), and so they need to find ways to distinguish between more productive and less productive noticing. This task has been made even more complex as mathematics teaching has increasingly become associated with in-the-moment decisions whereby teachers take into account the variety of students' conceptions that arise. Thus, the conceptualization and study of teacher noticing contributes to national efforts to decompose the practice of teaching into specific components that might be studied and learned (Grossman et al. 2009).

Current conceptualizations of mathematics teacher noticing have generally been associated with two components: *attending* and *making sense*. Researchers differ on what constitutes making sense, with some focusing exclusively on teachers' interpretations of events whereas others also include consideration of teachers' instructional responses. For example, Jacobs et al. (2010) consider instructional responses in conceptualizing professional noticing of

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children's mathematical thinking as comprised of three interrelated skills: (a) *attending* to children's strategies, (b) *interpreting* children's understandings, and (c) *deciding how to respond* on the basis of children's understandings. Another difference in how researchers conceptualize noticing is whether the focus is on documenting everything teachers find noteworthy or documenting whether teachers notice particular aspects of instruction identified as important by researchers, such as students' mathematical thinking or specific mathematical content knowledge.

Noticing differs from constructs such as *knowledge* and *beliefs* because noticing names an interactive, practice-based process rather than a category of cognitive resource. Specifically, the focus of mathematics-teacher noticing is on how teachers interact with a mathematical instructional situation, and this practice-based nature of noticing makes it complex and thereby challenging to develop. However, professional development has been found to support teachers while they learn to *notice* differently. One particularly promising approach to enhancing mathematics-teacher noticing has been through the work of video clubs (e.g., M.G. Sherin and van Es 2009), whereby teachers collaboratively view and analyze classroom videos.

Learning to notice in new, more sophisticated, ways supports teachers while they learn to teach more effectively. As such, learning to notice productively in an instructional setting is an important, but often hidden, teaching skill. Sherin et al. (2011) noted that classrooms are too complex for teachers to ever be able to notice everything before responding. They suggested that instead of focusing on all possible contingencies, professional developers focus on ways of helping teachers develop new understandings of their learning environments so that they can make more informed instructional decisions. In this way, teachers' changing practices are driven by enhanced teacher noticing whereby they are "seeing and making sense differently of things that are happening in the classroom" (p. 11).

## Cross-References

- ▶ Frameworks for Conceptualizing Mathematics Teacher Knowledge
- ▶ Mathematics Teacher Educator as Learner
- ▶ Questioning in Mathematics Education
- ▶ Teacher Beliefs, Attitudes, and Self-Efficacy in Mathematics Education

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# Number Lines in Mathematics Education

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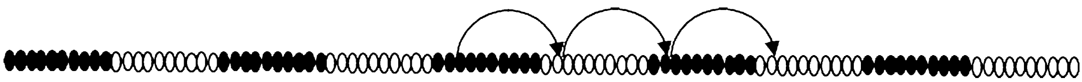
## Keywords

Addition and subtraction; Mental arithmetic; Visualization; Instruction theory; Modeling

## Characteristics

Number lines figure prominently in mathematics education. They may take various shapes and

**Number Lines in Mathematics Education, Fig. 1** Bead string



**Number Lines in Mathematics Education, Fig. 2** Jumps on the bead string

forms, from a clothesline with number cards in the early grades, to straight lines on paper representing rational numbers or integers. Number lines may feature all numbers under consideration or just a selection, depending on the function the number line has to fulfill. The 1st-grade number cards, for instance, are to support the learning of the number sequence. Whereas a more schematized number line may be used to illuminate the structure and magnitude of rational numbers and decimals. In this contribution we will focus on the empty number line, which is kept even more sparse than the latter in order to fulfill its role as a specifically designed instructional tool.

## The Empty Number Line

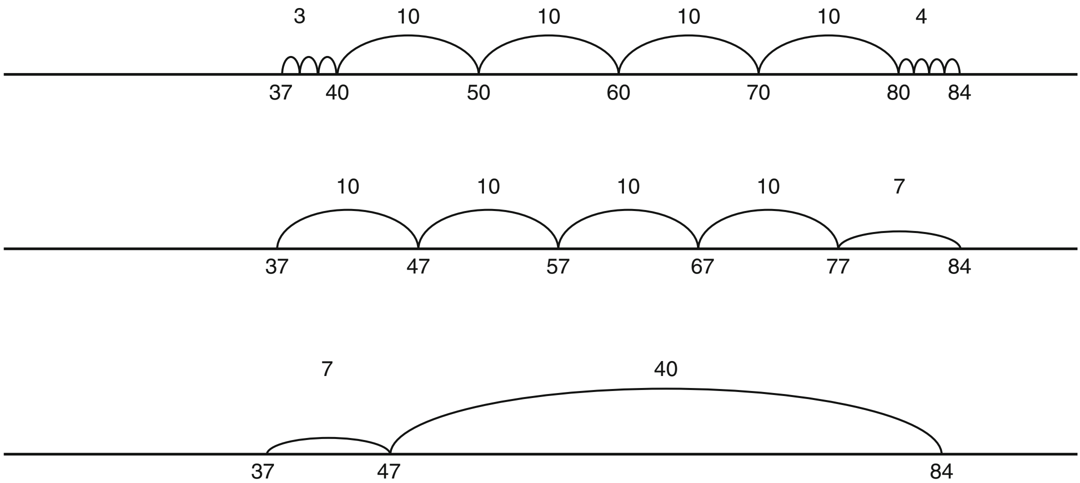
The idea of using the empty number line as a means of support for adding and subtracting numbers up to 100 was introduced by Whitney (1985) and elaborated and publicized by Treffers (1991), who linked it to the so-called domain-specific instruction theory for realistic mathematics education (RME) (see also Gravemeijer 2004). In doing so, he also adopted Whitney's suggestion of using a bead string, consisting of 100 beads that are grouped in a pattern of ten dark beads, ten light beads, ten dark beads, etc. (see Fig. 1), as a precursor to the number line.

The activities with the bead string consist of counting beads (starting from the left and marking the total with a clothespin), incrementing, decrementing, and comparing numbers of beads. The rationale for those activities is that students will start to use the color structure of the bead string, by curtailing the counting of beads to counting by tens and ones. Students may start using multiples of ten as reference points, both for identifying given numbers of beads (e.g., $63 = 6 \times 10 + 3$ or $68 = 7 \times 10 - 2$) and for adding and subtracting beads. Adding 30 to 42, for instance, may be carried out via "jumps of ten": "$42 + 10 = 52$, $52 + 10 = 62$, $62 + 10 = 72$" (see Fig. 2).

Next the activities with the bead string are symbolized on a number line, where small arcs signify jumps of one and bigger arcs jumps of ten. In this manner the number line may start to function as a way of scaffolding ten-referenced strategies for addition and subtraction up to 100. And the students may start curtailing the jumps in various manners (see Fig. 3), a method which can be expanded to numbers up to 1,000 (Selter 1998). Research showed that the empty number line is a powerful model for instruction (Klein et al. 1998).

## Flexible Solution Strategies

Note that number line the students start with is literally empty, and the students only mark the numbers that play a role in their calculation. The marks on the number line emanate from the student's own thinking. This allows for a wide variety of flexible solution strategies – which are compatible with a group of solution strategies that students develop spontaneously. Research shows that the informal strategies students develop to solve addition and subtraction problems up to 100 fall in two broad categories, "splitting tens and ones" and "counting in jumps" (Beishuizen 1993). An instance of splitting tens and ones would be solving $44 + 37 = \ldots$, for example, via $40 + 30 = 70$; $4 + 7 = 11$; and $70 + 11 = 81$. Counting in jumps would involve




**Number Lines in Mathematics Education, Fig. 3** Various strategies for 37 + 47 on the number line

solutions such as 44 + 37 = . . .; 44 + 30 = 74; 74 + 7 = 81 or 44 + 37 = . . .; 44 + 6 = 50; 50 + 10 = 60; 60 + 10 = 70; 70 + 10 = 80; and 80 + 1 = 81. According to Beishuizen (1993), procedures based on splitting lead to more errors, than solution procedures that are based on curtailed counting. Other researchers found that students tend to come up with a wide variety of counting solutions when confronted with "linear-type" context problems (see Gravemeijer 2004). Capitalizing on counting strategies therefore fits the reform mathematics idea of supporting students in constructing their own mathematical knowledge.

## Further Elaboration

Treffer's approach with the bead string as precursor to the empty number line is further elaborated in an instructional program that aims at teaching flexible solution strategies via a process of progressive schematizing, which proceeds along three levels of schematizing: informal/contextualized; semiformal/model supported; and formal/arithmetical. This process is supported by the training of subskills. The program consists of two parts, "numbers" and "operations with numbers." The former addresses the basic skills of counting, ordering and localizing, and jumping to given numbers. The latter addresses complementary skills, such as addition to 10, partitioning, jumps of 10, and relating subtraction and addition. This program has been integrated in a teaching and learning trajectory for calculation with whole numbers in primary school in the Netherlands (Heuvel-Panhuizen 2001). The latter advices to start with counting in jumps in grade 2 and to expand the repertoire of mental calculation techniques in grade 3 with de split method and "flexible" or "varying" strategies.

## An Alternative Approach

Parallel to this, an alternative approach has been developed in which the bead string is replaced by a series of measuring tools in an interactive inquiry classroom culture setting (Stephan et al. 2003; Gravemeijer 2004). This approach gives priority to unitizing and to developing a network of number relations. The rationale of the focus on number relations is that the students' knowledge of number relations forms the basis for what – from an observer's point of view – looks like the application of strategies. While what the students actually do is combining number facts which are ready to hand to them, in order to derive new number facts. According to this view, the construction of a network of number relations involves a shift from numbers that signify countable objects for the students, to numbers as entities in and of themselves. This idea is further elaborated with the emergent modeling design heuristic (Gravemeijer 1999) in design experiment in Nashville (Stephan et al. 2003). Here the choice for measuring is dictated by the ambiguity of the numbers on the number line. On the one hand, the numbers refer to *quantities*, and, on the other hand, they refer to *positions* on the number line. Most addition and subtraction problems that the students have to solve deal with quantities, while the solution methods involve the order of the numbers in the number sequence. Linear measurement offers the opportunity to address this ambiguity. A number on a ruler also signifies both a position and a quantity or a magnitude. And students may develop a deeper understanding of the relation between the two, when they come to see the activity of measuring as the accumulation of distance. The latter implies that each number word used in the activity of iterating signifies the *total measure* of the distance measured until that moment. From an emergent modeling perspective, the notion of a ruler can be construed as an overarching model. The ruler may be seen as a curtailment of iterating a measurement unit and thus emerge as a *model of* iterating some measurement unit, which is superseded by the empty number line as a more abstract ruler that functions as a *model for* mathematical reasoning with numbers up to 100.

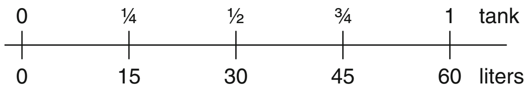
As a caveat, however, it should be noted that ample care has to be taken to avoid that the empty number line is seen as a simplified picture of a ruler. Instead, the jumps on the number line have to be perceived as means of describing one's arithmetical thinking. In contrast to the ruler, the empty number line should not be seen as proportional. For trying to strive for an exact proportional representation would severely hamper flexible use of the number line.

## Imagery

An issue of concern is what the number line and its precursors signify for the students. A teaching experiment in which the number line was not preceded by a bead string or a ruler showed the importance of "imagery." To come to grips with a new tool, the students have to be able to see an earlier activity with earlier tools in the activities with the new tool (Gravemeijer 1999). In the original approach, the actions on the number line are expected to signify corresponding activities on the bead string. In the sequence that is based on linear measurement, a series of transitions take place in which activities with new tools have to fulfill the imagery criterion: first, when the activity of measuring various lengths by iterating some measurement unit is curtailed to measuring with tens & ones; next, when the activity of iterating tens & ones is modeled with a ruler; then, when the activity shifts from measuring to reasoning about measures while incrementing, decrementing or comparing lengths; thereafter, when the arithmetical solution methods that may be supported by referring to the decimal structure of the ruler are symbolized with arcs on an empty number line; and finally, when this more abstract representation is used as a way of scaffolding and as a way of communicating solution methods for all sorts of addition and subtraction problems.

## Effect Studies

Most Dutch primary school textbooks are compatible with the way the empty number line approach is elaborated in the "teaching and learning trajectory for calculation with whole numbers in primary school in the Netherlands" (Heuvel-Panhuizen 2001) that was mentioned earlier. The results of national surveys halfway primary school, nevertheless, show that the Dutch students are not as proficient in subtracting two-digit numbers as might have been expected (Kraemer 2011). A follow-up study on the solution procedures of students (Kraemer 2011) shows that jumping is used frequently, and with good results, but the other methods generate many wrong answers. His data further reveal a strong tendency to solve contextual problems in two directions, via direct subtraction or indirect addition, and bare sums primarily in one direction, direct subtraction, which is not always efficient. Kraemer (2011) argues that the identified patterns suggest the students use what works for them. This is initially the combination of jumping and "subtract strategies." Over time, however, they start trying to combine these strategies with split and reasoning procedures. Then they run into problems because they




**Number Lines in Mathematics Education, Fig. 4** The amount of fuel in a gasoline tank

still miss important conceptual and instrumental building blocks for splitting and more sophisticated reasoning with numbers up to 100.

From these findings, we may conclude that careful attention has to be paid to fostering a conceptual understanding of splitting strategies and variable strategies and of the relations between the various strategies. Which is to show that the empty number line can be a powerful tool, but its success is very dependent of the way it is embedded in a broader instructional setting.

### The Double Number Line

Although the empty number line is well researched, a variant of it, the double number line, has not gotten that much attention. The double number line can be used as a means of support for coordinating two units of measure. This is particularly useful in the domains of fractions and percentages, where the units are often linked to numerosities or magnitudes (van Galen et al. 2008). Here we may think, for instance, of reasoning about the content of a petrol tank which can hold 60 l: half a tank contains 30 l, ¼ tank half of that, and ¾ tank the sum of the latter two (see Fig. 4).

Further research is needed to establish whether working with the double number line can – similarly to the empty number line – effectively foster more formal forms of mathematical reasoning.

## Cross-References

▶ Curriculum Resources and Textbooks in Mathematics Education
▶ Manipulatives in Mathematics Education
▶ Mathematical Representations
▶ Semiotics in Mathematics Education
▶ Visualization and Learning in Mathematics Education

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# Number Teaching and Learning

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## Keywords

Arithmetic; Irrational numbers; Natural numbers; Negative numbers; Rational numbers

## Characteristics

"Numbers" is one of the most important strands in the mathematics curricula worldwide. According to Verschaffel, Greer and De Corte (2007), there are several reasons for this: (a) The teaching of numbers is worldwide, and number operations and applications are connected and used in real life; (b) it relates and constitutes a foundation for all other topics in mathematics; and (c) it is one of the first topics students are formally taught in school, and students' disposition to mathematics often depends on this. Because of its importance and its nature, number learning and teaching has attracted enormous attention by mathematics education researchers, experimental psychologists, cognitive psychologists, developmental psychologists, and neuroscientists. Through the years, a better understanding has been developed regarding the components that constitute numerical understanding, the nature of its learning and development, the learning environments that facilitate this learning as well as appropriate tools for assessing the learning and teaching of numbers.

### Natural Numbers

Researchers identified three main strands of research on numbers: the behaviorist, the cognitive, and the situative (Greeno, et al. 1996). Research on numbers has been mostly cognitive (Bergeron and Herscovics 1990; Verschaffel et al. 2006, 2007), but since the 1990s the influence of situative theories such as social constructivism, ethnomathematics, and situated cognition made a strong impact (Verschaffel et al. 2006). Recently, the prevailing view is that cognitive and situated methodologies (e.g., tests, clinical research procedures, experimental teaching, design experiments, computer simulations, action research) may be combined to give a better picture of the various phenomena. In this section we will examine the way in which research on natural numbers has evolved over the years. This evolution also applies for research on rational and in some extent of other number systems.

#### Natural Numbers, Operations, and Estimation

Research on numbers has been under the focus of researchers since the end of the nineteenth century and has attracted enormous research attention. Dewey (1898) was one of the first researchers who provided an analysis of early number and presented methods for teaching arithmetic. A few years later, Thorndike (1922) published a book "The Psychology of Arithmetic" where he presented the nature, measurement, and construction of arithmetical abilities. Since then the learning and teaching of numbers has been a fundamental stream of mathematics education research. Until the 1950s most of the work on numerical understanding concentrated mainly on natural numbers, number sequence, counting, and subitizing. After the 1950s one of the most dominant theoretical and methodological approaches that guided research in numbers was Piaget's theory, which suggested that the construction of natural numbers is based on logical reasoning abilities (e.g., conservation of number, class inclusion, transitivity property, and seriation). The Piagetian tradition tended to disregard counting and subitizing. A second theoretical approach that guided research was the counting-based approach which suggested that numerical concepts evolve from counting skills which individuals develop through the quantification process (Bergeron and Herscovics 1990).

Most of the studies that flourished from the 1980s until the 1990s were mostly cognitive and rather local (Bergeron and Herscovics 1990). They described students' development, their strategies, and misconceptions as well as their conceptual structures of whole number concepts and operations (Verschaffel et al. 2006). Research has shown that children's understanding of numbers and their operations progresses successively in more abstract, complex, and general conceptual structures (Fuson 1992). Researchers also identified three main categories of strategies that students use when solving one-step addition and subtraction word problems: direct modeling with physical objects, verbal counting (counting all, counting on, or counting back), and mental strategies (derived facts and known facts). Students' strategies were also explored in multiplication and division although less extensively than in addition and subtraction. Researchers suggested that development in subtraction and division progresses in a different way from that of addition and multiplication.

Researchers also explored students' abilities in operations with multi-digit numbers. These studies suggested that a number of students often make procedural mistakes in algorithms since they get confused by the multistep procedures, while in other cases they have poor conceptual understanding of place value, grouping, and ungrouping. Researchers also claimed that without efficient knowledge of basic number facts, students are bound to have difficulties in multi-digit oral and written arithmetic (Kilpatrick et al. 2001). The relationship of strategies, principles, and number facts was also examined. One of the findings of these studies was that different strategies may be employed by students when dealing with different numbers (Kilpatrick et al. 2001).

Special attention was also given to the abilities, difficulties, and strategies of students with learning difficulties in numbers and their operations as well as of the appropriate teaching approaches for these students (e.g., Baroody 1999).

Apart from the emphasis on number operations, current reform documents call for emphasis on estimation. Three types of estimation were identified: numerosity, computational, and measurement (Sowder 1992). Although research on estimation is rather limited, researchers seem to agree that estimation is complex and difficult for students and often for adults. It develops over time and individuals use either self-invented or taught strategies to respond to estimation tasks.

After the 1990s research on numbers was affected by the situated theoretical perspective and more specifically the emerging theoretical frameworks of social constructivism, ethnomathematics, and situated cognition. According to Verschaffel et al. (2006) numerous studies focused on (a) the design, implementation, and evaluation of instructional programs, such as Realistic Mathematics Education and the impact of the socio-mathematical norms on mathematical learning; (b) teachers content knowledge, pedagogical content knowledge, actions and beliefs in the learning of numbers, and their impact on students' mathematical learning; (c) and the acquisition of numerical knowledge out of school and ways in which this knowledge may be exploited and used in the classrooms. Several researchers examined the impact that the environment where individuals grow and act may have on their abilities with numbers (e.g., ethnomathematics).

After the 1990s, the field of cognitive neuroscience also started making links to mathematics education research. Neuroscientific research examined students' mental structures of numbers and the way in which individuals internally represent and process numbers. The idea was that brain activation (e.g., using fMRI) might provide us with a more detailed picture of the cognitive sub processes that have an effect on mathematical thinking and learning.

After 2000 an increased research interest has been shown by mathematics educators on early childhood understanding of numbers and their operations (Sarama and Clements 2009). Researchers argued that young children's informal mathematical knowledge is strong, wide, and advanced. Researchers developed level of cognitive progressions, in various number domains, with the use of learning trajectories (Sarama and Clements 2009). Other research studies demonstrated that young children need to be engaged in sophisticated, purposeful, and meaningful mathematical activities which will support the development of various strategies (Sarama and Clements 2009) and students' conceptual understanding of number.

### Numbers and Problem Solving

In the 1980s–1990s, word problems followed a cognitive approach with emphasis on students' strategies, errors, and mental structures. Three types of additive number problems were identified (change, combine, and compare), two types of multiplicative problems (asymmetric (equal grouping, multiplicative comparison, rate) and symmetric (area, Cartesian product) (Greer 1992)), and two types of division problems (partitioning and measurement). Researchers also looked into intuitive models that may affect students' responses such as multiplication being repeated addition and division as partition.

In their review Verschaffel et al. (2006) argued that research on problem solving focused on different aspects each time: (a) on conceptual schemas that students possess while solving such problems, (b) on students' strategies when dealing with numbers in the context of mathematical problems, (c) on the different heuristic and metacognitive skills in the solution of numerical problems, and (d) on problem posing. However, after the 1990s, it became apparent that the cognitive perspective which guided this initial research was mainly related to problems which were not authentic. At this point the impact of situative theoretical perspective became stronger, and researchers started investigating students' misconceptions based on social, cultural, affective, and metacognitive factors such as, students' informal knowledge, teachers' mathematical and pedagogical knowledge, and teaching approaches. This move also led to an increased interest in the introduction of modeling problems from as early as primary schools as well "emergent modeling" activities related to numbers.

#### Arithmetic and Other Mathematical Domains

Early in the twentieth century, the teaching of arithmetic was restricted to performing the standard operations and algorithms. In the 1980s emphasis was given to the procedural and conceptual understanding of numbers (Hiebert 1986). This emphasis continued well into the 1990s. At this point the reform of mathematics curricula also yielded a shift towards the development of students' understanding of numbers and emphasized the importance for students to investigate the relationships, patterns, and connections. Extensive attention started to emerge regarding the connections between numbers and algebra. A number of researchers claimed that arithmetic is essentially algebraic and can set the ground for formal algebra. At the same time, they argued that algebra can strengthen the understanding of arithmetic structure (Verschaffel et al. 2007). Special emphasis was also given to the connections of numbers to other areas of mathematics such as measurement, geometry, and probability but especially data handling.

#### Number Sense

In recent years curricula reforms use extensively the term "number sense" and consider it a major essential outcome of school curricula. Although, its importance in mathematics curricula is recognized; its usefulness in research is controversial (Verschaffel et al. 2007). This arises from the fact that there is no catholic acceptance of what this term involves. Most often the term "number sense" encompasses the (a) use of different representations of numbers, (b) identification of relative and absolute magnitudes of numbers, (c) use system of benchmarks, (d) composition and decomposition of numbers, (e) conceptual understanding of operations, (f) estimation, (g) mental computations, as well as (h) the judgment about the reasonableness of results. Based on the descriptions and on the components of "number sense," there have been several attempts to construct tools to measure number sense. Furthermore, researchers also focused on designing intervention programs and examining their impact.

### Rational Numbers

There is a lot of research on students' understanding of rational numbers at different levels (from young learners to prospective and in-service teachers). These studies are mainly epistemological, cognitive, and situative. Most of them concentrated on the various interpretations and representations of rational numbers, students' abilities, and in a smaller extent on instructional programs. Recently Confrey, Maloney, and Nguyen (2008) identified eight major areas of research on rational numbers: (1) fractions; (2) multiplication and division; (3) ratio, proportion, and rate; (4) area; (5) decimals and percent; (6) probability; (7) partitioning; and (8) similarity and scaling. Based on this synthesis, they concluded that rational number is a complex concept and its teaching needs major revisions, especially regarding the sequence of the topics taught.

#### Epistemological and Cognitive View Rational Numbers Learning

Research studies that had taken an epistemological view tried to clarify the nature of rational number and its subconstructs. Kieren (1976) was the first to propose that fractions consist of four subconstructs: measure, ratio, quotient, and operator. Later Behr, Lesh, Post, and Silver (1983) extended this model and proposed that part-whole/partitioning is posited a fundamental subconstruct underlying the other four subconstructs previously suggested by Kieren (1976). According to researchers none of the subconstructs can stand alone. Each construct allows the consideration of rational numbers from a different perspective. Other studies discussed the different cognitive structures needed to understand the various subconstructs of rational numbers. Several researchers, international curricula, and textbooks are in favor of the inclusion of multiple fraction subconstructs and argued that students benefit from this.

Studies on rational numbers which are cognitive in nature concentrated on the investigation of the cognitive structures children bring into the understanding of rational numbers and the way in which these cognitive structures develop when the children are formally introduced to rational numbers. Such cognitive studies also concentrated on the development of the conceptual understanding of fractions and the obstacles to this learning. For instance, a number of studies concentrated on the way that the conceptualization of whole numbers may affect students understanding of rational numbers and make sense of decimal and fractions notations (Streefland 1991). Students often do not interpret fractions as numbers but view fractions as two numbers with a line between them. When adding fractions, they often add the numerators and denominators or are unable to order fractions from smaller to larger (e.g., Behr et al. 1992). Regarding the decimal representation of fractions, young students often believe that decimal numbers have a predictable order and that decimals with more digits after the decimal point are larger than decimals with fewer digits after the decimal point. Steffe and Olive (2010) have a rather different view. They argue that the mental operations necessary for the understanding of whole number should not be viewed as an obstacle to fractions understanding but as a foundation for fractional understanding (reorganization hypothesis).

Furthermore, research also indicated that individuals often have a procedural understanding of fraction operations which is attributed to the reliance of mechanical learning of rules. For instance, young students accept the representation of “a” parts of “b” unequal parts as fractions or that in the division of fractions one needs to reverse the second fraction and multiply. In addition to this, researchers seem to agree out of the four operations, division of fraction is the most difficult for individuals to understand.

### Teaching of Rational Numbers

According to Behr et al. (1992) until 1992 few research studies specifically targeted teaching of rational numbers. Lamon (2007) argued that this was due to the fact that the research domain including rational numbers, fractions, ratios, and proportions had not reached a level of maturity which could inform teaching practices. A number of researchers (e.g., Lamon 2007; Confrey et al. 2008) designed intervention programs by identifying learning trajectories and then tested their results.

The use of manipulatives (concrete or virtual) and of multiple representations are considered as very important in the teaching of fractions and especially in teaching operations with fractions. Research has shown positive impact of the use of visual representations on students’ conceptual understanding of fractions. A widely used representation is the area model representation (Saxe et al. 2007). However, according to Saxe et al. (2007), area models have some limitations. These researchers (Saxe et al. 2007) conducted research studies and designed programs in order to investigate the way in which number lines can help students develop understanding of the fraction concepts and their properties. The use of number line was also acknowledged as very important for the understanding of decimal. In addition, there is also considerable research on the investigation of virtual representations and more extensively of digital technologies in the learning of fractions.

## Negative Numbers

The concept of negative numbers is introduced when students have already learned to work with natural numbers. As a result, when the teaching of negative numbers begins, some properties concerning natural numbers turn out to be conflicting. Fischbein (1987) claimed that two intuitive obstacles affect students' understanding of negative numbers. First, the concept of negative numbers is intuitively contradictory to the concept of positive numbers defined as quantifiable entities. Secondly, negative numbers are a "by-product of mathematical calculations and not the symbolic expression of existing properties" (Fischbein 1987, p. 101). Another main obstacle identified is the difficulty to see the number line as one thing (unified number line) where the value of numbers is supported and not as two opposite semi-lines (divided line) where only the magnitude of numbers is supported. Students have to realize the difference between the magnitude and the value of the number. Despite the difficulties students might face while dealing with negative numbers, there is also evidence that students have intuitive knowledge about negative numbers and in some cases are able to perform operations with negative numbers before formal instruction. Thus, recently a number of researchers claimed that addition and/or subtraction with negative numbers may be introduced from younger ages if appropriate models are used. Still there is also evident that students face difficulty to move from concrete operations to formal operations.

Regarding the teaching of negative numbers, there is a long-standing debate whether they should be introduced through models (such as number lines, elevators, or the annihilation/creation model where two-color counters are used) (Verschaffel et al. 2006) or as formal abstractions (Fischbein 1987). Most of the researchers seem to adopt the model approach. There is no consensus regarding the model or representation which is most effective as well as the number of different models (multiplicity or not) that should be used. Opinions are also conflicting regarding the use of these models and whether they should be used only during the introduction of negative numbers or all the way through the teaching of integers.

### Irrational Numbers

Despite the importance of irrational numbers, only a small number of research studies have focused on this topic. The concept of irrational numbers is considered as one of the most difficult concepts in mathematics, especially since it does not present discrete countable quantities but refers to continuous quantities. It is a by-product of logical deduction and cannot be captured by our senses. Most studies emphasize the deficiencies in students' and teachers' understanding of irrational numbers; for instance, their difficulty to provide appropriate definition for irrational number or to recognize whether a number is rational or irrational. Students often develop the understanding of natural, rational, and irrational numbers as different systems and are unable to see them in a flexible whole (Zazkis and Sirotic 2010). This is often a source of students' difficulties and misconceptions. Other research studies showed that students' and prospective teachers' difficulties may also arise from the discreteness of natural numbers, which is a barrier to understanding the dense structure of the rational and irrational numbers. Attempts were also made to develop instructional material for secondary school students, preservice, and in-service teachers.

## Cross-References

▶ Affect in Mathematics Education
▶ Algorithms
▶ Data Handling and Statistics Teaching and Learning
▶ Early Algebra Teaching and Learning
▶ Early Childhood Mathematics Education
▶ Ethnomathematics
▶ Information and Communication Technology (ICT) Affordances in Mathematics Education
▶ Intuition in Mathematics Education
▶ Learning Difficulties, Special Needs, and Mathematics Learning
▶ Manipulatives in Mathematics Education
▶ Mathematical Modelling and Applications in Education
▶ Mathematical Representations
▶ Metacognition
▶ Misconceptions and Alternative Conceptions in Mathematics Education

▶ Number Lines in Mathematics Education
▶ Pedagogical Content Knowledge in Mathematics Education
▶ Probability Teaching and Learning
▶ Problem Solving in Mathematics Education
▶ Realistic Mathematics Education
▶ Shape and Space – Geometry Teaching and Learning
▶ Situated Cognition in Mathematics Education
▶ Sociomathematical Norms in Mathematics Education
▶ Teacher Beliefs, Attitudes, and Self-Efficacy in Mathematics Education
▶ Word Problems in Mathematics Education

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# P

## Pedagogical Content Knowledge in Mathematics Education

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### Keywords

Mathematical knowledge for teaching; Professional knowledge; Lee Shulman, Deborah Ball

### Characteristics

Intense focus on the notion of "pedagogical content knowledge" (PCK) within teacher education is attributed to Lee Shulman's 1985 AERA Presidential address (Shulman 1986) in which he referred to PCK as the "special amalgam of content and pedagogy" central to the teaching of subject matter. His widely cited follow-up paper (Shulman 1987) elaborated PCK as follows:

> the most powerful analogies, illustrations, examples, explanations, and demonstrations — [. . .] the most useful ways of representing and formulating the subject that make it comprehensible to others.... Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them. . . (p. 7)
>
> the particular form of content knowledge that embodies the aspects of content most germane to its teachability. (p. 9)

Immediate and widespread interest in the notion rested on Shulman's claim that PCK, combined with subject knowledge and curriculum knowledge, formed critical knowledge bases for understanding and improving subject-specific teaching. While subject matter knowledge (SMK) and PCK are frequently dealt with together in research studies, interest and contestation in the boundary lead to separate but related entries for them in this encyclopedia (see SMK entry). PCK studies in mathematics education indicate attempts at (a) sharpening theorizations of PCK, (b) measuring PCK, and (c) using notions of PCK to build practical skills within teacher education or combinations of these elements. This entry summarizes key work across these groups.

#### Theorizations of PCK

Key writings in the category of sharpening theorizations of PCK examine both the boundary between PCK and the broader field of subject-related knowledge – sometimes referred to as "mathematics knowledge for teaching" (MKT) – and inwards at subcategories within PCK.

Deborah Ball and the Michigan research group sharpened the distinctions between content knowledge and PCK in their theorization based on the classroom practices of expert teachers: "subject matter knowledge" (SMK) broke down into

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common content knowledge (CCK), specialized content knowledge (SCK), and horizon knowledge and PCK into knowledge of content and students (KCS), knowledge of content and teaching, and knowledge of curriculum (Ball et al. 2008).

Critiques of work drawing from Shulman's categorizations argue that the "static" conceptualization of MKT with separate components is unhelpful in relation to the interactive and dynamic nature of MKT. Centrally, these critiques argue that MKT is better interpreted as an attribute of pedagogic practices in specific contexts and related to specific mathematical ideas, rather than a generalized attribute of the teacher. Fennema and Franke's (1992) conceptualization of MKT as constituted by knowledge of mathematics, combined with PCK comprised of elements of knowledge of learners' mathematical cognition, pedagogical knowledge, and beliefs views this combination as a taxonomy that can identify the "context-specific knowledge" of a teacher, rather than a more generalized picture of the teacher's MKT. Rowland et al. (2003) similarly emphasize, in their "Knowledge Quartet" formulation consisting of Foundation, Transformation, Connection, and Contingency knowledge (the latter three relating to PCK), that the profile of MKT produced is a categorization of teaching situations, rather than of teachers.

While all of these models were developed from studies of practice, Fennema and Franke and Rowland et al.'s models include a beliefs component – which does not feature in Ball et al.'s conceptualization.

Other studies have looked at PCK in alternative formulations (e.g., Silverman and Thompson 2008), with the notion of "connections" within mathematics and with learning (Askew et al. 1997; Ma 1999) seen as critical. Petrou and Goulding (2011) provide an overview of key writings in the MKT field.

## Measuring PCK

Ball's research group shifted their attention into measuring MKT to verify assumptions about its relationship to teaching quality and student learning. The group developed multiple choice items based on specific MKT subcomponents that were administered to teachers, with data collected on their elementary grade classes' learning backgrounds and learning gains across a year. Hill et al.'s (2005) analysis showed content knowledge measures across the common and specialized categories as significantly associated with learning gains. While Ball's group conceptualizes CCK and SCK as part of content knowledge, the descriptions of SCK that are provided – e.g., understanding of representations and explanations – fall within other writers' conceptualizations of PCK.

Baumert et al. (2010), noting the absence of direct attention to teaching in Ball et al.'s measurement-oriented work, developed the COACTIV framework that distinguished content knowledge from PCK and examined the relationships between content knowledge, PCK, classroom teaching, and student learning gains in Germany. In the COACTIV (Professional Competence of Teachers, Cognitively Activating Instruction, and the Development of Students' Mathematical Literacy) model (focused on secondary mathematics teaching), content knowledge is understood as "a profound mathematical understanding of the mathematics taught at school" (p. 142), and PCK is subdivided into knowledge of mathematical tasks as instructional tools, knowledge of students' thinking and assessment of understanding, and knowledge of multiple representations and explanations of mathematical problems. With this distinction, separate content knowledge and PCK open response items were developed and administered to nearly 200 teachers in different tracks of the German schooling system. Mathematics test performance data were gathered from over 4,000 students in these teachers' classes. Instructional quality was measured through three data sources. The first encompassed selected class, homework, test and examinations tasks, and the degree of alignment between assessment tasks and the Grade 10 curriculum. The second source considered the extent of individual learning support, measured through student rating scales. The third source examined classroom management as degree of agreement between teacher and student perceptions about disciplinary climate.

Baumert et al.'s findings suggested that their theoretical division of content knowledge and PCK was empirically distinguishable, with their PCK variable showing more substantial associations with student achievement and instructional quality than their content knowledge variable.

### Using PCK to Support the Development of Pedagogic Practice

The third category of PCK literature links to studies of teacher development using PCK frameworks. This strand often uses longitudinal case study methodologies.

Fennema and Franke and Rowland's MKT models have associated development-focused studies. Turner and Rowland (2011) provide examples of the Knowledge Quartet's use in England to stimulate development of teaching, and Fennema and Franke, with colleagues, have produced studies on the longevity of the PCK aspects presented within professional development programs.

This category too contains other studies drawing on aspects of PCK. Kinach (2002) focuses on secondary mathematics teachers' development of instructional explanations – a key feature of PCK across different formulations. Learning studies interventions (Lo and Pong 2005) focus on building teachers' awareness of the relationship between particular objects of learning and students' work with these objects – a feature of the KCS terrain.

### Emerging Directions

Emerging work questions the assumption of basic coherence and connection in MKT that underlies much of the PCK writing (Silverman and Thompson 2008). Qualitative case studies of classroom teaching detail inferences relating to PCK (and SMK), and the consequences for mathematics learning in contexts of pedagogic fragmentation and disconnections are beginning to feature (Askew et al. 2012).

PCK as a field therefore continues to thrive, in spite of ongoing differences in nomenclature, underlying views about specific sub-elements, and the nature of their interaction.

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▶ Mathematical Knowledge for Teaching

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# Policy Debates in Mathematics Education

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## Definition

Policy in mathematics education concerns the nature and shape of the mathematics curriculum, that is, the course of study in mathematics of a school or college. This is the teaching sequence for the subject as planned and experienced by the learner. Four aspects can be distinguished, and these are the focuses of policy debates:

1. The aims, goals, and overall philosophy of the curriculum
2. The planned mathematical content and its sequencing, as in a syllabus
3. The pedagogy employed by teachers
4. The assessment system

## History

The New Math debate of the late 1950s to the mid-1960s was primarily about the content of the mathematics curriculum. At that time traditional school mathematics did not incorporate any modern topics. The content consisted primarily of arithmetic at elementary school, plus traditional algebra, Euclidean geometry, and trigonometry at high school. The New Math curriculum broadened the elementary curriculum to include other aspects of mathematics, and high school mathematics incorporated modern algebra (including sets, functions, matrices, vectors), statistics and probability, computer mathematics (including base arithmetic), and modern geometry (transformation geometry, topological graph theory). The launch of Sputnik, the first earth orbiting satellite, by the Soviet Union in 1957, during the Cold War led to fears that the USA and UK were being overtaken in technology and in mathematics and science education by the Soviets. Government funding became available, especially in the USA, to extend projects modernizing the mathematics curriculum in a bid to broaden and improve students' knowledge of mathematics, such as the Madison Project in 1957 and The School Mathematics Study Group in 1958 in the USA. In the UK independent curriculum projects emerged, including the School Mathematics Project (SMP) in 1961 and Nuffield Primary Mathematics in 1964. These projects did not cause much controversy at the national policy levels although there was a concern by parents that they did not understand the New Math their children were learning. The relatively muted debates concerned the changing content of the mathematics curriculum rather than its pedagogy or assessment.

In the mid to late 1960s onwards a new debate emerged about discovery learning. In the UK the Schools Council Curriculum Report No. 1 (Biggs 1965) on the teaching and learning of mathematics in primary school proposed practical approaches and "discovery learning" as the most effective ways of teaching mathematics. Sixty-five percent of all primary teachers in the UK read Biggs (1965), and it had a significant impact. Discovery learning was a central part of the 1957 Madison Project developed by Robert B. Davis. This and similar developments led to a major policy debate on discovery learning. Is discovery learning the most effective way to learn mathematics? Proponents of discovery contrasted it with rote learning. Self-evidently rote learning cannot be the best way to learn all but the simplest mathematical facts and skills since it means simply "learning by heart." However, educational psychologist Ausubel (1968) argued successfully that discovery and rote learning are not part of a continuum but on two orthogonal axes defined by pairs of opposites: meaningful versus rote learning and reception versus discovery learning. Meaningful learning is linked to existing knowledge; it is relational and conceptual. Rote learning is arbitrary, verbatim, and disconnected – unrelated to other existing knowledge of the learner. Knowledge learned by reception comes already formulated and is acquired through communication, such as in expository teaching or reading. Ausubel distinguishes this from discovered knowledge that has to be formulated by the learner herself.

The promotion of discovery learning led to heated debate on both sides of the Atlantic. Shulman and Keislar (1966) offered a review, but to this day the evidence remains equivocal. This debate was primarily about pedagogy – how best to teach mathematics. But underneath this debate one can discern battle lines being drawn between a child-centered, progressive ideology of education with roots going back to Rousseau, Montessori, Dewey, and a traditionalist teacher- and knowledge-centered ideology of education favored by some mathematicians and university academics.

The mid-1970s saw the birth of the back-to-basics movement promoting basic arithmetical skills as the central goal of the teaching and learning of mathematics for the majority. This was a reaction to the progressivism of the previous decade, most clearly defined in the aims of the Industrial Trainers group mentioned below, and became an important plank of the traditionalist position on the mathematics curriculum.

The early 1980s led to a further entrenchment in the progressive/traditional controversy. In the USA the influential National Council of Teachers of Mathematics (NCTM) recommended that "Problem solving must be the focus of school mathematics in the 1980s" (1980, pp. 2–4). In the UK the Cockcroft Inquiry (1982) recommended problem solving and investigational work be included in mathematics for all students. Thus the debate remained at the level of pedagogy but shifted to problem solving.

The progressivist versus traditionalist debate was born anew in the late 1980s (UK) and the 1990s (USA) but now encompassed the whole mathematics curriculum on a national basis.

## Analytical Framework

The British government developed and installed the first legally binding National Curriculum in 1988 for all students age 5–16 years in all state schools (excluding Scotland). The debate over the mathematics part of National Curriculum in became a heated contest between different social interest groups. Ernest (1991) analyzed this as a contest between five different groups with different broad ranging ideologies of education, the aims, and orientation of which are summarized in Table 1 (In the full treatment there are 14 different ideological components for each of these 5 groups).

These different social groups were engaged in a struggle for control over the National Curriculum in mathematics, since the late 1980s (Brown 1996). In brief, the outcome of this contest was that the first three more reactionary groups managed to win a place for their aims in the curriculum. The fourth group (progressive educators) reconciled themselves with the inclusion of a personal knowledge-application dimension, namely, the processes of "Using and Applying mathematics," constituting one of the National Curriculum assessment targets. However instead of representing progressive self-realization through creativity aims through mathematics, this component embodies utilitarian aims: the practical skills of being able to apply mathematics to solve work-related problems with mathematics. Despite this concession over the nature of the process element included in the curriculum, the scope of the element has been reduced over successive revisions that have occurred in the subsequent 20 years and has largely been eliminated. The fifth group, the public educators, found their aims played no part in the National Curriculum. The outcome of the process is a largely utilitarian mathematics curriculum developing general or specialist mathematics skills and capabilities, which are either decontextualized – equipping the learner with useful tools – or which are applied to practical problems. The contest between the interest groups was an ideological one, concerning not only all four aspects of curriculum but also about deeper epistemological theories on the nature of mathematics and the nature of learning.

**Policy Debates in Mathematics Education, Table 1** Five interest groups and their aims for mathematics teaching (based on Ernest 1991)

| Interest group | Social location | Orientation | Mathematical aims |
|---|---|---|---|
| **1. Industrial trainers** | Radical New Right conservative politicians and petty bourgeois | Authoritarian, basic skills centered | Acquiring basic mathematical skills and numeracy and social training in obedience |
| **2. Technological pragmatists** | Meritocratic industry-centered industrialists, managers, etc., New Labor | Industry and work centered | Learning basic skills and learning to solve practical problems with mathematics and information technology |
| **3. Old Humanist mathematicians** | Conservative mathematicians preserving rigor of proof and purity of mathematics | Pure mathematics centered | Understanding and capability in advanced mathematics, with some appreciation of mathematics |
| **4. Progressive educators** | Professionals, liberal educators, welfare state supporters | Child-centered progressivist | Gaining confidence, creativity, and self-expression through maths |
| **5. Public educators** | Democratic socialists and radical reformers concerned with social justice and inequality | Empowerment and social justice concerns | Empowerment of learners as critical and mathematically literate citizens in society |

During the period following the introduction of the National Curriculum in mathematics, pressure from various groups continued to be exerted to shift the emphasis of the curriculum. Mathematicians who can often be characterized as belonging to the Old Humanist grouping published a report entitled *Tackling the Mathematics Problem* (London Mathematical Society 1995), commissioned by professional mathematical organizations. This criticized the inclusion of "time-consuming activities (investigations, problem solving, data surveys, etc.)" at the expense of "core" technique and technical fluency. Furthermore, it claimed many of these activities are poorly focused and can obscure the underlying mathematics. This criticism parallels that heard in the "math wars" debate in the USA.

## "Math Wars"

In the USA the National Council of Teachers of Mathematics (NCTM) published its so-called Standards document in 1989 recommending a "Reform"-based (progressive) mathematics curriculum for the whole country. This emphasized problem solving and constructivist learning theory. The latter is not just discovery learning under a new name because constructivists acknowledge that learners need to be presented with representations of existing mathematical knowledge to reconstruct them for themselves. This initiated the savage debate in the USA called the Math Wars (Klein 2007).

The Standards influenced a generation of new mathematics textbooks in the 1990s, often funded by the National Science Foundation. Although widely praised by mathematics educators, particularly in California, concerned parents formed grassroots organizations to object and to pressure schools to use other textbooks. Reform texts were criticized for diminished content and lack of attention to basic skills and an emphasis on progressive pedagogy based on constructivist learning theory. Critics in the debate derided mathematics programs as "dumbed-down" and described the genre as "fuzzy math."

In 1997 Senator Robert Byrd joined the debate by making searing criticisms of the mathematics education reform movement from the Senate floor focusing on the inclusion of political and social justice dimensions in one mathematics textbook. In the spreading and increasingly polarized debate, the issues spread from traditional versus progressive content and pedagogy to left versus right political orientations and traditional objectivist versus constructivist (relativist) epistemology and philosophy of mathematics. This way the debate took on aspects of the parallel "science wars" also taking place, primarily in the USA. This is the heated debate between scientific realists, who argued that objective scientific knowledge is real and true, and sociologists of science. The latter questioned scientific objectivity and argued that all knowledge is socially constructed. This is an insoluble epistemological dispute that has persisted at least since the time of Socrates in philosophical debates between skeptics and dogmatists. Nevertheless, it fanned the flames of the Math Wars debate.

In 1999 the US Department of Education released a report designating 10 mathematics programs as “exemplary” or “promising.” Several of the programs had been singled out for criticism by mathematicians and parents. Almost immediately an open letter to Secretary of Education Richard Riley was published calling on him to withdraw these recommendations. Over 200 university mathematicians signed their names to this letter and included seven Nobel laureates and winners of the Fields Medal. This letter was repeatedly used by traditionalists in the debate to criticize Reform mathematics, and in 2004 NCTM President Johnny Lott posted a strongly worded denunciation of the letter on the NCTM website.

In 2006, President Bush was stirred into action by the heated controversy and created the National Mathematics Advisory Panel to examine and summarize the scientific evidence related to the teaching and learning of mathematics. In their 2008 report, they concluded that recommendations that instruction should be entirely “student centered” or “teacher directed” are not supported by research. High-quality research, they claimed, does not support the exclusive use of either approach. The Panel called for an end to the Math Wars, although its recommendations were still the subject of criticism, especially from within the mathematics education community for its comparison of extreme forms of teaching and for the criteria used to determine “high-quality” research.

## Defusing the Debates

Policy debates have raged over the mathematics curriculum throughout the past 50 years. They have been strongest in the USA and UK but have occurred elsewhere in the world as well. In Norway, for example, there is a much more muted but still heated debate as to whether mathematics or the child should be the central focus of the curriculum. Proponents of a child-centered curriculum promote general pedagogy in teacher education as opposed to a specifically mathematics pedagogy with its associated emphasis on teachers’ pedagogical content knowledge in mathematics.

The spread of policy debates has also become much wider following the impact of international assessment projects such as TIMSS. Politicians sometimes blame what is perceived to be poor national performance levels in mathematics on one or other aspect of the curriculum. Unfortunately policy debates too often become politicized and drift away from the central issues of determining the best mathematics curriculum for students. In becoming polarized, the debates become controversies that propel policy swings from one extreme to the other, like a pendulum. Ernest (1989) noted this pattern, but regrettably the pendulum-like swings from one extreme position to the opposite continue unabated to this day.

The fruitlessness of swings from traditional to progressive pedagogy in mathematics is illustrated in an exemplary piece of research by Askew et al. (1997). This project studied the belief sets and teaching practices of primary school teachers and their correlation with students’ numeracy scores over a period of 6 months. Three belief sets and approaches to teaching numeracy were identified in the teachers:

1. Connectionist beliefs: valuing students’ methods and teaching with emphasis on establishing connections in mathematics (mathematics and learner centered)
2. Transmission beliefs: primacy of teaching and view of maths as collection of separate routines and procedures (traditionalist)
3. Discovery beliefs: primacy of learning and view of mathematics as being discovered by students (progressivist)

The classes of teachers with a connectionist orientation made the greatest gains, so teaching for connectedness were measurably the most effective methods. This included attending to and valuing students’ methods as well as teaching with an emphasis on establishing connections in mathematics. Traditional transmission beliefs and practices were not shown to be as effective. Likewise, discovery beliefs and practices were equally ineffective, refuting the progressivist claim that the teaching and learning of mathematics by discovery is the most effective approach. Of course Askew et al. (1997) only report a small-scale, in-depth study of about 20 teachers and must be viewed with caution and needs replication. Nevertheless its results illustrate the futility of policy debates becoming overly ideological and losing contact with empirical measures of effectiveness from properly conducted research.

## Cross-References


- ▶ Authority and Mathematics Education
- ▶ Critical Mathematics Education
- ▶ Dialogic Teaching and Learning in Mathematics Education
- ▶ History of Mathematics Teaching and Learning
- ▶ Inquiry-Based Mathematics Education
- ▶ Mathematical Literacy
- ▶ Political Perspectives in Mathematics Education
- ▶ Realistic Mathematics Education
- ▶ Recontextualization in Mathematics Education
- ▶ Teacher-Centered Teaching in Mathematics Education


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# Political Perspectives in Mathematics Education


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## Keywords

Power; Politics; Modernity; Neutrality of mathematics; Social rationalities; In(ex)clusion; Mathematics for all; Credit system; Subjectivity


## Definition

A political perspective in mathematics education is a way of looking at how mathematics, education, and society relate to power. It stands on the critical recognition that mathematics is not only important in society due to its exceptional, intrinsic characteristics as the purest and most powerful form of abstract thinking but also and foremost, because of its functionality in the constitution of the dominant cultural project of Modernity. Thus, it assumes that the teaching and learning of mathematics are not neutral practices but that they insert people – be it children, youth, teachers, and adults – in socially valued mathematical rationalities and forms of knowing. Such insertion is part of larger processes of selection of people that schooling operates in society. It results in differential positioning of inclusion or exclusion of learners in relation to access to socially privileged resources such as further education, labor market, and cultural goods.

## History

The political perspectives of mathematics education became a concern for teachers and researchers in the 1980s. While the change from the nineteenth to the twentieth centuries was a time of inclusion of mathematics in growing, massive, national education systems around the world, the change from the twentieth to the twenty-first centuries has been a time for focusing on the justifications for the privileged role of mathematics in educational systems at all levels. The apparent failure of the New Math movement in different industrialized countries allowed to raise concerns about the need for mathematics teaching and learning that could reach as many students as possible and not only a selected few (Damerow et al. 1984). Questions of how mathematics education could be studied from perspectives that allowed moving beyond the boundaries of the mathematical contents in the school curriculum started to be raised. In mathematics education, the first book published in English as part of an international collection, containing the word politics in the title, was "The politics of mathematics education" by Stieg Mellin-Olsen (1987). However, "The mastery of reason: Cognitive development and the production of rationality" by Valerie Walkerdine (1988) is a seminal work in critical psychology discussing how school mathematics education subjectifies children through inscribing in them and in society, in general, specific notions of the rational child and of abstract thinking.

The political concern and involvement of many mathematics educators in their teaching and research practice was also an initial entry that allowed sensitivity and awareness for searching how mathematics education could be "political" (Lerman 2000). Such political awareness on issues such as how mathematics has played a role as gatekeeper to entry in further education, for example, has been important. However, a political "awareness" does not constitute the center of a political approach since there is a distinction between being sympathetic to how mathematics education relates to political processes of different type and making power in mathematics education the focus of one's research. In other words, not all people who express a political sympathy actually study the political in mathematics education (Gutierrez 2013; Valero 2004).

With this central distinction in mind, it is possible to differentiate a variety of political perspectives, some that could be called *weak* in the sense that they make a connection between mathematics education and power but do not concentrate on the study of it as a constituent of mathematics education but rather as a result or a simply associated factor. *Strong* political approaches in mathematics education are a variety of perspectives that do have a central interest in understanding mathematics education as political practices.

## Weak Political Perspectives

A general characteristic of weak political perspectives in mathematics education is the adherence to some of the positive features attributed to mathematics and mathematics education, particularly those that have to do with people's empowerment and social and economic progress. More often than not, these views assume some kind of intrinsic goodness of mathematics and mathematics education that is transferred to teachers and learners alike through good and appropriate education practices. In the decade of the 1980s and fully in the 1990s, the broadening of views on what constitutes mathematics education allowed for formulations of the aims of school mathematics in relation to the response to social challenges of changing societies and, in particular, in response to the consolidation of democracy. It was possibility to enunciate the idea that, as part of a global policy of "Education for all" by UNESCO, mathematics education had to contribute to the competence of citizens, but also to open access for all students. In many countries, both at national policy level and at the level of researchers and teachers, there was a growing concern for mathematics for all and mathematics for equity and inclusion. The study of how different groups – women, linguists, ethnic or religious minorities, and particular racial groups – of students systematically underachieve and how to remediate that situation grew extensively. Part of the weak political approaches also includes studies of how mathematics education practices are shaped by educational policies. South Africa, given the transition from apartheid to democracy at the beginning of the 1990s, has been a particularly interesting national case where deep changes of policy had been studied to see how and why mathematics education in primary and secondary school is transforming to contribute – or not – to the construction of a new society. Many of these studies have a weak political approach in the sense that they are justified and operate on some political assumptions on mathematics education and its role in society, but intend to study appropriate pedagogies and not how pedagogies in themselves effect the exclusion that the programs intend to remediate.

## Strong Political Perspectives

Strong political perspectives in mathematics education problematize the assumed neutrality of mathematical knowledge and provide new interpretations of mathematics education as practices of power. Ethnomathematics can be read as a political perspective in mathematics education in its challenge to the supremacy of Eurocentric understandings of mathematics and mathematical practices. The strong political perspectives of ethnomathematics is presented in studies that not only argue for how the mathematical practices of different cultural groups – not only indigenous or ethnic groups but also professional groups – are of epistemological importance and value but also how some of those cultural practices are inserted in the calculations of power so that they can construct a regime of truth around themselves and thus gain a privileged positioning in front of other practices (Knijnik 2012).

Critical mathematics education as a wide and varied political approach takes the study of power in relation to how mathematics is a formatting power in society through its immersion in the creation of scientific and technological structures that operate in society (Christensen et al. 2008). It also studies the processes of exclusion and differentiation of students when mathematics education practices reproduce the position of class and disadvantage of students (Frankenstein 1995); and when such reproduction is part of the way (school), mathematics is given meaning in public discourses and popular culture (Appelbaum 1995). It also offers possibilities for rethinking practices when democracy is thought as a central element of mathematics education (Skovsmose and Valero 2008).

The study of the political in the alignment of relation to the alignment of mathematics education practices with Capitalism is also a recent and strong political reading of mathematics education that offers a critical perspective on the material, economic significance of having success in mathematics education. Both educational practices (Baldino and Cabral 2006) and research practices (Lundin 2012; Pais 2012) lock students in a credit system where success in mathematics represents value.

In the USA, and as a reaction to endemic operation of race as a strong element in the classification of people's access to cultural and economic resources, the recontextualization of critical race theories into mathematics education has provided new understandings of mathematics education as a particular instance of a White-dominant cultural space that operates exclusion from educational success for African American learners (Martin 2011), as well as for Latino(a)s (Gutiérrez 2012).

The recontextualization of poststructural theories in mathematics education has also led to the study of power in relation to the historical construction of Modern subjectivities. The effects of power in the bodies and minds of students and teachers (Walshaw 2010), as well as in the public and media discourses on mathematics (Moreau et al. 2010), are studied in an attempt to provide insights into how the mathematical rationality that is at the core of different technologies in society shapes the meeting between individuals and their culture. Even though most research concentrates on the issue of identity construction and subjectivity, some studies attempting cultural histories of mathematics as part of Modern, massive educational systems are also broadening this type of political perspective (Popkewitz 2004; Valero et al. 2012).

## Cross-References

- ► Equity and Access in Mathematics Education
- ► Mathematization as Social Process
- ► Policy Debates in Mathematics Education
- ► Poststructuralist and Psychoanalytic Approaches in Mathematics Education
- ► Socioeconomic Class in Mathematics Education
- ► Sociological Approaches in Mathematics Education

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# Poststructuralist and Psychoanalytic Approaches in Mathematics Education


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## Keywords

Poststructuralism; Psychoanalysis

## Definition

Approaches that draw on developments within wider scholarly work that conceives of modernist thought as limiting.

## Characteristics

Poststructuralist and psychoanalytic approaches capture the shifts in scholarly thought that gained currency in Western cultures during the past 50 years. Conveying a critical and self-reflective attitude, both raise questions about the appropriateness of modernist thinking for understanding the contemporary social and cultural world. Since the publication of Lyotard's *The Postmodern Condition* (translated into English in 1984), poststructuralist and psychoanalytic thinking have provided an expression within the social sciences and humanities and, more recently, within mathematics education, for a loss of faith in the "grand narratives" of Western history and, in particular, enlightened modernity. A diverse set of initiatives in social and philosophical thought, originating from the work of Michel Foucault (e.g., 1970), Jacques Derrida (e.g., 1976), Julia Kristeva (e.g., 1986), and Jacques Lacan (1977), among others, helped crystallize poststructuralist and psychoanalytic ideas among researchers and scholars within mathematics education about how things might be thought and done differently.

Poststructuralist and psychoanalytic approaches provide alternatives to the traditions of psychological and sociological thought that have grounded understandings about knowledge, representation, and subjectivity within mathematics education. These traditions understand reality as characterized by an objective structure, accessed through reason. More specifically, the traditions are based on the understanding that reason can provide an authoritative, objective, true, and universal foundation of knowledge. They also assume the transparency of language. Epistemological assumptions like these, about the relationship between the knower and the known, are accompanied by beliefs about the kind of being the human is. Typically, the related ontologies are dualist in nature. They include such dichotomies as rational/irrational, objective/subjective, mind/body, cognition/affect, and universal/particular. Taken together, these characteristically modernist beliefs about ontology and epistemology have informed theories of human interaction, teaching, learning, and development within mathematics education.

Developments within psychology and sociology that began to question these understandings paved the way for a different perspective. Sociology has helped seed poststructuralist work that aims to draw attention to the ways in which power works within mathematics education, at any level, and within any relationship, to constitute identities and to shape proficiencies. Psychology has informed a psychoanalytical turn, designed to unsettle fundamental assumptions concerning identity formations. Postmodernists and psychoanalysts share some fundamental assumptions about the nature of the reality being studied, assumptions about what constitutes knowledge of that reality, and assumptions about what are appropriate ways of building knowledge of that reality.

Researchers in mathematics education who draw on this body of work have an underlying interest in understanding, explaining, and analyzing the practices and processes within mathematics education. Their analyses chart teaching and learning, and the way in which identities and proficiencies evolve; tracking reflections; investigating everyday classroom planning, activities, and tools; analyzing discussions with principals, mathematics teachers, students, and educators; mapping out the effects of policy, and so forth. In the process of deconstructing taken-for-granted understandings, they reveal how identities are constructed within discourses, they demonstrate how everyday decisions are shaped by dispositions formed through prior events, and they provide insights about the way in which language produces meanings and how it positions people in relations of power. The assumptions upon which these analyses are based enable an exploration of the lived contradictions of mathematics processes and structures.

These analyses are developed around a number of key organizing principles: language is fragile and problematic and constitutes rather than reflects an already given reality. Meaning is not absolute in relation to a referent, as had been proposed by structuralism. The notion of knowing as an outcome of human consciousness and interpretation, as described by phenomenology, is also rejected. Moreover, knowing is not an outcome of different interpretations, as claimed by hermeneutics. Instead, for poststructuralist and psychoanalytic scholars, reality is in a constant process of construction. What is warranted at one moment of time may be unwarranted at another time. The claim is that because the construction process is ongoing, no one has access to an independent reality. There is no "view from nowhere," no conceptual space not already implicated in that which it seeks to interpret. There is no stable unchanging world and no realm of objective truths to which anyone has access. The notion of a disembodied autonomous subject with agency to choose what kind of individual he or she might become also comes under scrutiny. The counter-notion proposed is a "decentered" self – a self that is an effect of discourse which is open to redefinition and which is constantly in process.

## Poststructuralist Approaches

Foucault's work is considered by many to represent a paradigmatic example of poststructuralist thought. His work raises critical concerns about how certain practices, and not others, become intelligible and accepted, and how identities are constructed. Foucauldian analyses centered within mathematics educational sites explore lived experience, not in the sense of capturing reality and proclaiming causes but of understanding the complex and changing processes by which subjectivities and knowledge production are shaped. In that sense, the focus shifts from examining the nature of identity and knowledge to a focus on how identity and knowledge are discursively produced. In these analyses, "discourse" is a key concept. Discourses sketch out, for teachers, students, and others, ways of being in the classroom and within other institutions of mathematics education. They do that by systematically constituting specific versions of the social and natural worlds for them, all the while obscuring other possibilities from their vision. Discursivity is not simply a way of organizing what people say and do; it is also a way of organizing actual people and their systems. It follows that "truths" about mathematics education emerge through the operation of discursive systems.

Discursive approaches within mathematics education draw attention to the impact of regulatory practices and discursive technologies on the constructions of teachers, students, and others. It reveals the contradictory realities of teachers, students, policy makers, and so forth and the complexity and complicity of their work. Such work emphasizes that teachers and students are the production of the practices through which they become subjected (e.g., Hardy 2009; Lerman 2009).

Power in these approaches envelopes everyone. What the analyses reveal is that, in addition to operating at the macro-level of the school, power seeps through lower levels of practice such as within teacher/student relations and school/teacher relations (see Walshaw 2010). Even in a classroom environment that provides equitable and inclusive pedagogical arrangements, poststructural approaches have shown that power is ever present through the classroom social structure, systematically creating ways of being and thinking in relation to class, gender, and ethnicity and a range of other social categories (see Walshaw 2001; Mendick 2006; Knijnik 2012).

In illuminating the impact of regulatory practices and technologies on identity and knowledge production, fine-grained readings of classroom interaction have revealed the regulatory power of teachers' discourse in providing students with differential access to mathematics (de Freitas 2010). Such readings shed light on how the discursive practices of teachers contribute to the kind of mathematical thinking and the kind of mathematical identities that are possible within the classroom.

### Psychoanalytic Approaches

Psychoanalytic analyses in mathematics education explore the question of identity. Lacan's (e.g., 1977) and Žižek's (e.g., 1998) explanations of how identities are constructed through an understanding of how others see that person have been influential in revealing that teachers, students, and others are not masters of their own thoughts, speech, or actions. Žižek's psychoanalytic position is that the self is not a center of coherent experience: "there are no *identities* as such. There are just *identifications* with particular ways of making sense of the world that shape that person's sense of his self and his actions" (Brown and McNamara 2011, p. 26). A person's identifications are not reducible to the identities that the person constructs of himself. Rather, the self is performed within the ambivalent yet simultaneous relationship of subjection/agency.

Psychoanalytic observations of identity formation are likely to reveal how identities develop through discourses and networks of power that shift continually in a very unstable fashion, changing as alliances are formed and reformed. When identities are formed in a very mobile space, what emerge are fragmented selves, layers of self-understandings, and multiple positionings within given contexts and time (see Hanley 2010). This psychoanalytic idea is fundamental to understanding that teachers and students (among others) negotiate their way through layered meanings and contesting perceptions of what a "good" teacher or student looks like. To complete a negotiation, there is a level at which the teacher or student invests, or otherwise, in a discursive position made available (see Bibby 2009).

A teacher's, for example, investments within one discourse rather than another is explained through the notion of affect and, more especially, through the notions of obligation and reciprocity. Affect, in the psychoanalytic analysis, is not a derivative aspect but a constitutive quality of classroom life (see Walshaw and Brown 2012). It is not an interior experience, but rather, it operates through processes that are historical in a way that is not entirely rational nor observable. Researchers in mathematics education who draw on psychoanalytic theory maintain that determinations exist outside of our consciousness and, in the pedagogical relation, for example, influence the way teachers develop a sense of self as teacher and influence their interactions in the classroom. The identities teachers have of themselves are, in a very real sense, "comprised," made in and through the activities, desires, interests, and investments of others. Understandings like these invite unknowingness, fluidity, and becoming, which, in turn, have the effect of producing different knowledge.

### Emancipatory Possibilities

Although both poststructuralist and psychoanalytic theorists question the modernist concept of enlightenment, in reconceptualizing emancipation away from individualist sensibilities, they highlight possibilities for where and in what ways mathematics educational practices might be changed (see Radford 2012). In addition to uncovering terrains of struggle, poststructuralist and psychoanalytic analyses foster democratic provision, enabling a vision of critical-ethical teaching where different material and political conditions might prevail. What is clarified in these approaches is that discourses are not entirely closed systems but are vehicles for reflecting on where mathematics education is today, how it has come to be this way, and the consequences of conventional thought and actions. Importantly, such analyses are a political resource for transforming the processes and structures that currently deny teachers, students, policy makers, and others the achievement of their ethical goals within mathematics education.

## Cross-References


- ▶ Psychological Approaches in Mathematics Education
- ▶ Sociological Approaches in Mathematics Education

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# Probability Teaching and Learning


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## Different Meanings of Probability

While the meaning of a typical mathematical object or operation (rectangles, division, etc.) is clear and not subject to interpretation, probability has received different meanings along history that still today are challenged. Although there are no contradictions in the probability calculus per se, different philosophical theories and the emerging conceptions of probability still persist, among which the most relevant for teaching are the classical, frequentist, subjectivist, and axiomatic or formal conceptions (Batanero et al. 2005) that we briefly analyze below.

Probability reveals a dual character since its emergence: a *statistical side* was concerned with finding the objective mathematical rules behind sequences of outcomes generated by random processes through data and experiments, while another *epistemic side* views probability as a personal degree of belief (Hacking 1975).

Progress in probability was linked to games of chance; it is not surprising that the pioneer interpretation was based on an assumption of equiprobability for all possible elementary events, an assumption which is reasonable in such games as throwing dice. In the *classical definition*, given by Abraham de Moivre in 1718 in the *Doctrine of Chances* and later refined by Laplace in 1814 in his *Philosophical essay on probability*, probability is simply a fraction of the number of favorable cases to a particular event divided by the number of all cases possible in that experiment. This definition was criticized since its publication since the assumption of equiprobability of the outcomes is based on subjective judgment, and it restricts the application from the broad variety of natural phenomena to games of chance.

In his endeavor to extend the scope of probability to insurance and life-table problems, Jacob Bernoulli justified to assign probabilities to events through a frequentist estimate by elaborating the Law of Large Numbers. In the frequentist approach sustained later by von Mises or Renyi, probability is defined as the hypothetical number towards which the relative frequency tends. Such a convergence had been observed in many natural phenomena so that the frequentist approach extended the range of applications enormously. A practical drawback of this conception is that we never get the exact value of probability; its estimation varies from one repetition of the experiments (called sample) to another. Moreover, this approach is not appropriate if it is not possible to *repeat* the experiment under exactly the *same* conditions.

While in the classical and in the frequentist approaches probability is an "objective" value we assign to each event, the Bayes's theorem, published in 1763, proved that the probability for a hypothetical event or cause could be revised in light of new available data. Following this interpretation, some mathematicians like Keynes, Ramsey, or de Finetti considered probability as a personal degree of belief that depends on a person's knowledge or experience. Bayes' theorem shows that an initial (prior) distribution about an unknown probability changes by relative frequencies into a posterior distribution. Consequently, from data one can derive an interval so that the unknown probability lies within its boundaries with a predefined (high) probability. This is another proof that relative frequencies converge and justifies using data to estimate unknown probabilities. However, the status of the prior distribution in this approach was criticized as subjective, even if the impact of the prior diminishes by objective data, and de Finetti proposed a system of axioms to justify this view in 1937.

Despite the fierce discussion on the foundations, progress of probability in all sciences and sectors of life was enormous. Throughout the twentieth century, different mathematicians tried to formalize the mathematical theory of probability. Following Borel's work on set and measure theory, Kolmogorov, who corroborated the frequentist view, derived in 1933 an axiomatic. This axiomatic was accepted by the different probability schools because with some compromise the mathematics of probability (no matter the classical, frequentist or subjectivist view) may be encoded by Kolmogorov's theory; the interpretation would differ according to the school one adheres to. However, the discussion about the meanings of probability and the long history of paradoxes is still alive in intuitions of people who often conflict with the mathematical rules of probability (Borovcnik et al. 1991).

## Probability in the School Curriculum

Students are surrounded by uncertainty in economic, meteorological, biological, and political settings and in their social activities such as games or sports. The ubiquity of randomness implies the student's need to understand random phenomena in order to make adequate decisions when confronted with uncertainty; this need has been recognized by educational authorities by including probability in the curricula from primary education to high school and at university level.

The philosophical controversy about the meaning of probability has also influenced teaching (Henry 1997). Before 1970, the classical view of probability based on combinatorial calculus dominated the school curriculum, an approach that was difficult, since students have problems to find the adequate combinatorial operations to solve probability problems. In the "modern mathematics" era, probability was used to illustrate the axiomatic method; however this approach was more suitable to justify theories than to solve problems. Both approaches hide the multitude of applications since the equiprobability assumption is restricted to games of chance. Consistently, many school teachers considered probability as a subsidiary part of mathematics, and either they taught it in this style or they left it out of class. Moreover, students hardly were able to apply probability in out-of-school contexts.

With increasing importance of statistics at school and progress of technology with easy access to simulation, today there is a growing interest in an experimental introduction of probability as a limit of stabilized frequencies (frequentist approach). We also observe a shift in the way probability is taught from a formula-based approach to a modern experiential introduction where the emphasis is on probabilistic experience. Students (even young children) are encouraged to perform random experiments or simulations, formulate questions or predictions about the tendency of outcomes in a series of these experiments, collect and analyze data to test their conjectures, and justify their conclusions on the basis of these data. This approach tries to show the students that probability is inseparable from statistics, and vice versa, as it is recognized in the curriculum.

Simulation and experiments can help students face their probability misconceptions by extending their experience with randomness. It is important, however, to clarify the distinction between ideally repeated situations and one-off decisions, which are also frequent or perceived as such by people. By exaggerating simulation and a frequentist interpretation in teaching, students may be confused about their differences or return to private conceptions in their decision making.

Moreover, a pure experimental approach is not sufficient in teaching probability. Though simulation is vital to improve students' probabilistic intuitions and in materialize probabilistic problems, it does not provide the key about how and why the problems are solved. This justification depends on the hypotheses and on the theoretical probability model on which the computer simulation is built, so that a genuine knowledge of probability can only be achieved through the study of some probability *theory*. However, the acquisition of such formal knowledge by students should be gradual and supported by experience with random experiments, given the complementary nature of the classic and frequentist approaches to probability. It is also important to amend these objective views with the subjectivist perspective of probability which is closer to how people think, but is hardly taken into account in the current curricula in spite of its increasing use in applications and that it may help to overcome many paradoxes, especially those linked to conditional probabilities (Borovcnik 2011).

When organizing the teaching of probability, there is moreover a need to decide what content to include at different educational levels. Heitele (1975) suggested a list of fundamental probabilistic concepts, which can be studied at various degrees of formalization, each of which increases in cognitive and linguistic complexity as one proceeds through school to university. These concepts played a key role in the history and form the base for the modern theory of probability while at the same time people frequently hold incorrect intuitions about their meaning or their application in absence of instruction. The list of fundamental concepts include the ideas of random experiment and sample space, addition and multiplication rules, independence and conditional probability, random variable and distribution, combinations and permutations, convergence, sampling, and simulation.

All these ideas appear along the curriculum, although, of course, with different levels of formalization. In primary school, an intuitive idea of probability and the ability to compute simple probabilities by applying the Laplace rule or via the estimation from relative frequencies using a simple notation seems sufficient. By the end of high school, students are expected to discriminate random and deterministic experiments, use combinatorial counting principles to describe the sample space and compute the associate probabilities in simple and compound experiments, understand conditional probability and independence, compute and interpret the expected value of discrete random variables, understand how to draw inferences about a population from random samples, and use simulations to acquire an intuitive meaning of convergence.

It is believed today that in order to become a probability literate citizen, a student should understand the use of probability in decision making (e.g., stock market or medical diagnosis) or in sampling and voting. In scientific or professional work, or at university, a more complex meaning of probability including knowledge of main probability distributions and even the central limit theorem seems appropriate.

## Intuitions and Misconceptions

For teaching, it is important to take into account informal ideas that people relate to chance and probability before instruction. These ideas appear in children who acquire experience of randomness when playing chance games or by observing natural phenomena such as the weather. They use qualitative notions (probable, unlikely, feasible, etc.) to express their degrees of belief in the occurrence of random events in these settings; however their ideas are too imprecise. Young children may not see stable properties in random generators such as dice or marbles in urns and believe that such generators have a mind of their own or are controlled by outside forces.

Although older children may realize the need of assigning numbers (probabilities) to events to compare their likelihood, probabilistic reasoning rarely develops spontaneously without instruction (Fischbein 1975), and intuitions are often found to be wrong even in adults. For example, the mathematical result that a run of four consecutive heads in coin tossing has no influence on the probability that the following toss will result in heads seems counterintuitive. This belief maybe due to the confusion between hypotheses and data: when we deal with coin tossing, we usually assume that the experiment is performed *independently*. In spite of the run of four heads observed, the model still is used and, then, the probability for the next outcome remains half for heads; however intuitively these data prompt people to abandon the assumption of independence and use the pattern of past data to predict the next outcome.

Piaget and Inhelder (1951) investigated children's understanding of chance and probability and described stages in the development of probabilistic reasoning. They predicted a mature comprehension of probability at the formal operational stage (around 15 years of age), which comprises that adolescents understand the law of large numbers – the principle that explains simultaneously the global regularity and the particular variability of each randomly generated distribution. However, later research contradicted some of their results; Green's (1989) investigation with 2,930 children indicates that the percentage of students recognizing random distributions decreases with age.

Moreover, research in Psychology has shown that adults tend to make erroneous judgments in their decisions in out-of-school settings even if they are experienced in probability. The well-known studies by Kahneman and his collaborators (see Kahneman et al. 1982) identify that people violate normative rules behind scientific inference and use specific *heuristics* to simplify the uncertain decision situation. According to them, such heuristics reduce the complexity of these probability tasks and are in general useful; however, under specific circumstances, heuristics cause systematic errors and are resistant to change.

For example, in the *representativeness heuristics*, people estimate the likelihood of an event taking only into account how well it represents some aspects of the parent population neglecting any other information available, no matter how relevant it is for the particular decision. People following this reasoning might believe that small samples should reflect the population distribution and consistently rely too much on them. In case of discrepancies between sample and population, they might even predict next outcomes to reestablish the alleged similarity. Other people do not understand the purpose of probabilistic methods, where it is not possible to predict an outcome with certainty but the behavior of the whole distribution, contrary to what some people expect intuitively. A detailed survey of students' intuitions, strategies, and learning at different ages may be found in the different chapters of Jones (2005) and in Jones et al. (2007).

Another fact complicates the teaching of probability (Borovcnik and Peard 1996): whereas in other branches of mathematics counterintuitive results are encountered only at higher levels of abstraction, in probability counterintuitive results abound even with basic concepts such as independence or conditional probability.

Furthermore, while in logical reasoning – the usual method in mathematics – a proposition is true or false, a proposition about a random event would only be true or false after the experiment has been performed; beforehand we only can consider the *probability* of possible results. This explains that some probability theorems (e.g., the central limit theorem) are expressed in terms of probability.

## Challenges in Teaching Probability

The preceding philosophical and psychological debate suggests that teachers require a specific preparation to assure their competence to teach probability. Unfortunately, even if prospective teachers have a major in mathematics, they usually studied only probability *theory* and consistently lack experience in designing investigations or simulations (Stohl 2005). They may be unfamiliar with different meanings of probability or with frequent misconceptions in their students. Research in statistics education has shown that textbooks lack to provide sufficient support to teachers: they present an all too narrow view of concepts; applications are restricted to games of chance; even definitions are occasionally incorrect or incomplete.

Moreover, teachers need training in pedagogy related to teaching probability as general principles valid for other areas of mathematics are not appropriate (Batanero et al. 2004). For example, in arithmetic or geometry elementary operations can be reversed and reversibility can be represented by concrete materials: when joining a group of three marbles with another group of four, a child always obtains the same result (seven marbles); if separating the second set from the total, the child always returns to the original set provided that the marbles are seen as equivalent (and there is hardly a dispute on such an abstraction). These experiences are vital to help children progressively abstract the structure behind the concrete situation, since they remain closely linked to concrete situations in their mathematical thinking. However, with a random experiment such as flipping a coin, a child obtains different results each time the experiment is performed, and the experiment cannot be reversed. Therefore, it is harder for children to understand (and acknowledge) the structure behind the experiments, which may explain why they do not always develop correct probability conceptions without instruction.

Our previous discussion also suggests several important questions to be considered in future research: How should we take advantage of the multifaceted nature of probability in organizing instruction? How to conduct children to gradually view probability as an a priori degree of uncertainty, as the value to which relative frequencies tend in random experiments repeated under the same conditions, and as a personal degree of belief, where "subjectivist" does not mean arbitrariness, but use of expert knowledge? How to make older students realize that probability should be viewed as a mathematical model, and not a property of real objects? And finally, how best to educate teachers to become competent in the teaching of probability?

## Cross-References

► Data Handling and Statistics Teaching and Learning

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# Problem Solving in Mathematics Education

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## Keywords

Problem solving; Inquiring approach; Modeling activities; Representations; Curriculum; Digital technology

## Introduction

The core and essence of a problem solving approach to learn mathematics is summarized in the following quotation: Problem solving is a lifetime activity. Experiences in problem solving are always at hand. All other activities are subordinate. Thus, the teaching of problem solving should be continuous. Discussion of problems, proposed solutions, methods of attacking problems, etc. should be considered at all times (Krulik and Rudnick 1993, p. 9).

## Characteristics

### What Does Mathematical Problem Solving Involve?

Mathematical problem solving is a research and practice domain in mathematics education that fosters an inquisitive approach to develop and comprehend mathematical knowledge Santos-Trigo 2007. As a research domain, the problem-solving agenda includes analyzing cognitive, social, and affective components that influence and shape the learners' development of problem-solving proficiency. As an instructional approach, the agenda includes the design and implementation of curriculum proposals and corresponding materials that enhance problem-solving activities. Key elements in both the research and practice endeavors are the characterization of problems and what the problem-solving processes entail. Often, a list of routine and nonroutine problems is chosen as a means to elicit and develop students' problem-solving competencies. Also, the same mathematical contents to be learned and textbooks problems are seen as opportunities for learners to engage in problem-solving activities. These activities involve making sense of concepts or problem statements; looking for different ways to represent, explore, and solve the tasks; extending the tasks' initial domain; and developing a proper language to communicate and discuss results. The ways to organize and implement problem-solving activities might take different routes depending on the instructor's aims, educational level, and students' background.

In university and graduate levels, the Moore method, a variant of problem-solving approach, might involve the selection of a list of theorems and course problems that students are asked to unpack, explain, and prove within a learning community that foster the members' participation including the instructor as a moderator (Halmos 1994). Other problem-solving approaches rely on promoting scaffolding activities to gradually guide students' construction of problem-solving abilities. Instructional strategies involve fostering and valuing students' small group participation, plenary group discussions, the instructor presentations through modeling problem-solving behaviors, and the students' constant mathematical reflection. Lesh and Zawojewski (2007) identify modeling activities as essential for students to develop knowledge and problem-solving experiences. They contend that in modeling processes, interactive cycles represent opportunities for learners to constantly reflect on, revise, and refine tasks' models. Thus, the multiplicity of interpretations of problem solving has become part of the identity of the field.

## Problem-Solving Developments: Frameworks, Focus, and Current Themes

The most salient feature of the problem-solving research agenda is that the themes, questions, and research methods have changed perceptibly and significantly through time. Shifts in research themes are intimately related to shifts in research designs and methodologies (Lester and Kehle 2003). Early problem-solving research relied on quantitative methods and statistical hypothesis testing designs; later, approaches were, and continue to be, based mostly on qualitative methodologies. In addition, the development and systematic use of digital technologies not only has offered new paths to represent and explore mathematical situations (through the use of dynamic models); also the students' appropriation of these tools becomes an important issue in the research and instructional agenda (Hoyles and Lagrange 2010).

Research programs structured around problem solving have made significant contributions to the understanding of the complexity involved in developing the students' deep comprehension of mathematics ideas, in using research results in the design and structure of curricular frameworks, and in directing mathematical school practices.

The cumulative findings in problem solving provide useful information on how the field has evolved during the last 40 years in terms of themes, research designs, curriculum proposals, and mathematical instruction. Of course, there are traces of mathematical problem-solving activities throughout the history of human civilization that have contributed to the development of the problem-solving agendas. For example, the same year that Polya published his *How to Solve It* book, Hadamard published *Essay on the Psychology of Invention in the Mathematics Field*. Hadamard asked 100 physicists/mathematicians how they performed their work and identified a four-step model that describes their problem-solving experiences: preparation, incubation, illumination, and verification. However, in the 1970s, the discussion of previous problem-solving developments, including Krutetskii's (1976) work on the study of mathematical abilities of gifted children, became an important issue in the mathematics education agenda.

In the following sections, the goal is to provide an account of the main problem-solving developments that appeared during the last 40 years. This account includes some examples of problem-solving directions in specific countries, identifies current issues, and possible future directions in the field.

### Focus on Problem-Solving Activities and Mathematicians' Work

It is recognized that mathematical problems and their solutions are a key ingredient in the making and development of the discipline. What does the process of formulating and solving mathematical problems entail? The discussion of this type of question, within the mathematics community, provided valuable information to characterize problem-solving processes and to think of the students' construction of mathematical knowledge in terms of problem-solving activities. In mathematics education, the mathematicians' work and developments in disciplines as psychology became relevant to relate problem-solving activities and the students' learning of mathematics. Schoenfeld (1985) suggests that open critiques (Kline 1973) to the new math and the back-to-basic reforms in the USA were important to focus on problem-solving activities as a way for students to learn mathematics. Polya (1945) reflected on his own experience as a mathematician to write about the process involved and ways to be successful in problem-solving activities. (Polya used retrospection (looking back at events that already have taken place) and introspection (self-examination of one's conscious thought and feelings) methods to write about problem solving and ways to teach it.) He proposed a general framework that describes four problem-solving stages (understanding the problem, devising a plan, carrying out the plan, looking back). He also discussed the role and importance of using heuristic methods in students' construction of mathematical knowledge. His ideas not only shaped initial research programs in problem solving but also appeared in curriculum proposals and teaching scenarios (Krulik and Reys 1980).

Polya's ideas were found in curriculum materials and in the ways the development of mathematical instruction was organized and structured. The use of heuristic methods was deemed relevant, and instruction or teaching activities were organized and centered on the teacher who was in charge of modeling problem-solving behaviors for the students.

Research studies included quantitative designs to document and contrast groups or classes of students' problem-solving behaviors exposed to differential approaches. Research results indicated that the identification of problem-solving strategies and the process of modeling their use in instruction was not sufficient for students to foster their comprehension of mathematical knowledge and problem-solving approaches. This recognition allowed the mathematics education community to reflect on ways to characterize and explain the students' development of mathematical thinking and problem-solving approaches. Specifically, it was important to analyze in detail how students could gradually develop a way of thinking consistent with mathematical practice in terms of problem-solving activities (Schoenfeld 1992).

During this period, problem-solving perspectives appeared explicitly in curriculum discussions, and in mathematical instruction, the Polya's model was a predominant approach to guide teaching strategies. The need to do research to support, structure, and implement problem-solving activities became crucial, and the development of qualitative methods was relevant to complement and extend previous quantitative problem-solving analysis.

### The Importance of Problem-Solving Frameworks and the Design of Curriculum Proposals

The use of qualitative tools provided a means to analyze and discuss both features of mathematical thinking and the process involved in problem-solving approaches. Schoenfeld (1985) implemented a research program that focused on analyzing students' development of mathematical ways of thinking consistent with current mathematics practices. A key issue in his program was to characterize what it means to think mathematically and to document how students become successful or develop proficiency in solving mathematical tasks. He used a set of nonroutine tasks to engage first year university students in problem-solving activities. As a result, Schoenfeld proposed a framework to explain and document students' problem-solving behaviors in terms of four dimensions: the use of basic mathematical resources, the use of cognitive or heuristic strategies, the use of metacognitive or self-monitoring strategies, and students' beliefs about mathematics and problem solving.

This framework has been used extensively not only to document the extent to which problem solvers succeed in their problem-solving attempts but also to organize and foster students' development of problem-solving experiences in the classrooms. Schoenfeld (1992) also shed light on the strengths and limitations associated with the use of Polya's heuristics. Schoenfeld (1992, p. 353) pointed out that “Polya’s characterization did not provide the amount of detail that would enable people who were not already familiar with the strategies to be able to implement them.” In this period, the shift from using quantitative to qualitative methods appeared in research studies, and students’ interactions were valued and promoted in mathematical instruction. The importance of students’ previous knowledge was recognized when engaging themselves in problem-solving discussions; students became the center of instruction that valued their active participation as a part of a learning community. In addition, the NCTM (1989) launched a curriculum framework structured around problem-solving approaches. This framework was updated in 2000 (NCTM 2000). This is the curriculum proposal that best promotes the students development of mathematical experiences based on problem-solving approaches. Recently, the Common Core State Mathematics Standards (CCSMS) (2010) also identified problem solving as one of the standard processes to develop students’ mathematical proficiency. Through all grades, students are encouraged to engage in problem-solving practices that involve making sense of problems, and persevere in solving them, to look for and express regularity in repeated reasoning, to use appropriate tools strategically, etc.

Problem-solving frameworks offer valuable information regarding the main aspects that influence the development of problem-solving competencies. In addition, they provided basis to analyze in deep the role of metacognition and beliefs systems in learners’ comprehension of mathematics.

### Regional Problem-Solving Developments and the Use of Digital Tools

Developments in mathematical problem solving have gone hand in hand with development and discussions in mathematics education. For example, a situated cognition perspective links the learning process to problem-solving activities within specific contexts, and a community of practices perspective emphasizes student’s social interactions as a way to make sense and work on mathematical problems.

Regional or country mathematics education traditions also play a significant role in shaping and pursuing a problem-solving agenda. Artigue and Houdement (2007) summarized the use of problem solving in mathematics education, and they pointed out that in France problem solving is conceptualized through the lens of two influential and prominent theoretical and practical frameworks in didactic research: the theory of didactic situations (TDS) and the anthropological theory of didactics (ATD).

In the Netherlands, the problem-solving approach is associated with the theory of Realistic Mathematics that pays special attention to the process involved in modeling the real-world situations. They also recognized a strong connection between mathematics as an educational subject and problem solving as defined by the PISA program Doorman et al. 2007.

Cai and Nie (2007) pointed out that problem-solving activities in Chinese mathematics education have a long history and are viewed as a goal to achieve and as an instructional approach supported more on experience than a cognitive analysis. In the classroom, teachers stress problem-solving situations that involve discussion: *one problem multiple solutions*, *multiple problems one solution*, and *one problem multiple changes*. “The purpose of teaching problem solving in the classroom is to develop students’ problem solving skills, help them acquire ways of thinking, form habits of persistence, and build their confidence in dealing with unfamiliar situations” (Cai and Nie 2007, p. 471).

Recently, Schoenfeld (2011) updated his 1985 problem-solving framework to explain how and why problem solvers make decisions that shape and guide their problem-solving behaviors. He proposes three constructs to explain in detail what problem solvers do on a moment-by-moment basis while engaging in a problem-solving process: the problem solver’s resources, goals, and orientations. He suggests that these constructs offer teachers, and problem solvers in other domains, tools for reflecting on their practicing decisions. In his book, he uses the framework to analyze and predict the behaviors of mathematics and science teachers and a medical doctor.

The development and use of digital technology opened up new paths for problem-solving approaches. For example, basis, frameworks, and instructional approaches that emerged from analyzing students' problem-solving experiences centered on the use of paper and pencil should be reexamined in accordance to what the use of the tools brings in to play. That is, they need to be adjusted or extended to incorporate and document ways in which the use of digital technology fosters new methods of representing and reasoning about problem situations (Hoyles and Lagrange 2010). The systematic use of technology not only enhances what teachers and students do with the use of paper and pencil but also extends and opens new routes and ways of reasoning for students and teachers to develop mathematics knowledge (Santos Trigo and Reyes-Rodriguez 2011). Thus, emerging reasoning associated with the use of the tools needs to be characterized and made explicit in curriculum and conceptual frameworks in order for teachers to incorporate it and to foster its development in teaching practices. In terms of curriculum materials and instruction, the use of several digital technologies could transform the rigid and often static nature of the content presentation into a dynamic and flexible format where learners can access to several tools (dynamic software, online encyclopedias, widgets, videos, etc.) while dealing with mathematical tasks.

The advent and use of computational technology in society and education influence and shape the academic problem-solving agenda. The learners' tools appropriation to use them in problem-solving activities involves extending previous frameworks and to develop different methods to explain mathematical processes that are now enhanced with the use of those tools.

### Directions for Future Research

In retrospective, research in problem solving has generated not only interesting ideas and useful results to frame and discuss paths for students to develop mathematical knowledge and problem-solving proficiency; it has also generated ways to incorporate this approach into the design of curriculum proposals and instructional approaches. However, it is not clear how teachers implement and assess their students' development of problem-solving competences. In this context, teachers, together with researchers, need to be engaged in problem-solving experiences where all have an opportunity to discuss and design problem-solving activities and ways to implement and evaluate them in actual classroom settings. In addition, there are different paths for students to develop mathematical thinking, and the use of tools shapes the ways they think of, represent, and explore mathematical tasks or problems. Then, theoretical frameworks used to explain learners' construction of mathematical knowledge need to capture or take into account the different ways of reasoning that students might develop as a result of using a set of tools during the learning experiences. As a consequence, there is a need to develop or adjust current problem-solving frameworks to account not only the students' processes of appropriation of the tools but also the need to characterize the ways of reasoning, including the use of new heuristics, for example, dragging in dynamic representations, with which students construct learning as a result of using digital tools in problem-solving approaches. In addition, it is important to develop methodological tools to observe, analyze, and evaluate group's problem-solving behaviors that involve the use of digital technology.

## Cross-References


- ▶ Affect in Mathematics Education
- ▶ Collaborative Learning in Mathematics Education
- ▶ Communities of Practice in Mathematics Education
- ▶ Competency Frameworks in Mathematics Education
- ▶ Critical Thinking in Mathematics Education
- ▶ Inquiry-Based Mathematics Education
- ▶ Learning Practices in Digital Environments
- ▶ Mathematical Ability
- ▶ Mathematical Modelling and Applications in Education

▶ Mathematical Representations
▶ Metacognition
▶ Problem Solving in Mathematics Education
▶ Realistic Mathematics Education
▶ Scaffolding in Mathematics Education
▶ Task-based Interviews in Mathematics Education
▶ Technology and Curricula in Mathematics Education
▶ Visualization and Learning in Mathematics Education

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# Professional Learning Communities in Mathematics Education

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## Characteristics

During the past decade, professional learning communities have drawn the attention of educationists interested in school leadership, school learning, and teacher development. Professional learning communities aim to establish school cultures, which are conducive to ongoing learning and development, of students, teachers, and schools as organizations (Stoll et al. 2006). Professional learning communities refer to groups of teachers collaborating to inquire into their teaching practices and their students' learning with the aim of improving both. In order to improve practice and learning, professional learning communities interrogate their current practices and explore alternatives in order to refresh and re-invigorate practice (McLaughlin and Talbert 2008). Exploring alternatives is particularly important in mathematics education where a key goal of teacher development is to support teachers' orientations towards understanding and engaging students' mathematical thinking in order to develop conceptual understandings of mathematics among students.

A key principle underlying professional learning communities is that if schools are to be intellectually engaging places, all members of the school community should be intellectually engaged in learning on an ongoing basis (Curry 2008). Professional learning communities are "fundamentally about learning – learning for pupils as well as learning for teachers, learning for leaders, and learning for schools" (Katz and Earl 2010, p. 28). Successful learning communities are those that challenge their members to reconsider taken-for-granted assumptions in order to generate change, for example, challenging the notion that working through procedures automatically promotes conceptual understandings of mathematics. At the same time, not all current practices are problematic, and successful professional learning communities integrate the best of current practice with ideas for new practices.

A number of characteristics of successful professional learning communities have been identified: they create productive relationships through care, trust, and challenge; they de-privatize practice and ease the isolation often experienced by teachers; they foster collaboration, interdependence, and collective responsibility for teacher and student learning; and they engage in rigorous, systematic enquiry on a challenging and intellectually engaging focus. Professional learning communities in mathematics education focus on supporting teachers to develop their own mathematical knowledge and their mathematical knowledge for teaching, particularly in relation to student thinking (Brodie 2011; Curry 2008; Jaworski 2008; Katz et al. 2009; Little 1990).

The notion of collective learning in professional learning communities is important. The idea is that teachers who work together learn together, making for longer-term sustainability of new practices and promoting community-generated shifts in practice, which are likely to provide learners with more coherent experiences across the subject or school (Horn 2005; McLaughlin and Talbert 2008). Professional learning communities support teachers to "coalesce around a shared vision of what counts for high-quality teaching and learning and begin to take collective responsibility for the students they teach" (Louis and Marks 1998, p. 535). Ultimately, a school-wide culture of collaboration can be promoted, although working across subject disciplines can distract from a focus on subject knowledge (Curry 2008). Networked learning communities, where professional learning communities come together across schools in networks, provide further support and sustainability for individual communities and improved teacher practices (Katz and Earl 2010).

There is differing terminology for learning communities, which illuminate subtle but important differences in how communities are constituted. These include "communities of practice," "communities of enquiry," and "critical friends groups." The key emphasis in the notion of *professional* learning is that it signals the focus of the community and the learning as both data-informed and knowledge-based.

## Data-Informed and Knowledge-Based Enquiry

Professional learning communities can be established within or across subjects, and in each case the communities would choose different focuses to work on. Working within mathematics suggests that the focus would be on knowledge of and intellectual engagement with mathematics and the teaching and learning of mathematics. Effective communities focus on addressing student needs through a focus on student achievement and student work, joint lesson and curriculum planning, and joint observations and reflection on practice, through watching actual classroom lessons or videotaped recordings of classroom practice. Mathematics learning communities support teachers to focus on learner thinking through examples of learners solving rich problems (Borko et al. 2008; Whitcomb et al. 2009) or through teachers' analyzing learner errors (Brodie 2011).

In many cases data comes from national tests, and teachers work together to understand the data that the tests present and to think about ways to improve their practice that the data suggests. Working with data as a mechanism to improve test scores can be seen as a regulatory practice, with external accountability to school managers and education department officials. Proponents of teacher-empowered professional learning communities argue strongly that the goal of such data analysis must be to inform teachers' conversations in the communities, as a form of internal accountability to knowledge and learning (Earl and Katz 2006). Data can also include teachers' own tests, interviews with learners, learners' work, and classroom observations or videotapes.

The professional focus of professional learning communities requires that the learning in these communities be supported by a knowledge base as well as by data. As teachers engage with data, their emerging ideas are brought into contact with more general findings from research. Jackson and Temperley (2008) argue for a model where practitioner knowledge of the subject, learners, and the local context meets public knowledge, which is knowledge from research and best practice. The interaction between data from classrooms and wider public knowledge is central in creating professional knowledge, for two reasons. First, without outside ideas coming into the communities' conversations, they can become solipsistic and self-preserving and may continue to maintain the status quo rather than invigorate practice. Second, data and knowledge work together to promote internal accountability, to the learners and teachers and to support the creation of new professional knowledge, which is research-based, locally relevant, and collectively generated. (Data-informed practice is different from evidence-based practice. Evidence-based practice suggests that only research-based evidence is good enough to inform teacher professional development. Data-informed professional development suggests that teachers themselves, with some expert guidance, can and should interpret data that is available to them and integrate research knowledge with their local circumstances).

## Leadership

Leadership in professional learning communities is central, particularly in helping to bring together data from practice and the findings of research. Leaders can be school-based or external, for example, district officials or teacher-educators from universities. For long-term sustainability, there should be leadership within the school, or within a cluster of schools.

Two key roles have been established as important for leaders in professional learning communities. The first is promoting a culture of inquiry and mutual respect, trust, and care, where teachers are able to work together to understand challenges in their schools more deeply and support each other in the specific challenges that they face as teachers. The second is to support teachers to focus on their students' knowledge and subsequently their own knowledge and teaching practices. The second role is crucial in supporting professional learning communities where subject-specific depth is the goal, depth in learning and knowledge for both teachers and learners.

It is important for leaders in professional learning communities to also be learners and to be able to admit their own weaknesses (Brodie 2011; Katz, et al. 2009). At the same time, it is important for leaders to have and present expertise, which helps the community to move forward. In mathematics, leaders need to recognize opportunities for developing mathematical knowledge and knowledge of learning and teaching mathematics among teachers, for example, what counts as appropriate mathematical explanations, representations, and justifications and how these can be communicated with learners. Other functions for leaders in professional learning communities are developing teachers' capacities to analyze classroom data; supporting teachers to observe and interpret data rather than evaluate and judge practice; supporting teachers to choose appropriate problem of practices to work on, once the data has been interpreted; and helping teachers to work on improving their practice and monitoring their own and progress in doing this, as well as their learners' progress (Boudett and Steele 2007). So leadership in professional learning communities is a highly specialized task.

### Impact and Research

There is a growing body of research that shows that professional learning communities do promote improved teacher practices and improved student achievement (Stoll et al. 2006). However, the evidence is mixed depending on which aspects of learning different studies choose to focus on. Research into professional learning communities invariably must confront how to recognize and describe learning, both in the conversations of the community and in classrooms. It is well known from situated theory that learning does not travel untransformed between sites, rather it is recontextualized and transformed as it travels from classrooms to communities and back again.

A second issue that research into professional learning communities must confront is the relationship between group and individual learning. While the focus of the community is on group learning and interdependence, ultimately each person contributes in particular ways to the community and brings particular expertise, and different people will learn and grow in different ways. Kazemi and Hubbard (2008) suggest a situated framework for research into how the individual and the group coevolve in mathematics professional learning communities. Group and individual trajectories can be examined in relationship to each other, through a focus on particular practices and artifacts of practice discussed by the community. How particular practices travel from the classroom into the community and back again can be traced through linking what happens in the community to what happens in teachers' classrooms.

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# Psychological Approaches in Mathematics Education

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## Keywords

Behaviorism; Collaborative learning; Constructivism; Concept development; Teaching experiments; Design research; Technology-Based learning environments; Abstraction; Socio-mathematical norms

## Characteristics

Cognitive psychology, developmental psychology, and educational psychology are general fields of research for which mathematics education naturally seems one among many domains of application. However, the history of these domains of research and of the development of research in mathematics education is much more complex, and not at all hierarchical. For example, in their monumental *Human Problem Solving* (Newell and Simon 1972), Newell and Simon acknowledged that many of their ideas (which became among the fundamentals of Cognitive Psychology) were largely inspired from George Pólya's *How to Solve It* (Pólya 1945). Another prestigious link is of course Piaget's *Episté mologie Génétique* – his theory of human development: the theory was based on memorable experiments in which Piaget designed conservation tasks in which mathematical entities were focused on (number, quantity, length, proportions, etc.). Also Cole's Cultural Psychology (1996) is largely based on the comparison between mathematical practices in different societies. The reasons for these ties are profound, and beyond the very different approaches adopted, mathematics represents a domain through which human cognition, cognitive development, or human development can be studied. We focus here on some psychological approaches adopted in mathematics education. Although these approaches have come out at different times, approaches were not merely replaced and each of them is still vibrant in the community of researchers in mathematics education.

## The Constructivist Approach

Our review of psychological approaches in mathematics is not exhaustive. We mention the approaches that contribute to our understanding of learning and teaching processes and that can help in what we consider as their improvement. For this reason, we overlooked behavioristic approaches. We will begin with constructivism – a learning theory with a very long history that can be traced to John Dewey. The simple and general idea according to which learning occurs when humans actively engage in tasks has been understood very differently by different psychologists. For some, constructivism means discovery-based teaching techniques, while for others, it means self-directedness and creativity. Wertsch (1998) adopts a social version of constructivism – socioculturalism – to encourage the learner to arrive at his or her version of the truth, influenced by his or her background, culture, or embedded worldview. Historical developments and symbol systems, such as language, logic, and mathematical systems, are inherited by the learner as a member of a particular culture, and these are learned throughout the learner's life. The fuzziness and generality of the definition of constructivism led to inconsistent results. It also led to the memorable "math wars" controversy in the United States that followed the implementation of constructivist-inspired curricula in schools with textbooks based on new standards. In spite of many shortcomings, the constructivist approach had the merit to lead scientists to consider the educational implications of the theories of human development of Piaget and Vygotsky in particular in mathematics education (von Glasersfeld 1989; Cobb and Bauersfeld 1995).

## The Piagetian Approach: Research on Conceptions and Conceptual Change

The impact of Piaget's theory of human development had and still has an immense impact on research on mathematics education. Many researchers adapted the Piagetian stages of cognitive growth to describe learning in school mathematics. Collis' research on formal operations and his notion of closure (Collis 1975) are examples of this adaptation. With the multi-base blocks (also known as Dienes blocks), Dienes (1971) was also inspired by Piaget's general idea that knowledge and abilities are organized around experience to sow the seeds of contemporary uses of manipulative materials in mathematics instruction to teach structures to young students.

Since the 1970s researchers in science education realized that students bring to learning tasks alternative frameworks or misconceptions that are robust and difficult to extinguish. The idea of misconception echoed Piagetian ideas according to which children consistently elaborate understandings of reality that do not fit scientific standards. Researchers in mathematics education adopted these ideas in terms of tacit models (Fischbein 1989) or of students' concept images (Tall and Vinner 1981). These frameworks were seen as theories to be replaced by the accepted, correct scientific views. Bringing these insights into the playground of learning and development was a natural step achieved through the idea of conceptual change. This idea is used to characterize the kind of learning required when new information comes in conflict with the learners' prior knowledge usually acquired on the basis of everyday experiences. It is claimed that then a major reorganization of prior knowledge is required – a conceptual change. The phenomenon of conceptual change was first identified for scientific concepts and then in mathematics (e.g., the acquisition of the concept of fraction requires radical changes in the preexisting concept of natural number, Hartnett and Gelman 1998). Misconceptions were thought to develop when new information is simply added to the incompatible knowledge base, producing synthetic models, like the belief that fractions are always smaller than the unit. Learning tasks, in which students were faced with a cognitive conflict, were expected to replace their misconceptions by the current accepted conception. Researchers in mathematics education continue studying the discordances and conflicts between many advanced mathematical concepts and naïve mathematics. Intuitive beliefs may be the cause of students' systematic errors (Fischbein 1987; Stavy and Tirosh 2000; Verschaffel and De Corte 1993). Incompatibility between prior knowledge and incoming information is one source of students' difficulties in understanding algebra (Kieran 1992), fractions (Hartnett and Gelman 1998), and rational numbers (Merenluoto and Lehtinen 2002). The conceptual change approach is still vivid because of its instructional implications that help to identify concepts in mathematics that are going to cause students great difficulty, to predict and explain students' systematic errors, to understand how counterintuitive mathematical concepts emerge, to find the appropriate bridging analogies, and more generally, to develop students as intentional learners with metacognitive skills required to overcome the barriers imposed by their prior knowledge (Schoenfeld 2002). However, harsh critiques pointed out that cognitive conflict is not an effective instructional strategy and that instruction that "confronts misconceptions with a view to replacing them is misguided and unlikely to succeed" (Smith et al. 1993, p. 153). As a consequence, misconceptions research in mathematics education was abandoned in the early 1990s. Rather, researchers began studying the knowledge acquisition process in greater detail or as stated by Smith et al. (1993) to focus on "detailed descriptions of the evolution of knowledge systems" (p. 154) over long periods of time.

## Departing from Piaget: From Research on Concept Formation to Teaching Experiments

The fine-grained description of knowledge systems in mathematics education was initiated as an effort to adapt his theory to mathematics education (Skemp 1971). Theories of learning in mathematics were elaborated, among them the theory of conceptual fields (Vergnaud 1983), the notion of tool-object dialectic (Douady 1984, 1986), and theories of process-object duality of mathematical conceptions (Sfard 1991; Dubinsky 1991). Van Hiele's theory of development of geometric thinking (Van Hiele 2004) seems at a first glance to fit Piaget's view of development with its clear stages. However, it clearly departed from Piaget's theory in the sense that changes result from teaching rather than from independent construction on the part of the learner. The method of the teaching experiment was introduced to map trajectories in the development of students' mathematical conceptions. Steffe et al. (2000) produced fine-grained models of students' evolving conceptions that included particular types of interactions with a teacher and other students. It showed that learning to think mathematically is all but a linear process, but that what can be seen as mistakes or confusions may be essential in the learning process. Moreover, "misconceptions" often resist teacher's efforts, but they eventually are necessary building blocks in the learning of conceptions. In the same vein, Schwarz et al. (2009) elaborated the RBC model of abstraction in context to identify the building blocks of mathematical abstraction which are often incomplete or flawed. Such studies invite considering alternative approaches to understand the development of mathematical thinking. The RBC model takes into account the impressive development of sociocultural approaches in mathematics education.

## Sociocultural Approaches

Descriptions of students learning in teaching experiments stressed the importance of the social plane – of the interactions between teacher and students. Vygotsky's theory of human development was a natural source of inspiration for researchers in mathematics education in this context. A series of seminal studies on street mathematics (e.g., Nunes et al. 1993) on the ways unschooled children used mathematical practices showed the situational character of mathematical activity. Rogoff's (1990) integration of Piagetian and Vygotskian theories to see in guided participation a central tenet of human development fitted these developments in research in mathematics education. Rogoff considered learning and development as changes of practice. For her, learning is mutual as the more knowledgeable (the teacher) as well as students learn to attune their actions to each other. Cobb and colleagues took the mathematics classroom in its complexity as the natural context for learning mathematics (Cobb et al. 2001; Yackel and Cobb 1996). He introduced the fundamental notion of social and socio-mathematical norms to point at constructs that result from the recurring enactment of practices in classrooms (an embryonic version of this notion had already been elaborated by Bauersfeld (1988)). Cobb and colleagues showed that those norms are fundamental for studying individual and group learning: learning as a change of practice entails identifying the establishment of various norms. Vygotsky's *intersubjectivity* as the necessary condition for maintaining communication was replaced by Cobb and colleagues by *taken-as-shared beliefs*. Cobb also considered the mathematical practices of the classrooms (standards of mathematical argumentation, ways to reasoning with tools and symbols) as other general collective constructs to be taken into account to trace learning. Norms are constructed in the mini-culture of the classroom in which researchers are not only observers but actively participate in the establishment of this mini-culture. Cobb adopts here a new theoretical approach in the Learning Sciences – Design Research (Collins et al. 2004). This interesting approach led to many studies in mathematics education, but also raised the tough issue of generalizability of design experiments.

Although according to Cobb and his followers, learning is highly situational, knowledge that emerges in the classroom is presented in a decontextualized form that fits (or not) accepted mathematical constructs. The writings of influential thinkers challenged this view. In *L'archéologie du savoir*, Michel Foucault (1969) convincingly traced the senses given to ideas such as "madness" along the history through the analysis of texts. Instead of identifying knowledge as a static entity, he forcefully claimed that human knowledge should be viewed as "a kind of discourse" – a special form of multimodal communication. Leading mathematics education researchers adopted this perspective (Lerman 2001; Kieran et al. 2002). In her theory of commognition, Sfard (2008) viewed discourse as what changes in the process of learning, and not the internal mental state of an individual learner. From this perspective, studying mathematics learning means exploring processes of discourse development. The methodology of the theory of commognition relies on meticulous procedures of data collecting and analysis. The methods of analysis are adaptations of techniques developed by applied linguists or by discursively oriented social scientists. The discourse of the more knowledgeable other is for Sfard indispensable, not only as an ancillary help for the discovering student but as a discourse to which he or she should persist to participate, in spite of the fact its nature is incommensurable with the nature of his or her own discourse. Sfard's theory and Cobb's theory, which stemmed from research in mathematics education, have become influential in the Learning Sciences in general.

## Open Issues

Leading modern thinkers such as Bakhtin have headed towards *dialogism*, a philosophy based on dialogue as a symmetric and ethical relation between agents. This philosophical development has yielded new pedagogies that belong to what is called *Dialogic Teaching*, and new practices, for example, (un-)guided small group collaborative and argumentative practices, or teacher's facilitation of group work. A good example of dialogic teaching enacted in mathematics classrooms is *Accountable Talk* (Michaels et al. 2009). Dialogic Teaching raises harsh psychological issues as in contrast with sociocultural approaches for which adult guidance directs emergent learning, dialogism involves symmetric relations.

Numerous technological tools have been designed by CSCL (Computer-Supported Collaborative Learning) scientists to facilitate (un-)guided collaborative work for learning mathematics. These new tools enable new discourse practices with different synchronies and enriched blended multimodalities (oral, chat, computer-mediated actions, gestures). Virtual Math Teams (Stahl 2012) is a representative project which integrates powerful dynamic mathematics applications such as GeoGebra in a multiuser platform for (un)guided group work on math problems, so that small groups of students can share their mathematical explorations and co-construct geometric figures online. In a recent book, *Translating Euclid*, Stahl (2013) convincingly shows how collaborating students can reinvent Euclidean geometry with minimal guidance and suitable CSCL tools. The possibilities opened by new technologies challenge the tenets of sociocultural psychology: the fact that students can collaborate during long periods without adult guidance challenges neo-Vygotskian approaches for which adult guidance is central for development. To what extent can it be said that the designed tools embody adult discourse? In spite of the fact that the teacher is often absent, new forms of participation of the teacher fit dialogism (e.g., moderation as caring but minimally intrusive guidance). The psychological perspective that fit changes in participation and the role of multiple artifacts in these changes is an extension of the Activity Theory, the theory of Expansive Learning (Engeström 1987) to the learning of organizations rather than the learning of individuals. The mechanisms of the emergent learning of the group are still mysterious, though. It seems then, that, again, mathematics education pushes psychology of learning to unconquered lands.

## Cross-References


- ▶ Abstraction in Mathematics Education
- ▶ Activity Theory in Mathematics Education
- ▶ Argumentation in Mathematics Education
- ▶ Collaborative Learning in Mathematics Education
- ▶ Concept Development in Mathematics Education
- ▶ Constructivism in Mathematics Education
- ▶ Constructivist Teaching Experiment
- ▶ Design Research in Mathematics Education
- ▶ Information and Communication Technology (ICT) Affordances in Mathematics Education
- ▶ Learning Environments in Mathematics Education
- ▶ Sociomathematical Norms in Mathematics Education


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Q

## Quasi-empirical Reasoning (Lakatos)

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### Keywords

Lakatos; Maverick traditions in the philosophy of mathematics; Fallibilism; Proofs and refutations; Proof; Philosophy of mathematics

### Definition

This entry examines Lakatos' assertion that the nature of mathematical knowledge is quasi-empirical, in attempting to describe the growth of mathematical knowledge and its implications for mathematics education.

### Characteristics

The Hungarian philosopher Imre Lakatos (1976) considered mathematics to be a quasi-empirical science in his famous book "*Proofs and Refutations: The Logic of Mathematical Discovery*." The book, popularized within the mathematics community by Reuben Hersh (1978) after this paper "Introducing Imre Lakatos" (Hersh 1978), might also be considered as Lakatos' response to the claims on the methodology of mathematics, related to explaining how it is that mathematical knowledge qualifies for superlative epistemological qualities such as certainty, indubitability, and infallibility.

Lakatos' attempted to illustrate the fallibility of mathematics. Written as a fictionalized classroom dialogue, Lakatos' book (1976) presented an innovative, captivating, and powerful context for a reconstructed historical debate and proof of the Descartes-Euler theorem for polyhedral, as a generic example of the development of mathematical knowledge. Lakatos appealed to the history of the theorem, by embedding what he had discovered in his dissertation (dissertation topic suggested to him by George Polya). The Descartes-Euler theorem asserts that for a polyhedron p we have $V - E + F = 2$, where V, E, and F are, respectively, the number of vertices, edges, and faces of p. He showed how Descartes-Euler's theorem and the concepts involved in it evolved through proofs, counterexamples, and proofs modified in light of the counterexamples, thereby illustrating the fallibility of mathematics.

The core of Lakatos' philosophy of mathematics is that mathematical theorems are defeasible and subject to refutations not unlike claims in empirical sciences. Lakatos (1976, 1978) attempted to establish an analogy between Popper's (1962) conjectures and refutations in science and the logic of attempts at deductive proofs and refutations in mathematics and to describe the rational growth of mathematical

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knowledge (p. 5). In extending Popper's (1958, 1962) critical philosophy of science to mathematics, Lakatos claims that mathematical theorems are not irrefutably true statements, but conjectures, since we cannot know that a theorem will not be refuted. While in science, bold conjectures can be a starting point of the growth of knowledge, in mathematics presenting tentative proofs can be the starting point of the growth of knowledge, even if they contain hidden assumptions or lemmas that have not been proved yet.

Lakatos' approach in the philosophy of mathematics resulted in the argumentation that mathematics, like the sciences, is a quasi-empirical theory. Such theories have their "crucial truth-value injection" at the bottom. The logical flow in quasi-empirical theories is not the transmission of truth, but rather the retransmission of falsity. The term quasi-empirical describes the nature of the truth-value transmission in a particular deductive system, like mathematics, not whether the system is empirical. Lakatos argues that "from special theorems at the bottom ("basic statements") up towards the set of axioms . . . a quasi-empirical theory – at best -[can claim] to be well – corroborated, but always conjectural" (pp. 33–34). He further explains that informal, quasi-empirical, mathematics does not grow through a monotonous increase in the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism and by the logic of proofs and refutations. His opinion that mathematics is conjectural is in contrast to the view that mathematics is Euclidean in nature. According to Lakatos the efforts of Russell and Hilbert to Euclideanize mathematics failed: "the Grande Logiques cannot be proved true or even consistent; they can only be proved false or even inconsistence" (p. 15).

Although Lakatos described his work as a study of "mathematical methodology," much writing since then has used it as a font of suggestion concerning mathematics education, including school mathematics education. Researchers (e.g., Sriraman 2006) claim that Lakatos adopts the philosophical position of fallibilism and studies the implications of this view as a means of developing a model of mathematical inquiry, in attempting to relate this epistemological framework to actual classroom situations. During the last 30 years, a significant number of philosophers and mathematics educators alike have appropriated his ideas in *Proofs and Refutations* and inferred great meanings for the classroom practices of both teachers and students (Sierpinksa and Lerman 1996).

Lakatos' (1976, 1978) gave substantial impetus to developments in the sociology of mathematical knowledge. Lakatos' work can well serve as a basis for a social constructivist philosophy of mathematics, which in turn can be used to develop a theory of learning, such as constructivism. A social constructivist perspective clearly prefer the "Lakatosian" conception of mathematical certainty as being subject to revision over time to put forth a fallible and non-Platonist viewpoint about mathematics (Ernest 1991).

Ernest (1991) claimed that the fallibilist philosophy and social construction of mathematics presented by Lakatos not only had educational implications but that Lakatos was even aware of these implications (p. 208). Various examples propose a classroom discourse that conveys the thought-experimental view of mathematics as that of continual conjecture-proof-refutation that involves rich mathematizing experiences for students. Ernest argued that school mathematics should take on the socially constructed nature presented by Lakatos and also that teacher and students should engage in ways identical to those in his dialogue, specifically posing and solving problems, articulating and confronting assumptions, and participating in genuine discussion (p. 208). In line with Ernest's recommendations, Agassi (1980) identified that mathematics education could be benefited by a Lakatos' method of inspired teaching. Agassi proposed a Lakatosian method for the classroom, which had "the merit of taking the student from where he stands and using his interruptions of the lecture as a chief vehicle of his progress, rather than worrying about the teacher's progress" (p. 30). Likewise, Fawcett (1938) attempted to conduct a classroom situation like the one presented in

Lakatos' book. In a 2-year teaching experiment that highlighted the role of argumentation in choosing definitions and axioms, the students in Fawcett's study created suitable definitions, choose relevant axioms when necessary, and created a Euclidean geometry system by using the available mathematics of Euclid's time period.

How would mathematics teaching and learning have changed in a Lakatosian perspective? If a quasi-empirical view is taken, students no longer need to ignore their common sense, their experiences. Students' explorations can become a central aspect of teaching. The didactic possibilities of Lakatos' thought experiment abound but not much is present in the mathematics education literature in terms of teaching experiments that try to replicate the "ideal" classroom conceptualized by Lakatos. Sriraman (2006) suggests the use of combinatorial problems involving the use of sophisticated counting strategies with high school students to explore the Lakatosian possibilities of furthering mathematical discourse. Further with the advent of technology for mathematics learning which support students' explorations of visual representations, students' creation of mathematical statements based on exploration becomes a feasible and legitimate classroom activity.

Lakatos' work is situated within the philosophy of science and clearly not intended for nor advocates a didactic position on the mathematics education, but it has implications for teaching and learning of mathematics (Sriraman 2006). The legacy of Lakatos is not restricted to counterexamples and fallibility (Larvor 1998), but rather implies for a program based on sensitivity to the history of mathematics, an appreciation for the dynamics of its concepts and standards, its relation with other fields, and on the central role students might play in developing mathematical concepts.

## Cross-References

▶ Argumentation in Mathematics
▶ Argumentation in Mathematics Education

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# Questioning in Mathematics Education

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## Keywords

Questioning; Asking; Telling; Listening-to; Listening-for; Prompts; Focusing; Attention; Being mathematical

John Mason has retired

## Definition

Questioning means here the use of questions and other prompts offered to students so as to help them get unstuck or to direct their attention in a potentially useful way so that they make mathematical progress.

## Introduction

On the face of it, being taught mathematics consists mainly of responding to mathematical questions posed either by a text or a teacher. Support for how to respond comes from worked examples and exposition in the text and from questions and exposition by the teacher. But whether something said or written is actually a genuine question or a question masquerading as instruction is not always easy to discern. Furthermore, student responses to apparent questions may themselves be questions rather than answers.

### Mathematical Perspectives

James Stigler and James Hiebert (1999) observed that whereas in American (and indeed in most English speaking) mathematics classrooms, students are asked to obtain the answers to mathematical questions, in Japan it is more usual to be asked "in how many different ways can you find the answer?" Asking a complex and thought provoking question to initiate work on a topic makes assumptions about student competence and engagement. This pedagogical stance can be described as "deep end" or "complexity-oriented," as opposed to "shallow end" or "simplicity-oriented" teaching. The two approaches are based on entirely different assumptions about students as human beings. The first sees people as having demonstrated the powers necessary to tackle complexity, to make sense of mathematics, and as willing to persevere in the use those powers when challenged, frequently "folding back" (Pirie and Kieren 1994) in a spiral of frequent returns to the same ideas in increasingly complex ways (Bruner 1966). The second is based on a "staircase" theory that learning proceeds in careful, simple but inexorable steps in order to build up to complexity and that students have to have their hand held as they negotiate these steps.

If a teacher is frightened that students will not be able to address a problem, what they offer their students will reinforce lack of challenge and hence lack of resilience and resourcefulness (Claxton 2002). Their students are likely to develop the view that they will always be given simple tasks, and so with little or no experience of how unfamiliar challenges can be tackled, they are likely to balk when asked an unusual or challenging question. Worse, they may be reinforced in believing that their intelligence is bounded and so try to stay away from failure, which means refusing even relatively simple challenges (Dweck 2000). One of the biggest obstacles to student success is the assumptions made by the teacher about the capabilities of their students; another is the impression formed by students from their teachers, parents, and the institution of what they are capable of achieving. Even when a challenging task is used from a textbook, there is evidence that when students get stuck it is a natural tendency for the teacher to "dumb down" the question, essentially engaging in mathematical funnelling so that their students can succeed (Stein et al. 1996).

By contrast, students who experience challenges, whose teacher strives to be mathematical with and in front of the students so that they are exposed to "what to do when you are stuck" (Mason et al. 1982/2010), are more likely to develop resilience and resourcefulness and to be reinforced in the belief that they can succeed if they try hard enough and cleverly enough. Students, whose teacher strives to be mathematical with and in front of their students, are likely to gain insight into mathematics as a constructive and creative enterprise. Students who see their teacher sometimes getting stuck, and then unstuck, and who experience being stuck but then are encouraged to become aware of how they managed to get unstuck again are more likely to learn "what to do when you get stuck" (Mason et al. 1982/2010) and to develop resilience and resourcefulness. They are likely to learn to "know what to do when they don't know what to do" (Claxton 2002). Students whose teacher challenges them appropriately but significantly are likely to develop flexibility and creativity in their thinking.

### Asking as Telling

Many apparent questions are actually rhetorical: simply placing an interrogative voice tone at the end of an utterance does not guarantee that a question is being asked. For example, "what do we do with our rulers?" is actually drawing attention to inappropriate behavior and is not a genuine question (Ainley 1987). It is intended to focus attention on the behavior, and it is telling the student(s) to change their behavior.

A great deal of spontaneous classroom questioning is actually "telling" masquerading as "asking." In the flow of the classroom, the teacher has something come to mind and then asks a question which is intended to direct or focus student attention on what has come to mind. The question interrupts and structures students' attention. Students may experience the question as genuine, and try to respond, but usually students experience the question as a shift of attention into an instruction to "guess what is in my mind," while the teacher expects students to be "attending the way the teacher is (now) attending." Often it is only a student's "inappropriate" or unexpected reply to a question that provokes an awareness that there is an expected, even an intended, answer in the teacher's mind. These "telling" questions can be very subtle, but almost always plunge students into "guess what the teacher wants to hear," which may not advance their learning.

When you find yourself having asked such a question, you can either keep going or bail out. If you keep going, you are likely to find yourself asking another even more focusing question leading to a sequence of ever more precise and focused questions until eventually the student can answer without any effort. Although the teacher is following a train of increasing particularity or detail, the student is experiencing a sequence of interventions. Even though the teacher has followed a train of thought, the student has no access to that thinking, simply waiting until a question is asked that can be answered. Unfortunately the sequence of questions is entirely ephemeral, and no learning has taken place. John Holt (1964, p. 24) describes such an incident beautifully, and Heinrich Bauersfeld (1988, p. 36) called this pedagogic trap "funnelling," because the questions funnel student attention more and more narrowly, becoming simpler and simpler (see also Wood 1998). An alternative strategy is to exit from the interaction by acknowledging being caught in "guess what is in my mind" and telling students what "came to mind" or more extremely, taking a different approach or abandoning the issue altogether in order to return at a later date.

### Asking as Enquiring

Some questions are genuine, in the sense that the person asking does not know the answer and is presumably seeking that answer. For example, drawing attention to the status of an utterance with a question like "Is that a conjecture, or a fact or what?" or frequently asking students "How do you know ...?", students can respond to genuine enquiry, and the teacher can be genuinely interested in the students' response. The difference between "asking as telling" and "asking as asking" lies in a distinction made by Brent Davis (1996) between listening-for an expected response and listening-to what students are saying (and watching what students are doing).

Listening-to what students are saying and doing rather than listening-for what you expect is a form of "teaching by listening" (Davis 1996) which sounds paradoxical at first but is certainly possible, by setting up tasks (asking questions) that encourage students to reveal their thinking. A good way to get students to reveal their thinking is to ask them "how do you know?" when they make an assertion, for it reinforces the awareness that mathematics is about making and justifying conjectures. Another good way is to ask students "will that always be the case" when they make an actual or an implied generality and to ask "when else might that be the case" to jolt them out of the particular into widening their scope of generality.

Another good way to get students to reveal their thinking is to ask them to construct mathematical problems for themselves. For example, asking students to construct a problem (like the ones in a set of exercises, say) can be very revealing about the scope of generality that they perceive in those exercises. Variants include "a really simple example of a problem of this type," "a complicated example," or even "a general example" and can be augmented with "an example that will challenge other students" or "an example that shows you know how to tackle problems like these." Not only do these reveal dimensions of possible variation (Watson and Mason 2005) of which the student is aware, but it is also a good study technique to pose and then solve your own problems. Furthermore it is much more engaging to work on problems you have posed than on well-worked-over problems in a standard text. Students can respond with a degree of self-challenge that they feel comfortable with, and even if they do not completely successfully solve the problems they pose, they are learning something.

Another good way to get students to reveal their accessible example space (Watson and Mason 2005) is to get them to construct examples of mathematical objects meeting various constraints. By carefully choosing the constraints so as to force students to think beyond the first (usually rather simple) example that comes to mind enriches their example space while revealing the dimensions they are aware of that can be changed, and even something about the range of permissible change in those dimensions. For example, asking students to write down three pairs of numbers that differ by two often reveals a preference for whole numbers, even when students are told they will not be asked to do anything with those numbers. The same "construction" can be used with a pair of fractions, a pair of numbers whose logarithms differ by 2, a pair of trig functions that look different, a pair of integrals, and so on. As an example of increasing constraints, asking for a decimal number between 2 and 3, and without using the digit 5, and with at least one digit a 7 is highly revealing about students' appreciation of how decimals are constructed.

When listening-to students justifying conjectures, constructing problems or constructing objects, care must be taken not to confuse "absence of evidence" from "evidence of absence": just because a student does not vary something that can be varied, or change something in a particular way, does not mean that they did not think of it, only that they did not reveal it.

It is important to be clear here that "teaching by listening" is only one form of pedagogic strategy and is unlikely to succeed as the sole mode of interaction. Even Socrates asked questions and made the occasional observation!

## Intention and Effect

It is evident that all classroom questions (and many outside the classroom) are interventions in the flow of students' mentation and as such have two aspects: the *intention*, which is to focus or direct attention, to re-orient perspective, and the *effect*, which is either to re-orient student behavior or to reveal an as-yet unknown answer. Even a genuine question is an intervention, an interruption. Even asking a question when the student is immersed in being stuck and "not thinking about anything" except being stuck is an interruption in the student's state. Too many interventions, too frequent intervention, too intrusive an intervention may result in students coming to depend on the teacher rather than developing resilience and resourcefulness. What constitutes too much, too many, or too intrusive is a delicate matter which cannot be automated or even taught: it is a judgement that comes from experience, both as a learner and as a teacher.

## Classroom Ethos

The sociocultural-mathematical norms of a classroom (the classroom rubric) have a significant affect on what is possible in the way of asking questions and getting thoughtful responses (Yackel and Cobb 1996). In a classroom in which mathematical questions are asked which have simple answers that are either correct or incorrect, students can become dependent on the teacher asking appropriate questions. Uncertainty as to the correctness of an answer is likely to lead to increasing reticence in answering, of fear of being wrong and looking foolish. This in turn can lead to an instrumental and intelligence-testing view of mathematics. By contrast, in a classroom in which everything said (by students and the teacher) is taken to be a conjecture that needs to be tested and justified, students can be encouraged to try to articulate what they do understand, certain that they will be helped to modify their conjectures without being ridiculed. In a conjecturing atmosphere those who are certain hold back or ask helpful questions, while those who are uncertain try to articulate their uncertainty.

### Questions as Typifying What Mathematics Is About

Since students' experience of mathematics is dominated by the questions they are asked, their impression of what mathematics is about, of what the mathematics enterprise is about, is likely to be formed by the nature and content of the questions they are asked. Anne Watson and John Mason (1998; see also a primary version Jeffcoat et al. 2004) built on a collection of mathematically structured question types developed by Zygfryd Dyrslag (1984) to provide a wide-ranging collection of questions that draw attention to mathematical ways of thinking. They used a list of verbs of mathematics including

Exemplifying, Specializing, Completing, Deleting, Correcting, Comparing, Sorting, Organizing, Changing, Varying, Reversing, Altering, Generalizing, Conjecturing, Explaining, Justifying, Verifying, Convincing, and Refuting

and types of mathematical statements including

Definitions, Facts, Properties, Theorems, Examples, Counterexamples, Techniques, Instructions, Conjectures, Problems, Representation, Notation, Symbolization, Explanations, Justifications, Proofs, Reasoning, Links, Relationships, and Connections

to generate a grid of mathematically fruitful and pedagogically effective questions which are founded in the mathematical practices of experts.

### Internalizing Questions

If students are always asked the same question or type of question whenever they get stuck, whenever a new topic is being presented, or whenever a topic is being reviewed, then most students are likely to come to depend on the teacher asking that question. The question remains associated with the classroom rather than being internalized by students. By contrast, if the teacher begins by using a question type repeatedly and effectively and then gradually makes their prompts less and less explicit, students' attention can be directed to the types of questions that the teacher is asking and eventually to students spontaneously asking themselves the question. For example, asking questions like "what question am I going to ask you?" or "What did you do yesterday when you were stuck?" provides a metacognitive shift, an impetus for students to become aware of what they have been asked rather than remaining immersed in their task and simply responding to the question (Bauersfeld 1995). Meanwhile the teacher can begin introducing a different question or prompt.

The term "scaffolding" (Wood et al. 1976) is often used to refer to the temporary support that a teacher can provide for students, in which the teacher acts as "consciousness for two" (Bruner 1986). This notion applies both to "horizontal mathematization" in which students are prompted to become aware of other situations in which their thinking could be used ("utility" in the sense of Ainley and Pratt 2002) and to "vertical mathematization" in which students are prompted to become aware of what they have been doing as instances of some more general or "abstract" action (Treffers 1987). Shifting between levels of thinking is not entirely natural for many or even most students. It is a major role for teachers of mathematics.

However, teacher interventions, whether as reminders or as re-orientations of attention are likely to go unnoticed, because the student is immersed in the action. In order that students become aware of the questions they are asked, the prompts they are given that serve to redirect their attention usefully, it is usually necessary for the teacher to engage in what Brown et al. (1989) called fading or, in other words, to use increasingly indirect, even metacognitive prompts, so that eventually the students internalize the prompts for themselves (Love and Mason 1992). Learning and independence can really only be said to have been achieved when students spontaneously question themselves and each other. A good example can be found in Brown and Coles (2000) regarding the question "what is the same and what is different about . . .".

### Students Asking "Good Questions"

It must be every competent teacher's dream that students will ask "good" mathematical questions and a potential nightmare to be asked a lot of questions beyond the teacher's competence. Every teacher needs strategies to deal with the unexpected and difficult or challenging question. Displacement and deferral strategies include inviting students to record their conjecture for discussion later, having a public place reserved for current conjectures and questions, and seeking assistance from colleagues in the same or other institutions or on the web. But such questions are unlikely to come out unless students are being encouraged to pose questions and to make conjectures. The best way to stimulate genuine mathematical questions from students is to ask genuine questions oneself, to be seen to be enquiring, to have strategies (specializing and generalizing, representing, and transforming) to use, and to be satisfied to leave an enquiry as a conjecture for later (even much later) consideration. This is what is meant by "being mathematical with and in front of students," and it is the best way to offer students experience of the thrill and pleasure of thinking mathematically.

### Further Investigation

It might be tempting to research questions of the form "which form of questioning is the most effective?" or "Which order of questions is most effective?", but in mathematics education, any assertion of a generality has counterexamples (Tahta, personal communication, 1990). There is no universality, because so much of what happens depends on the rapport and relationship between teacher and students and between teacher and mathematics (Kang and Kilpatrick 1992; Handa 2011).

What is worthy of further investigation are questions of the following form:

What is it about a situation that brings certain questions or prompts to my mind? How might this inform ways of working with students so that they begin to come to mind for the students?

What is it about a situation that could bring certain useful questions or prompts to mind?

What blocks or deters me from asking certain types of questions?

What is it about some questions and prompts that attracts teachers to try to use them, while other questions and prompts do not?

## Cross-References

- ▶ Affect in Mathematics Education
- ▶ Curriculum Resources and Textbooks in Mathematics Education
- ▶ Hypothetical Learning Trajectories in Mathematics Education
- ▶ Inquiry-Based Mathematics Education
- ▶ Manipulatives in Mathematics Education
- ▶ Mathematical Representations
- ▶ Noticing of Mathematics Teachers
- ▶ Problem Solving in Mathematics Education
- ▶ Sociomathematical Norms in Mathematics Education
- ▶ Zone of Proximal Development in Mathematics Education

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# R

## Realistic Mathematics Education

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### Keywords

Domain-specific teaching theory; Realistic contexts; Mathematics as a human activity; Mathematization

### What is Realistic Mathematics Education?

Realistic Mathematics Education – hereafter abbreviated as RME – is a domain-specific instruction theory for mathematics, which has been developed in the Netherlands. Characteristic of RME is that rich, "realistic" situations are given a prominent position in the learning process. These situations serve as a source for initiating the development of mathematical concepts, tools, and procedures and as a context in which students can in a later stage apply their mathematical knowledge, which then gradually has become more formal and general and less context specific.

Although "realistic" situations in the meaning of "real-world" situations are important in RME, "realistic" has a broader connotation here. It means students are offered problem situations which they can imagine. This interpretation of "realistic" traces back to the Dutch expression "zich REALISEren," meaning "to imagine." It is this emphasis on making something real in your mind that gave RME its name. Therefore, in RME, problems presented to students can come from the real world but also from the fantasy world of fairy tales, or the formal world of mathematics, as long as the problems are experientially real in the student's mind.

### The Onset of RME

The initial start of RME was the founding in 1968 of the Wiskobas ("mathematics in primary school") project initiated by Edu Wijdeveld and Fred Goffree and joined not long after by Adri Treffers. In fact, these three mathematics didacticians created the basis for RME. In 1971, when the Wiskobas project became part of the newly established IOWO Institute, with Hans Freudenthal as its first director and in 1973 when the IOWO was expanded with the Wiskivon project for secondary mathematics education; this basis received a decisive impulse to reform the prevailing approach to mathematics education.

In the 1960s, mathematics education in the Netherlands was dominated by a mechanistic teaching approach; mathematics was taught

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directly at a formal level, in an atomized manner, and the mathematical content was derived from the structure of mathematics as a scientific discipline. Students learned procedures step by step with the teacher demonstrating how to solve problems. This led to inflexible and reproduction-based knowledge. As an alternative for this mechanistic approach, the "New Math" movement deemed to flood the Netherlands. Although Freudenthal was a strong proponent of the modernization of mathematics education, it was his merit that Dutch mathematics education was not affected by the formal approach of the New Math movement and that RME could be developed.

## Freudenthal's Guiding Ideas About Mathematics and Mathematics Education

Hans Freudenthal (1905–1990) was a mathematician born in Germany who in 1946 became a professor of pure and applied mathematics and the foundations of mathematics at Utrecht University in the Netherlands. As a mathematician he made substantial contributions to the domains of geometry and topology.

Later in his career, Freudenthal (1968, 1973, 1991) became interested in mathematics education and argued for teaching mathematics that is relevant for students and carrying out thought experiments to investigate how students can be offered opportunities for guided re-invention of mathematics.

In addition to empirical sources such as textbooks, discussions with teachers, and observations of children, Freudenthal (1983) introduced the method of the didactical phenomenology. By describing mathematical concepts, structures, and ideas in their relation to the phenomena for which they were created, while taking into account students' learning process, he came to theoretical reflections on the constitution of mental mathematical objects and contributed in this way to the development of the RME theory.

Freudenthal (1973) characterized the then dominant approach to mathematics education in which scientifically structured curricula were used and students were confronted with ready-made mathematics as an "anti-didactic inversion." Instead, rather than being receivers of ready-made mathematics, students should be active participants in the educational process, developing mathematical tools and insights by themselves. Freudenthal considered mathematics as a human activity. Therefore, according to him, mathematics should not be learned as a closed system but rather as an activity of mathematizing reality and if possible even that of mathematizing mathematics.

Later, Freudenthal (1991) took over Treffers' (1987a) distinction of horizontal and vertical mathematization. In horizontal mathematization, the students use mathematical tools to organize and solve problems situated in real-life situations. It involves going from the world of life into that of symbols. Vertical mathematization refers to the process of reorganization within the mathematical system resulting in shortcuts by using connections between concepts and strategies. It concerns moving within the abstract world of symbols. The two forms of mathematization are closely related and are considered of equal value. Just stressing RME's "real-world" perspective too much may lead to neglecting vertical mathematization.

## The Core Teaching Principles of RME

RME is undeniably a product of its time and cannot be isolated from the worldwide reform movement in mathematics education that occurred in the last decades. Therefore, RME has much in common with current approaches to mathematics education in other countries. Nevertheless, RME involves a number of core principles for teaching mathematics which are inalienably connected to RME. Most of these core teaching principles were articulated originally by Treffers (1978) but were reformulated over the years, including by Treffers himself.

In total six principles can be distinguished:

- The *activity principle* means that in RME students are treated as active participants in the learning process. It also emphasizes that mathematics is best learned by doing mathematics, which is strongly reflected in Freudenthal's interpretation of mathematics as a human activity, as well as in Freudenthal's and Treffers' idea of mathematization.

- The *reality principle* can be recognized in RME in two ways. First, it expresses the importance that is attached to the goal of mathematics education including students' ability to apply mathematics in solving "real-life" problems. Second, it means that mathematics education should start from problem situations that are meaningful to students, which offers them opportunities to attach meaning to the mathematical constructs they develop while solving problems. Rather than beginning with teaching abstractions or definitions to be applied later, in RME, teaching starts with problems in rich contexts that require mathematical organization or, in other words, can be mathematized and put students on the track of informal context-related solution strategies as a first step in the learning process.
- The *level principle* underlines that learning mathematics means students pass various levels of understanding: from informal context-related solutions, through creating various levels of shortcuts and schematizations, to acquiring insight into how concepts and strategies are related. Models are important for bridging the gap between the informal, context-related mathematics and the more formal mathematics. To fulfill this bridging function, models have to shift – what Streefland (1985, 1993, 1996) called – from a "model of" a particular situation to a "model for" all kinds of other, but equivalent, situations (see also Gravemeijer 1994; Van den Heuvel-Panhuizen 2003).

  Particularly for teaching operating with numbers, this level principle is reflected in the didactical method of "progressive schematization" as it was suggested by Treffers (1987b) and in which transparent whole-number methods of calculation gradually evolve into digit-based algorithms.
- The *intertwinement principle* means mathematical content domains such as number, geometry, measurement, and data handling are not considered as isolated curriculum chapters but as heavily integrated. Students are offered rich problems in which they can use various mathematical tools and knowledge. This principle also applies within domains. For example, within the domain of number sense, mental arithmetic, estimation, and algorithms are taught in close connection to each other.
- The *interactivity principle* of RME signifies that learning mathematics is not only an individual activity but also a social activity. Therefore, RME favors whole-class discussions and group work which offer students opportunities to share their strategies and inventions with others. In this way students can get ideas for improving their strategies. Moreover, interaction evokes reflection, which enables students to reach a higher level of understanding.
- The *guidance principle* refers to Freudenthal's idea of "guided re-invention" of mathematics. It implies that in RME teachers should have a proactive role in students' learning and that educational programs should contain scenarios which have the potential to work as a lever to reach shifts in students' understanding. To realize this, the teaching and the programs should be based on coherent long-term teaching-learning trajectories.

## Various Local Instruction Theories

Based on these general core teaching principles, a number of local instruction theories and paradigmatic teaching sequences focusing on specific mathematical topics have been developed over time. Without being exhaustive some of these local theories are mentioned here. For example, Van den Brink (1989) worked out new approaches to addition and subtraction up to 20. Streefland (1991) developed a prototype for teaching fractions intertwined with ratios and proportions. De Lange (1987) designed a new approach to teaching matrices and discrete calculus. In the last decade, the development of local instruction theories was mostly integrated with the use of digital technology as investigated by Drijvers (2003) with respect to promoting students' understanding of algebraic concepts and operations. Similarly, Bakker (2004) and Doorman (2005) used dynamic computer software to contribute to an empirically grounded instruction theory for early statistics education and for differential calculus in connection with kinematics, respectively.

The basis for arriving at these local instruction theories was formed by design research, as elaborated by Gravemeijer (1994), involving a theory-guided cyclic process of thought experiments, designing a teaching sequence, and testing it in a teaching experiment, followed by a retrospective analysis which can lead to necessary adjustments of the design.

Last but not least, RME also led to new approaches to assessment in mathematics education (De Lange 1987, 1995; Van den Heuvel-Panhuizen 1996).

## Implementation and Impact

In the Netherlands, RME had and still has a considerable impact on mathematics education. In the 1980s, the market share of primary education textbooks with a traditional, mechanistic approach was 95 % and the textbooks with a reform-oriented approach – based on the idea of learning mathematics in context to encourage insight and understanding – had a market share of only 5 %. In 2004, reform-oriented textbooks reached a 100 % market share and mechanistic ones disappeared. The implementation of RME was guided by the RME-based curriculum documents including the so-called Proeve publications by Treffers and his colleagues, which were published from the late 1980s, and the TAL teaching-learning trajectories for primary school mathematics, which have been developed from the late 1990s (Van den Heuvel-Panhuizen 2008; Van den Heuvel-Panhuizen and Buys 2008).

A similar development can be seen in secondary education, where the RME approach also influenced textbook series to a large extent. For example, Kindt (2010) showed how practicing algebraic skills can go beyond repetition and be thought provoking. Goddijn et al. (2004) provided rich resources for realistic geometry education, in which application and proof go hand in hand.

Worldwide, RME is also influential. For example, the RME-based textbook series "Mathematics in Context" Wisconsin Center for Education Research & Freudenthal Institute (2006) has a considerable market share in the USA. A second example is the RME-based "Pendidikan Matematika Realistik Indonesia" in Indonesia (Sembiring et al. 2008).

## A Long-Term and Ongoing Process of Development

Although it is now some 40 years from the inception of the development of RME as a domain-specific instruction theory, RME can still be seen as work in progress. It is never considered a fixed and finished theory of mathematics education. Moreover, it is also not a unified approach to mathematics education. That means that through the years different emphasis was put on different aspects of this approach and that people who were involved in the development of RME – mostly researchers and developers of mathematics education and mathematics educators from within or outside the Freudenthal Institute – put various accents in RME. This diversity, however, was never seen as a barrier for the development of RME but rather as stimulating reflection and revision and so supporting the maturation of the RME theory. This also applies to the current debate in the Netherlands (see Van den Heuvel-Panhuizen 2010) which voices the return to the mechanistic approach of four decades back. Of course, going back in time is not a "realistic" option, but this debate has made the proponents of RME more alert to keep deep understanding and basic skills more in balance in future developments of RME and to enhance the methodological robustness of the research that accompanies the development of RME.

## Cross-References

▶ Didactical Phenomenology (Freudenthal)

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# Recontextualization in Mathematics Education

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## Keywords

Anthropological theory of didactics; Classification; Didactic transposition; Discursive saturation; Domains of action; Emergence; Framing; Institutionalisation; Noosphere; Pedagogic device; Recontextualisation; Social activity method; Sociology; Strategic action

## Characteristics

Recontextualization refers to the contention that texts and practices are transformed as they are moved between contexts of their reading or enactment. This simple claim has profound implications for mathematics education and for education generally. There are three major theories in the general field of educational studies that directly and explicitly concern recontextualization: the Theory of Didactic Transposition (later the Anthropological Theory of Didactics) of Yves Chevallard, Basil Bernstein's pedagogic device, and Paul Dowling's Social Activity Method. These are all complex theories, so their presentation here of necessity entails substantial simplification.

The Theory of Didactic Transposition (TDT) proposes, essentially, that constituting a practice as something to be taught will always involve a transformation of the practice. This is a general claim that can be applied to any practice and any form of teaching, but Chevallard's (1985, 1989) work and that of many of those who have worked with the TDT is most centrally concerned with the teaching of mathematics in formal schooling (primary, secondary, or higher education phases). The work of the didactic transposition is carried out, firstly, by agents of what Chevallard referred to as the noosphere and involves the production of curricula in the form of policy documents, syllabuses, textbooks, examinations, and so forth constituting the "knowledge to be taught." The first task in this work is the construction of a body of source knowledge as the referent practice of the "knowledge to be taught." In the case of school mathematics, this source or "scholarly knowledge" has been produced by mathematicians over a very long historical period and in diverse contexts. In its totality, then, it is not a practice that is currently enacted by mathematicians, but is compiled in the noosphere. The next task is the constitution of the "knowledge to be taught" from this "scholarly knowledge," and it is the former that is presented to teachers as the curriculum. There is a further move, however, as the teacher in the classroom must, through interpretation and the production and management of lessons, transpose the "knowledge to be taught" into "knowledge actually taught." Even this knowledge is not necessarily equivalent to the knowledge acquired by the student, which is the product of a further transposition. The precise nature of the transposition at each stage is a function of the nature of the knowledge (scholarly, to be taught, actually taught) being recontextualized and of historical, cultural, and pedagogic specificities. TDT – which has been developed in terms of conceptual complexity as the Anthropological Theory of Didactics (ATD, Chevallard 1992) – invites researchers to investigate the precise processes whereby the recontextualizations have been achieved in particular locations and in respect of particular regions of the curriculum, so revealing the conditions and constraints on the teaching of mathematics in these contexts. This has been attempted in, for example, the topics of calculus (Bergsten et al. 2010), statistics (Wozniak 2007), and the limits of functions (Barbé et al. 2005).

Bernstein describes the "pedagogic device" as "the condition for culture, its productions, reproductions and the modalities of their interrelations" (1990; see also Bernstein 2000). It is a central feature of a highly complex theory that was developed over a period of some 40 years, so its representation here is of necessity radically simplified. Whereas Chevallard's theory is concerned with the epistemological and cultural constraints on didactics, Bernstein's interest lies in the manner in which societies are reproduced and transformed. Pedagogy and, in particular, transmission occur in all sociocultural institutions, although much of the work inspired by Bernstein has focused on formal schooling. An important exception to this is his early dialogue with the anthropologist, Mary Douglas (see Douglas 1996/1970), which contributed to Douglas's cultural theory and Bernstein's fundamental concepts, *classification* (regulation between contexts) and framing (regulation within a context). The *pedagogic device* regulates what is transmitted to whom, when, and how and consists of three sets of rules, hierarchically organized: *distribution*, *recontextualization*, and *evaluation*. Recontextualization rules, in particular, regulate the delocation of discourses from the fields of their production – the production of physics discourse in the university, for example – and their relocation as pedagogic discourse. This is achieved by the embedding of these *instructional discourses* in *regulatory discourses* involving principles of selection, sequencing, and pacing. Recontextualization is achieved by agents in the *official recontextualizing field* – policy makers and administrators – and the *pedagogic recontextualizing field* (teacher educators, the authors of textbooks, and so forth) that together might be taken to coincide with Chevallard's noosphere in terms of membership. Superficially, there might seem to be similarities between Bernstein's and Chevallard's theories. A crucial distinction, however, is that recontextualization for Bernstein, but not for Chevallard, is always governed by distribution. This entails that pedagogic discourse is always structured by the social dimensions of class, gender, and race. Bernstein's is a sociological theory, while Chevallard's might reasonably be described (in English) as an educational theory. Through the sociological concept, relative autonomy, Bernstein also allows for the possibility of the transformation of culture and, ultimately, of society. A further distinction lies in that Bernstein describes pedagogic discourse in terms of his fundamental categories, classification, and framing, which enables a description of form but not of content. Further resources for the description of the form of discourses are available in Bernstein's (2000) work on *horizontal* and *vertical discourses* and on *knowledge structures* where he describes mathematics as a vertical discourse having horizontal knowledge structure and a strong grammar. In this description he seems to be making no epistemological distinction between mathematics in its field of production, on the one hand, and school mathematics, on the other.

Dowling's (2009, 2013) *Social Activity Method* (SAM) presents a sociological organizational language that takes seriously lessons from constructionism and poststructuralism. As is the case with Chevallard's TDT, Dowling's work began with an interest in mathematics education (see Dowling 1994, 1995, 1996, 1998) but is more fundamentally sociological, giving a degree of priority to social relations over *cultural practices*. For Dowling, the sociocultural is characterized by *social action* that is directed at the formation, maintenance, and destabilizing of *alliances* and *oppositions*. These alliances and oppositions, however, are emergent upon the totality of social action rather than being the deliberate outcomes of individual actions. Alliances are visible in terms of regularities of practice that give the appearance of regulating who can do, say, think what, though, as emergent outcomes, they might be thought of, metaphorically, as advisory rather than determinant. Another feature of Dowling's theory is that it has a fractal quality, which is to say, the same language can be applied at any level of analysis and the language is also capable of describing itself. School mathematics is an example of what might be taken to exhibit a regularity of practice including the institutionalization of expression (signifiers) and content (signifieds) in texts. The strength of institutionalization varies, however, between strong and weak, giving rise to the scheme of *domains of practice* in Fig. 1, which constitutes part of the structure of all contexts, which is to say, of all alliances. Human agents might be described as seeing the world in terms of the scheme in Fig. 1 or, more precisely, from the perspective of the *esoteric domain*. Where the particular context is school mathematics, the agent may cast a *gaze* beyond school mathematics onto, for example, domestic practices such as



| | Content (signifieds) | |
|---|---|---|
| Expression (signifiers) | $I^+$ | $I^-$ |
| $I^+$ | *esoteric domain* | *descriptive domain* |
| $I^-$ | *expressive domain* | *public domain* |

$I^{+/-}$ represents strong/weak institutionalisation.

**Recontextualization in Mathematics Education, Fig. 1** Domains of action (Source: Dowling 2009)

**Recontextualization in Mathematics Education, Fig. 2** Modes of recontextualization (Source: Dowling 2013)

| Representation | Practice | |
|---|---|---|
| | $DS^-$ | $DS^+$ |
| $DS^-$ | *improvising* | *de-principling* |
| $DS^+$ | *rationalising* | *re-principling* |

shopping. The deployment of principles of recognition and realization that are specific to school mathematics will result in the recontextualization of domestic shopping as mathematized shopping. This contributes to the *public domain* of school mathematics, which thereby appears to be about something other than mathematics. This contrasts with *esoteric domain* text that is unambiguously about mathematics, the *descriptive domain* – the domain of mathematical modelling – that appears to be about something other than mathematics but that is presented in the language of mathematics, and the *expressive domain* (the domain of pedagogic metaphors) that appears to be about mathematics but that is presented in the language of other practices (an equation is a balance, and so forth). This scheme enables the description of complex mathematical texts and settings in terms of the distribution of the different domains of mathematical practice to different categories of student (e.g., in terms of social class). It can also reveal distinctions between modes of pedagogy that take different trajectories around the scheme. It should be emphasized that *public domain* shopping is not the same thing as domestic shopping; the recontextualization of practice always entails a transformation as is illustrated by Brantlinger (2011) in respect of critical mathematics education. The *gaze* of mathematics education is described (Dowling 2010) as *fetching* practices from other activities and recontextualizing them as mathematical practice. This is, in a sense, a didactic necessity in the production of apprentices to mathematics who must, initially, be addressed in a language that is familiar to them. A danger, however, lies in the *pushing* of the results back out of mathematics as the result no longer has ecological validity. The scheme in Fig. 1 is reproduced in all activities that can be recognized as exhibiting regularity of practice and at all levels within any such practice. Chung (2011), for example, has directed an elaborated version of the scheme at literary studies.

Another category from SAM is *discursive saturation*, which refers to the extent to which a practice makes its principles linguistically available. To the extent that an activity or part of an activity can be described as high or low discursive saturation ($DS^+$ or $DS^-$), then another scheme is generated that describes modes of recontextualization. This scheme is shown in Fig. 2. If school mathematics can generally be described as $DS^+$ and domestic shopping as $DS^-$, then the recontextualizing of domestic shopping as school mathematics public domain – the representation of shopping by mathematics – can be described as *rationalizing* and the recontextualizing of, say, banking by school mathematics as *re-principling*.

These three theories of *recontextualization* – those of Chevallard, Bernstein, and Dowling – offer different possibilities to researchers, and practitioners in mathematics education and themselves draw on different theoretical and disciplinary antecedents. They are, however, not in competition as much as being complementary. All three present languages that can be and have been deployed far more widely than mathematics education, though Chevallard's and Dowling's theories certainly have their roots in this field of research. Naturally, all three theories have undergone more or less transformative action in respect of their recontextualization for the purposes of this entry.

## Cross-References


▶ Anthropological Approaches in Mathematics Education, French Perspectives
▶ Calculus Teaching and Learning
▶ Critical Mathematics Education

► Curriculum Resources and Textbooks in Mathematics Education
► Didactic Transposition in Mathematics Education
► Gender in Mathematics Education
► Language Background in Mathematics Education
► Mathematical Knowledge for Teaching
► Mathematical Language
► Mathematics Curriculum Evaluation
► Metaphors in Mathematics Education
► Semiotics in Mathematics Education
► Socioeconomic Class in Mathematics Education
► Sociological Approaches in Mathematics Education

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# Reflective Practitioner in Mathematics Education

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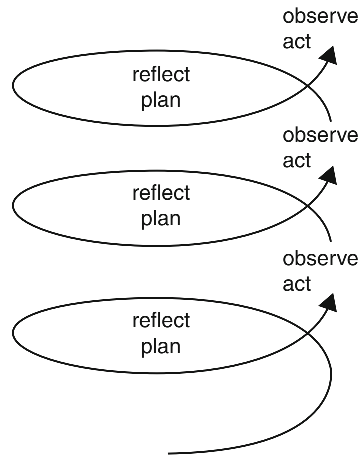
*Reflective practice* is a commonly used term in mathematics education, often without careful definition, implying a contemplative reviewing of learning and/or teaching in mathematics in order to approve, evaluate, or improve practice. A feedback loop is often suggested in which reflective practice feeds back into the design or initiation of practice providing possibilities for improved practice. More precise definitions often draw on Dewey, who wrote:

> Active, persistent and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it and the further conclusions to which it tends constitutes reflective thought (1933, p. 9)
>
> . . . reflective thinking, in distinction to other operations to which we apply the name of thought, involves (1) a state of doubt, hesitation, perplexity, mental difficulty, in which thinking originates, and (2) an act of searching, hunting, inquiring, to find material that will resolve the doubt and dispose of the perplexity (p. 12).
>
> . . . Demand for the solution of a perplexity is the steadying and guiding factor in the entire process of reflection. (p. 14)

Rather than a perspective just of contemplative thought, Dewey emphasizes the important element of *action* in reflection and the goal of an *action outcome*. This has led to a linking of reflective practice with so-called *action research* which is research conducted by practitioners into aspects of (their own) professional practice. Stephen Kemmis a leading proponent of action research spoke of reflection as "meta-thinking," thinking about thinking. He wrote:

> We do not pause to reflect in a vacuum. We pause to reflect because some issue arises which demands that we stop and take stock or consider before we act. . . . We are inclined to see reflection as something quiet and personal. My argument here is that reflection is action-oriented, social and political. Its product is praxis (informed committed action) the most eloquent and socially significant form of human action. (Kemmis 1985, p. 141)

Kemmis conceptualized action research with reference to a critically reflective spiral in action research of plan, act and observe, and reflect (Kemmis and McTaggart 1981; Carr and Kemmis 1986), and other scholars have adapted this subsequently (e.g., McNiff 1988) (Fig. 1).




**Reflective Practitioner in Mathematics Education, Fig. 1** Action-reflection cycle (McNiff (1988), pp 44, Fig 3.7)

More recent scholars relate ideas of reflection, seminally, to the work of Donald Schön who has written about the *reflective practitioner* in professions generally and in education particularly (Schön 1983, 1987). Schön relates reflection to *knowing* and describes *knowing-in-action* and *reflection-in-action*. With reference to Dewey, he writes about *learning by doing*, the importance of action in the process of learning, and relates *doing* and *learning* through a reflective process.

> Our knowing is ordinarily tacit, implicit in our patterns of action and in our feel for the stuff with which we are dealing. It seems right to say that our knowing is in our action (1983, p. 49).

Schön refers to *knowing-in-action* as "the sorts of know-how we reveal in our intelligent action – publicly observable, physical performances like riding a bicycle and private operations like instant analysis of a balance sheet" (1997, p. 25). He claims a subtle distinction between knowing-in-action and reflection-in-action. The latter he links to moments of *surprise* in action: "We may reflect *on* action, thinking back on what we have done in order to discover how our knowing-in-action may have contributed to an unexpected outcome" (p. 26). "Alternatively," he says, "we may reflect in the midst of action without interrupting it . . . our thinking serves to reshape what we are doing while we are doing it" (p. 26). Schön distinguishes reflection-on-action and reflection-in-action. The first involves looking back *on* an action and reviewing its provenance and outcomes with the possibility then of modifying future action; the second is especially powerful, allowing the person acting to recognize a moment in the action, possibly with surprise, and to act, there and then, differently. John Mason has taken up this idea in his *discipline of noticing*: we *notice*, in the moment, something of which we are aware, possibly have reflected *on* in the past and our noticing afford us the opportunity to act differently, to modify our actions in the process of acting (Mason 2002).

Michael Eraut (1995) has criticized Schön's theory of reflection-in-action where it applies to teachers in classrooms. He points out that Schön presents little empirical evidence of reflection-in-action, especially where teaching is concerned. The word *action* itself has different meanings for different professions. In teaching, *action* usually refers to action in the classroom where teachers operate under pressure. Eraut argues that time constraints in teaching limit the scope for reflection-in-action. He argues that there is too little time for considered reflection as part of the teaching act, especially where teachers are responding to or interacting with students. Where a teacher is walking around a classroom of children quietly working on their own, reflection-in-action is more possible but already begins to resemble time *out* of action. Thus Eraut suggests that, in teaching, most reflection is reflection-*on*-action, or reflection-*for*-action. He suggests that Schön is primarily concerned with reflection-*for*-action, reflection whose purpose is to affect action in current practice.

In mathematics education research into teaching practices in mathematics classrooms, Jaworski (1998) has worked with the theoretical ideas of Schön, Mason, and Eraut to characterize observed mathematics teaching and the thinking, action, and development of the observed teachers. The research was undertaken as part of a project, the Mathematics Teachers' Enquiry (MTE) Project, in which participating teachers engaged in forms of action research into their own teaching. Jaworski claims that the three prepositions highlighted in the above discussion, *on*, *in*, and *for*, "all pertain to the thinking of teachers at different points in their research" (p. 9) and provides examples from observations of teaching and conversations with teachers. To some degree, all the teachers observed engaged in action research in the sense that they explored aspects of their own practice in reflective cycles. However, rather than the theorized *systematicity* of action research (e.g., McNiff 1988), Jaworski described the cyclic process of growth of knowledge for these teachers as *evolutionary*, as "lurching" from time to time, opportunity to opportunity, as teachers grappled with the heavy demands of being a teacher and sought nevertheless to reflect on and in their practice. As Eraut suggested, the nature of teaching in classrooms is demanding and complex for the teacher, as is the ongoing life in a school and the range of tasks a teacher is required to undertake. Teachers' reflection *on* their practice, evidenced by reports at project meetings and observations of teacher educator researchers, led to noticing *in* the moment in classrooms, reflection-*in*-action, and concomitant changes in action resulting from such noticing.

A question that arises in considering reflective practice in mathematics education concerns what difference it makes (to reflective practice) that it is being used in relation to mathematics and to the learning and teaching of mathematics. Although in the mathematics education literature there are many references to the reflection of practitioners, there is a singular lack of relating reflective practice directly to mathematics. We see writings by mathematics educators referring, for example, to mathematics teachers who are reflective practitioners, reflecting on their practice of teaching mathematics; however, the mathematics is rarely addressed per se. We read about specific approaches to teaching mathematics and to engagement in reflective practice, for example, the identification of "critical incidents," or the use of a "lesson study approach." To a great extent, the same kinds of practices and issues might be reported if the writers were talking about science or history teaching. There is also a dearth of research in which mathematics students are seen as reflective practitioners.

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# Rural and Remote Mathematics Education

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## Keywords

Rural mathematics education; Remote mathematics education; Distance education

## Definition(s)

Definitions of rural and remote mathematics contexts differ considerably from country to country and region to region – nevertheless most definitions consider geographical position, population density, and distance from the nearest urban area. The Organisation for Economic Co-operation and Development (OECD) classifies regions within its member countries into three groups based on population density – predominantly urban, intermediate, or predominantly rural. A region is considered rural if it meets three methodology criteria: (1) “local units” within a region are rural if they have a population density of less than 150 inhabitants per square kilometer, (2) more than 50 % of the population in the region live in rural local units, and (3) they will not contain an urban center of over 200,000 people (OECD 2010a).

Developing regions around the world, in particular Africa and Asia, are still mostly rural. However, by 2030 these regions will join the developed world in having a mostly urban population. Although the developed world has been predominantly urban since the early 1950s, some countries have a relative high proportion of the population outside major cities (e.g., Australia, 34 %; Canada, 19 %) (Australian Bureau of Statistics [ABS] 2012; Statistics Canada 2008). Social indicators show that people living in rural areas have less access to a high quality of life than do those living in urban areas, based on factors such as employment, education, health, and leisure (UN 2011). To some degree, research in this area has been considered from a deficit perspective, often perceived as backward, attached to tradition, and anti-modern (Howley et al. 2010).

### Differences in Student Performance

Students in large urban areas tend to outperform students in rural schools by the equivalent of more than one year of education (OECD 2012). Severe poverty, often exacerbated in rural areas due to a lack of employment, education opportunities, and infrastructure, manifests the situation (Adler et al. 2009). Although socioeconomic background accounts for part of the difference, performance difference remain even when socioeconomic background is removed as a factor (OECD 2012). In other situations, severe environmental conditions, including drought and flood, heighten the challenging nature of educational opportunities in rural areas (Lowrie 2007). Differences in students’ success in mathematics are often correlated with the size of their community, along with its degree of remoteness (Atweh et al. 2012). Rural, and especially remote, communities face challenges of high staff turnover, reduced professional learning opportunities, and difficulty in accessing quality learning opportunities for students (Lyons et al. 2006). The capacity to attract teachers with strong mathematics pedagogical content knowledge – already a challenge in many countries – is heightened in rural and remote areas with students having limited opportunities to study higher levels of mathematics (Kitchenham and Chasteauneuf 2010; Ngo 2012). As the OECD (2010b, p. 13) highlights, "...disadvantaged schools still report great difficulties in attracting qualified teachers... Findings from PISA suggest that, in terms of teacher resources, many students face the double liability of coming from a disadvantaged background and attending a school with lower quality resources."

### Opportunities in Rural and Remote Settings

From a pedagogical perspective, communication technologies provide opportunities for enhanced mathematics engagement (Lowrie 2006). In fact, distance education often leads the way in communication initiatives and technological advances (Guri-Rosenblit 2009). A benefit can be that rural/remote schools and students have access to current and innovative technologies that are not yet being used in mainstream metropolitan schools. In this sense, remote settings provide opportunities for mathematics pedagogy to be differently contextualized (Lowrie and Jorgensen 2012).

Distance education features strongly in the organization structuring of education in remote areas – with students afforded the opportunity to study mathematics without leaving their home community. Such situations change the nature and role of teaching – with the student having to be more self-reliant since face-to-face engagement with their teacher is minimal. High-quality teaching and learning are fostered through well-designed resources and strong home-school partnerships (Lowrie 2007). The shared decision-making that is negotiated and established in distance education contexts is highly influential in the students' numeracy development (Goos and Jolly 2004) and can be looked upon in reshaping the practices of more traditional mathematics classrooms.

## Cross-References


- ▶ Equity and Access in Mathematics Education
- ▶ Socioeconomic Class in Mathematics Education
- ▶ Urban Mathematics Education


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# S

## Scaffolding in Mathematics Education

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### Keywords

Support system; Help; Zone of proximal development; Educative strategy

### Definition

Scaffolding is generally conceived as an interactional process between a person with educational intentions and a learner, aiming to support this learner's learning process by giving appropriate and temporary help. Scaffolding in mathematics education is the enactment of this purposive interaction for the learning of mathematical actions and problem solving strategies.

A number of clarifying corollary postulates are usually added for the completion of this general definition of scaffolding in a specific situation:

- Scaffolding is an intentional support system based on purposive interactions with more competent others, which can be adults or peers; the support can be individualized (one teacher scaffolding one student) or collective (a group scaffolding its members in a distributed way).
- The support consists of employing instructional means that are supposed to help learners with the accomplishment of a new (mathematical) task by assisting him/her to carry out the required activity through providing help at parts of the activity that aren't yet independently mastered by the learner; this is to be distinguished from just simplifying the task by cutting it down into a collection of isolated elementary tasks.
- Scaffolding aims at providing learners help that is contingent on the learner's prior qualities and contributes to the development of knowledge, skills, and confidence to cope with the full complexity of the task; as such scaffolding is to be distinguished from straightforward instruction in correct task performance.
- As a support system scaffolding is essentially a temporary construction of external help that is supposed to fade away in due time.

### Characteristics

#### Tutoring Learning

The notion of educational support systems for the appropriation of complex activities was first introduced by Bruner in the 1950s in his studies of language development in young children. In opposition to the Chomskyan explanation of language development resulting from an inherent

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Language Acquisition Device (LAD), Bruner advocated a theory of language development that holds that parent–infant interactions constitute a support system for children in their attempts to accomplish communicative intentions. In Bruner's view it is this Language Acquisition Support System (LASS) that "scaffolds" children's language development.

In a seminal article on adult tutoring in children's problem solving, Bruner and his colleagues generalized the idea of learning support systems to the domain of problem solving in general and explicitly coined the notion of scaffolding as a process of tutoring children for the acquisition of new problem solving skills (see Wood et al. 1976). They point out that scaffolding "consists essentially of the adult 'controling' those elements of the task that are initially beyond the learner's capacity, thus permitting him to concentrate upon and complete only those elements that are within his range of competence" (Wood et al. 1976, p. 90). In the elaboration of the scaffolding process, Wood et al. (1976, p. 98) identify several scaffolding functions:

1. *Recruitment*: scaffolding should get learners actively involved in relevant problem solving activity.
2. *Reduction in* degrees of freedom, i.e., keeping students focused on those constituent acts that are required to reach a solution and that they can manage while preventing them from being distracted by acts that are beyond their actual competence level; these latter actions are supposed to be under the control of the scaffolding tutor.
3. *Direction maintenance*: The tutor has the role of keeping students in pursuit of a particular objective and keeps them motivated to be self-responsible for the task execution.

Without explicitly mentioning the Vygotskian notion of the zone of proximal development, the formulations used by Bruner and his colleagues (see quote above) unequivocally refer to one of Vygotsky's operationalizations of this notion (see Vygotsky 1978, p. 86) as the discrepancy between what a learner can do independently and the learner's performance with help (support) from more knowledgeable others. In later explanations and elaborations of scaffolding, most authors have taken this notion of the zone of proximal development as a point of reference.

Using a Vygotskian theoretical framework, the work of Stone and Wertsch has contributed significantly to the understanding of scaffolding. Stone and Wertsch (1984) have examined scaffolding in a one-to-one remedial setting with a learning-disabled child. They could show how adult language directs the child to strategically monitor actions. Their analyses articulated the temporary nature of the scaffold provided by the adult. Close observation of communicative patterns in the adult–child interactions showed a transition and progression in the source of strategic responsibility from adult (or other-regulated) actions to child (self-regulated) actions. The gradual reduction of the scaffolding ("fading") is possible through the child's interiorization of the external support system (transforming it into "self-help").

Stone (1993) made a critical analysis of the use of the scaffolding concept as a purely instrumental teaching strategy. He pointed out that until the early 1990s most conceptions of scaffolding were missing an important Vygotskian dimension that has to do with the finality of scaffolding for the learner. Especially the learner's understanding of how the scaffolding and learning make sense beyond the narrow achievement of a specific goal adds personal sense to the cultural meaning of the actions to be learned through scaffolding. Stone refers to this dimension with the linguistic notion of "prolepsis" which can be seen here as an understanding in the learner of the value of the scaffolded actions in a future activity context. Until today many applications of the scaffolding strategy are still missing this proleptic dimension and neglect the process of personal sense attachment to the scaffolded actions.

The use of scaffolding in various contexts has led to different educative strategies for implementing scaffolding in classrooms with varying levels of explicitness as to the help given (for an excellent, recent, and very informative overview and empirical testing of scaffolding strategies, see van de Pol 2012). The most used scaffolding strategies with increasingly specific help are *modeling* (showing the task performance), *giving advice* (providing learners with suggestions that might help them to improve their performance), and providing *coaching* in the accomplishment of specific actions (giving tailored instructions for correct performance). Following Stone's critique on current scaffolding conceptions, however, it is reasonable to add, as a useful educative strategy, *embedding*, which entails luring the learner in familiar sociocultural practices in which the new knowledge, actions, operations, and strategies to be learned are functional components for a full participation in that practice. This embedding in familiar sociocultural practices helps students to discover the sense of both these learning goals and the teacher's scaffolding.

Attempts at employing scaffolding strategies in mathematics education can be generated from the above summarized general theory of scaffolding, provided that it is clear what kind of mathematical learning educators try to promote. If the formation of mathematical proficiency is reduced to learning to perform mathematical operations rapidly and correctly, then scaffolding should include *embedding* to make clear how the mastery of these operations may help students to participate autonomously in future practices. The choice for *coaching* on these specific actions in order to take care that they are mastered in correct form may be an important way of scaffolding the learning by repetition and practicing. If, however, the focus is on learning mathematics for understanding and hence on developing the ability of concept-based communication and problem solving with mathematical tools, a broader range of scaffolding strategies is needed. First of all the strategy of *embedding* is important: helping students to connect the actions to be learned with a sociocultural practice that is recognizable and accessible for them. One may think of practices like being a member of the mathematical community, but most of the time this scaffolding strategy consists in embedding the mathematical problem solving process in cultural practices like industrial design (e.g., designing a tricycle for toddlers), or practicing a third-world shop in the upper grade of primary school, or enacting everyday life practices (going to the supermarket or calculating your taxes). In a process of collaborative problem solving (and exploratory talk, see Mercer 2000) under guidance of the teacher, the teacher has to take care of the contingency of the actions and solutions on all participants' prior understandings but also of tailoring the scaffolding to the varying needs of the students: *modeling* general solutions (if necessary, when the students have problems to find the direction of where to find the solution of the mathematical problem at hand), giving hints (i.e., *giving advice*, if necessary, when the group's problem solving seems to go astray), or even stepwise *coaching* the execution of complex new actions when these actions are important for the resolution of the problem but go beyond the actual level of the participants' competences. In this latter case it is important that the teacher sensitively monitors the contingency of the steps in the learning process in the students.

Scaffolding in mathematics education that aims at mathematical understanding is basically a language-based (discursive) process in which students are collectively guided to a shared solution of mathematical problems and learn how this contributes to their understanding of the mathematical concepts that are being employed. Although there is as yet a growing body of (evidence-based) arguments for this discursive approach to the development of mathematical thinking (see, e.g., Pimm 1995; Sfard 2008), a number of unresolved issues are still waiting for elaboration:

- How to reconcile dialogical agreements in a group of students with the extensive body of proofs and understandings in the wider professional mathematical community? How can a teacher scaffold the students' processes of becoming a valid and reliable mathematics user in a variety of cultural contexts?
- How to scaffold the emergence of mathematical thinking in young children that opens a broad and reliable basis for the development of rich and valid mathematical thinking? How can we meaningfully scaffold the process of learning to talk, informed by mathematical concepts? Although practical and theoretical

know-how is currently being expanded (see van Oers 2010; Fijma 2102), further ecologically valid empirical studies are needed.

- How can teachers scaffold the process of mastery of automatization in mathematics while maintaining the foundations of this process in understanding and meaningful learning?
- How can teachers support the gradual fading of the teacher's scaffolding and turn this external (interpersonal) scaffolding into a personal quality of self-scaffolding? For this it is necessary to encourage the students to make and discuss their own personal verbalizations of the shared concepts and solutions. More study is needed into this formation of personalized regulatory abilities on the basis of accepted mathematical understandings, using a combination of dialogue (interpersonal exploratory talk) and polylogue (critical discourse with the wider mathematical community).

## Cross-References

▶ Collaborative Learning in Mathematics Education
▶ Inquiry-Based Mathematics Education
▶ Zone of Proximal Development in Mathematics Education

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# Semiotics in Mathematics Education

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## Keywords

Signs; Semiosis; Semiotics; Mathematical objects; Semiotic representations; Communicating mathematically; Decontextualization; Contextualization; Signifier; Signified; De Saussure; Triads; Charles Sanders Peirce; Object; Representamen; Interpretant; Iconic; Indexical; Symbolic; Intensional interpretant; Effectual interpretant; Communicational Interpretant; Cominterpretant; Commens; Epistemological triangle; Semiotic bundles; Diagrammatic reasoning; Abduction; Onto-semiotic theoretical model; Semiotic mediation

## Definitions and Background

Because mathematical objects cannot be apprehended directly by the senses (e.g., Otte 2006), their ontological status requires *signs* such as symbols and diagrams for their communication and learning. A *sign* (from ancient Greek semeion, meaning sign) is described by Colapietro (1993) as "something that stands for something else" (p. 179). Then *semiosis* is "a term originally used by Charles S. Peirce to designate any sign action or sign process; in general, the activity of a sign" (p. 178). Semiotics is "the study or doctrine of signs; the systematic investigation of the nature, properties, and kinds of sign, especially when undertaken in a self-conscious way" (p. 179). Both Duval (2006) and Otte (2006) stressed that mathematical objects should not be confused with their semiotic representations, although these signs provide the only access to their abstract objects. Ernest (2006) suggested that there are three components of semiotic systems (clearly illustrated by the systems of mathematics), namely, a set of signs, a set of relationships between these signs, and a set of rules for sign production.

Semiotics is particularly suited to investigation of issues in mathematics teaching and learning because it has the capacity to account for both the general and the particular. Mathematicians and teachers employ different symbolic practices in their work, while sharing the goal of communicating mathematically: mathematicians aim for *decontextualization* in reporting their research whereas teachers recognize a need for *contextualization* in students' learning of mathematical concepts (Sáenz-Ludlow and Presmeg 2006). Semiosis is essential in both of these practices. Further, as Fried (2011) pointed out, tensions between public and private realms arise in a persistent way in discussions connected with semiotics in mathematics education, reflecting "the division between students' own inner and individual understandings of mathematical ideas and their functioning within a shared sociocultural world of mathematical meanings" (p. 389).

## Semiotic Lenses and Their Uses

Semiotics has been a fruitful theoretical lens used by researchers investigating diverse issues in mathematics education in recent decades, as attested by Discussion Groups held at conferences of the International Group for the Psychology of Mathematics Education (PME) in 2001, 2002, 2003, and 2004 (Sáenz-Ludlow and Presmeg 2006) and at conferences with a focus on semiotics in mathematics education (Radford et al. 2011). Some theoretical formulations are described briefly in this section, along with the mention of semiotic investigations in which these lenses have proved useful in mathematics education.

Ferdinand de Saussure, working in linguistics, put forward a dyadic model of semiosis in which a *signifier* (such as the word "tree") stands for a *signified* (the concept of tree). Note that in this example, both the word and its concept are mental constructs, not objects accessible to the senses. Saussure's model allows for a chaining of signifiers that was used in mathematics education research by Walkerdine (1988) and Presmeg (1998). The need to acknowledge the human subject involved in such semiosis led Presmeg to the triadic model of Peirce (1992, 1998) and to a nested chaining model that includes interpretation of signs (Hall 2000; Presmeg 2006). Charles Sanders Peirce used triads extensively in his model of semiosis. His main triad involved the components of *object*, *representamen* that stands for the object in some way, and *interpretant*, involving the meaning assigned to the object-representamen pair. An illustration used by Whitson (1997) is as follows: *object*, it will rain; *representamen*, the barometer is falling; and *interpretant*, take an umbrella. Peirce used the term *sign* sometimes to designate the representamen and at other times to refer to the whole triad. In any case, the model allows for a nested chaining that may be continued indefinitely, as each interpretant in turn (and implicitly thereby the whole triad) may become an object that is represented by a new representamen and interpreted. Sáenz-Ludlow (2006) used this chaining property to illustrate the meanings emerging in the *language games* of interactions in an elementary mathematics classroom, involving the translation of signs into new signs.

Each of the relationships comprised in the Peircean triad were analyzed by him into further triads, e.g., the relationship of the representamen to its object could be *iconic* (like a picture), *indexical* (pointing to it in some way, e.g., smoke to fire), or *symbolic* (a conventional relationship, e.g., the numerals to their corresponding natural numbers). This model also includes the need for expression or communication: "Expression is a kind of representation or signification. A sign is a third mediating between the mind addressed and the object represented" (Peirce 1992, p. 281). In an act of communication, then, there are three kinds of interpretant, as follows: the "*Intensional* Interpretant, which is a determination of the mind of the utterer"; the "*Effectual* Interpretant, which is a determination of the mind of the interpreter"; and the "*Communicational* Interpretant, or say the *Cominterpretant*, which is a determination of that mind into which the minds of utterer and interpreter have to be fused in order that any communication should take place" (Peirce 1998, p. 478). The latter fused mind Peirce designated the *commens*, a notion that is useful in interpreting developments in the history of mathematics through the centuries (Presmeg 2003). The numerous triads introduced by Peirce provide lenses for various larger or smaller grains of analysis in research in mathematics education (Bakker 2004; Hoffman 2006).

With regard to mathematical communication, a different theory is provided by the *social* semiotics of linguist Michael Halliday, as used in the research of Morgan (2006), who analyzed the mathematical texts produced by secondary school students. Halliday emphasized "the ways in which language functions in our construction and representation of our experience and of our social identities and relationships" (Morgan 2006, p. 219). A fine grain is provided in this theory by the differentiation of *context of situation*, involving various kinds of specific goals, and *context of culture*, involving organizing concepts that participants hold in common, and by his notions of *field* (institutional setting of an activity), tenor (relations among the participants), and *mode* (written and oral forms of communication).

An independent model is provided by Steinbring (2005), who took the position that mathematical signs have both semiotic and epistemological functions. With regard to a particular mathematical concept, he argued that there is a reciprocally supported and balanced system, which he called the *epistemological triangle*. The three reference points of this triangle are the mathematical sign/symbol, the object/reference context, and the mathematical concept. He provided extensive examples of interaction of learners in elementary mathematics classrooms (Steinbring 2005, 2006) to show that the meaning of signs for individual learners is part and parcel of the semiotic and epistemological functions inherent in sign interpretation.

Elaboration and combination of constructs from semiotic theories have been necessary in research addressing the complexity of elements involved in mathematics teaching and learning. For instance, Arzarello introduced the construct *semiotic bundles* and Arzarello and Sabena (2011) integrated Toulmin's *structural description of arguments*; Peirce's notions of *sign*, *diagrammatic reasoning*, and abduction; and Habermas's model for *rational behavior*. Several research studies have used the inclusive *onto-semiotic* theoretical model of Godino and colleagues (e.g., Santi 2011). There is also the important independent branch of semiosis known as *semiotic mediation*, based on the theoretical formulations of Vygotsky, and used extensively in research by Mariotti (e.g., Falcade et al. 2007) and Bartolini Bussi (e.g., Maschietto and Bartolini Bussi 2009). Hoffman (2006) does not consider this variety of theoretical formulations of semiosis to be a problem, as long as the terminology is consistently defined and used in each instance. The various research questions being investigated demand different tools and lenses, according to the various semiotic traditions.

## Questions for Research on Semiosis in Learning and Teaching Mathematics

Following the publication of papers from two PME discussion groups in 2001 and 2002, Sáenz-Ludlow and Presmeg (2006) identified semiotic "windows through which to explain the teaching-learning activity while opening the gates for new avenues of research in mathematics education" (p. 9) by addressing questions such as the following:

- What exactly is entailed in the interpretation of signs? Are signs things and/or processes? When are signs interpreted as things and when are signs interpreted as processes by the learner?
- What is the role of speech and social interaction in the interpretation of signs? What is the role of writing in this interpretation?

- Are there different levels of sign interpretation? Do interpretations and the level of interpretations change with respect to different contexts? What is the role of different contexts in sign interpretation?
- Is it important for the teacher and the student to differentiate the variety of semiotic systems involved in the teaching-learning activity?
- Is there a dialectical relationship between sign use and sign interpretation? Is there a dialectical relationship between sign interpretation and thinking?
- Is it possible to involve students in creative acts of sign invention and sign combination to encapsulate the oral or written expression of their conceptualizations?
- Under what conditions do students attain the ability to express themselves flexibly in the conventional semiotic systems of mathematics?
- Can various semiotic theories be applied to analyze data gathered using different methodologies?
- Would it be possible to have a unified semiotic framework in mathematics education?

The latter remains an open question. However, some of the potential light thrown by using semiotic lenses in mathematics education research has been demonstrated in investigations already undertaken.

## Cross-References


▶ Argumentation in Mathematics Education
▶ Concept Development in Mathematics Education
▶ Discursive Approaches to Learning Mathematics
▶ Mathematical Language
▶ Mathematical Representations
▶ Theories of Learning Mathematics


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## Shape and Space – Geometry Teaching and Learning

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### Keywords

Shape's critical attributes; Euclidean geometry; Intuitive to formal; Visualization; Mathematization of the reality; Deduction; Concept definition; Concept image; Child's representational space; Internalization; Justification; Prototypical example; Prototypical judgment; Dragging operation; Uncertainty conditions

### Definition and Teaching Situation

**Geometry** (Ancient Greek: *γεωμετρία*; *geo* "earth," *metron* "measurement") is a mathematical area concerned with the space around us, with the shapes in the space, their properties, and different "patterns" and "thinking patterns" for which they serve as trigger and basis. As Freudenthal (1973) states it: "Geometry can only be meaningful if it exploits the relation of geometry to the experienced space... Geometry is one of the best opportunities that exist to learn how to mathematise reality" (p. 407).

From its very beginning, more than two and a half thousand years ago, geometry was developed along a few main aspects:

(a) Interacting with shapes in a space. This aspect arose independently in a number of early cultures as a body of practical knowledge concerning *lengths, areas*, and *volumes* and concerning *shapes' attributes* and the *relationships* among them (the practical-intuitive aspect).
(b) Shapes, their attributes, and their changes in space as fundamental ingredients for *constructing a theory* (the formal logic approach). Elements of a formal mathematical geometry emerged in the west as early as Thales (sixth century BC). By the third century BC, this aspect of geometry was put into *an axiomatic structure* by Euclid (*Euclidean geometry*).
(c) Shapes as basis for reflecting on *visual* information by representing, describing, generalizing, communicating, and documenting such information, e.g., for better understanding concepts, processes, and phenomena in different areas of mathematics and science and as a framework for realizing the contribution of mathematics to domains such as painting, sculpture, and architecture in which beauty can be generated through aesthetic configurations of geometrical shapes.

There is a classic "consensus that the first two aspects are linked because some levels of geometry as the science of space are needed for learning geometry as a logical structure" (Hershkowitz et al. 1990, p. 70). These two aspects seemed to be expressed explicitly in teaching and learning geometry in schools and in the research work which follows it. For quite many years the most acceptable way to teach geometry in K-12 was and in a sense still is hierarchical division of the themes and teaching approaches from intuitive (Aspect a) to formal (Aspect b) along the school's years, where the intuitive-interactive approach was the basis for elementary and preschool geometry and the formal one was left to high school. Seldom, the formal approach was also used for designing a learning environment for high school and/or universities in which learners developed an understanding of geometrical structures as abstract systems not necessarily linked to referents of a real environment, e.g., the non-Euclidian geometries.

Approaches towards the role of visualization in learning and teaching geometry and mathematics as a whole (Aspect c) varied according to the observers' eyes and interest. But, as geometry engaged with shapes in space, which are seen, presented, and documented visually, the role of visualization can't be ignored. Two aspects of visualization which are interweaved together are relevant to teaching and learning geometry:

(a) Visualization as one of the ways for mathematical thinking
(b) Visualization as a representation or as "a language" by which mathematical thinking, including a visual one, might be developed, limited, expressed, and communicated (Presmeg 2006)

Visual constructs are considered as a potential support for learning other mathematical constructs, but what about geometrical constructs? Visualization seems to be the entrance into geometry, the first internalization steps of the learner while she/he begins to mathematize the reality into geometrical constructs. There were quite many research works which were involved with visualization, but not very many that tried to investigate to what extent geometrical thinking is visual, or is interweaved with visual thinking, or affected by visual thinking? For example, when the learner is engaged in deductive proving, what is the effect or the role of visual thinking if any? Or, the opposite, when students are engaged in a visual problem solving, what *semiotic support* they need and may have for expressing their problem solving process and products? This third aspect concerning the role of visualization is the most neglected one, either because of the lack of awareness or because of the naïve assumption that the visual abilities and understanding are developed in a natural way and the learners do not need a special teaching.

The teaching and learning of geometry in preschool and elementary school was neglected in many places around the world: For example, at the preface to their book concerning "learning environments for developing understanding of geometry and space," Lehrer and Chazan (1998) writes: " . . .geometry and spatial visualization in school are often compressed into a caricature of Greek geometry, generally reserved for the second year of high school." Indeed in many states in the USA, this 1-year-course in Euclid geometry was taught in high school without any geometry's instruction before it.

This unfortunate situation was discussed intensively in the last few decades and as a result instructional and research efforts are done in order to improve it. For example, in the US NCTM curriculum standards, it is claimed that "the study of geometry in grades 5–8 links the informal explorations, begun in K-4, to the more formalized processes studied in grades 9-12" (NCTM 1989, p. 112). This intentional claim (which unfortunately does not mention the visual aspect) is strengthened by the hierarchical levels' structure of van Hiele's theory (1958), which is discussed in the next section.

## Theories Concerning Geometry Teaching and Learning

***Piaget***: In his developmental theories of the child's conception of space (Piaget and Inhalder 1967) and child's conception of geometry (Piaget et al. 1960), Piaget and his colleagues describe the development of the *child's representational space*. This is defined as the mental image of the real space in which the child is acting, where mental representation is an active reconstruction of an object at the symbolic level. Piaget in his typical way was interested in the mental transformations from the real space to the child's representational space and in those attributes of real objects that are invariant under these transformations and how they develop with age. This approach is a trigger to some sub-theories. For example, the distinction between the concept and the concept image. The concept is derived from its mathematical definition and the *concept image* which is the collection of the – mental images the student has concerning the concept, or the concept as it is reflected in the individual mind (Vinner 1983).

This dichotomy served as a basis for many research works in mathematics.

***van Hiele***: Whereas Piagetian theory relates mainly to geometry as the science of space, van Hiele's theory combines geometry as the science of space and geometry as a tool with which to demonstrate mathematical structure. The theory identifies a sequence of levels of geometrical thought from *recognition* and *visualization* up to rigor (for details on the theory as a whole and on the levels in particular, see Van-Hiele and van-Hiele-Geldof 1958; Hershkowitz et al. 1990). The most relevant feature for geometry instruction and learning is van Hiele's claim that the development of the individual's geometrical thinking, from one level to the next, is due to teaching and learning experiences and does not depend much on maturity.

## Geometry in Preschool and Elementary School

The aspect of interaction with shape and space (a) is the main component in elementary and preschool geometry learning in which the classic main goals are constructing knowledge about basic Euclidean geometric figures and simple relationships among them. Around the middle of the twentieth century, it was common to find a "pre-formal" course for elementary school, conceptualizing geometry as the science of space. The focus of this course was mostly on the identification and drawing of the regular shapes, Euclidean properties of these shapes, relationships among shapes, and a variety of measurement activities. Since the 1960s this type of course has come under severe criticism: mainly because it lacks inductive activities related to search of patterns, because there is no implicit and explicit focus on geometrical argumentation, and above all because the learners are passive in constructing their geometrical knowledge. It is worth to note that teaching visual skills and visual thinking which is highly recommended (c) is still very limited.

***Freedom in Selecting Geometrical Context, Content, and Teaching/Learning Paradigms***: As a result from the above criticism, we are witnesses in the last decades to a trend of refreshing projects in teaching and learning geometry as a whole, but mostly at the preschool and elementary school. These projects, which express democracy in choosing contexts, and approaches towards teaching and learning geometry, emerged from holistic vision of what shape and space could be, rather of what they often are in schools (see the RME entry in this encyclopedia). The book edited by Lehrer and Chazan (1998) is a paradigmatic example for this trend. The book describes a variety of attractive and productive environments for learning about space and geometry. In most of the designed learning environments, described in the book, students play active role in constructing their own geometrical knowledge. The designers'/researchers' description of student's learning shares a collective emphasis on *internalization, mathematization*, and *justification* (Hershkowitz 1998). *Internalization* is used in a Vygotskian spirit, as the transformation of external activity into internal activity, e.g., the change from "what I see?" to "how I see?" in accordance with the change of the observer's position in the RME curriculum (Gravemeijer 1998) and the dragging mode in dynamic geometry projects. M*athematization* is consistent with Freudenthal's philosophy of mathematics as *human activity* in which mathematizing is seen as a sort of an organizing process by which elements of a context are transformed into mathematical objects and relations. *Justification* is taken in a broad sense, meaning the variety of actions that students take in order to explain to others, as well as to themselves, what they see, do, think, and why. This broad sense is expressed in the mathematical and cognitive freedom towards legitimate kinds of justifications.

***Some Comments on Difficulties and Relevant Research***: Research has shown that common difficulties of learning geometry at the elementary level emerged mostly from the unique mathematical structures, in which figures are represented in learning geometry: From a mathematical point of view a geometrical concept, like other mathematical concepts, is derived from its definition which includes a minimal (necessary and sufficient) set of the concept's critical attributes that an instance should have in order to be the concept's example. Hence these critical attributes may be used by students as a criterion to classify instances. In contrast, students very often use one special example of the concept, the prototype/s as a criterion for classifying other examples. The prototypes are attained first and therefore are found in the concepts' image of quite young learners. The prototypical example has the "longest" list of critical attributes (Rosch and Mervis 1975), e.g., the squares are prototypical example of quadrilaterals, and indeed they have all the quadrilaterals' critical attributes plus their own critical attributes, like the sides' equality. This leads to a *prototypical judgment* by learners and to a creation of biased *concept image*s, like identifying a segment as a triangle altitude, only if it is an interior segment (Hershkowitz et al. 1990; Fujita and Jones 2007). The prototypes' phenomenon is understood better if we analyze it in the context of the typical structure of basic geometrical figures, *"the opposing directions inclusion relationships"* (Hershkowitz et al. 1990), among the sets of figures (concepts) at one direction and among their critical attributes at the opposite direction. This structure explains also other obstacles in learning geometry: e.g., the *figure-drawing obstacle* (Laborde 1993, p. 49), in which learning difficulty emerges in situations where an isolated drawing is the only representative of *a figure*, where the *figure* is the geometrical concept as a whole. Laborde made it clear that there is always a gap between the figure and a drawing which represents it, because (1) some properties of the drawing are irrelevant (non-critical attributes of the figure) (it becomes an obstacle when students try to impose these attributes as critical attributes on all figures' examples), (2) the elements of the figure have a variability which is absent in a single drawing, and (3) a single drawing may represent various figures (Yerushalmi and Chazan 1990). Dynamic geometry softwares enable students to overcome the abovementioned difficulties and more. By dragging elements of a drawing which was constructed geometrically on the computer screen, students may provide an infinite set of drawings of the same figure. This *variable method* of displaying a geometrical entity stresses the critical attributes, which become the invariants of the entity under dragging. Research indicates that students engaged in dynamic geometry tasks are able to capitalize on the ambiguity of drawings in the learning of geometrical concepts.

## High-School Geometry or Shapes in Space as Ingredients for Constructing a Theory

The two classical roles of teaching high-school geometry are still experiencing deductive reasoning and proofs as part of human culture and human thinking and verifying the universality of proved geometric statements. According to extreme classical approach, experimenting, visualizing, measuring, inductive reasoning, and checking examples are not counted as valid arguments and might be that this is the reason for neglecting them both in the elementary school and in the high-school level. Geometrical proofs are considered to be on a high level of the argumentative thinking continuum at school, and the traditional high-school geometry is the essence of the secondary school geometry in many places. It starts from what can be seen with the eyes, where space and shapes provide the environment, in which the learner gets the feeling of mathematical theory (Freudenthal 1973). At more advanced stage, it acquires a more abstract aspect. But, even in the most abstract stage, we still deal with some sorts of shapes and spaces, even when they can be seen with the "minds' eye" only. Nevertheless the trends of freedom towards the meaning of justifications and towards paradigms of teaching and learning mathematics as a whole and geometry in particular are taking their way into the traditional high-school course in geometry. Euclidian geometry is no longer discussed in terms of "Euclid must stay" or "must go," as if it is the only representative for a proper argumentation on the stage. This trend is accelerated due to several reasons:

i. ***The difficulty of teaching proving tasks in school and of understanding the role of proof***: The teaching of mathematical proof appears to be a failure in almost all countries (Balacheff 1991; Mariotti 2006). Moreover, students rarely see the point of proving. Balacheff (1991) claims that if students do not engage in proving processes, it is not so much because they are not able to do so, but rather they do not see any reason for it (p. 180). High-school students, even in advanced mathematics and science classes, don't realize that a formal proof confers *universal validity* to a statement. A large percentage of students states that checking more examples is desirable (Fischbein and Kedem 1982). Many do not distinguish between evidence and deductive proof as a way of knowing that a geometrical statement is true. After a full course of deductive geometry, most students don't see the point of using deductive reasoning in geometrical constructions and remain still naive empiricists whose approach to constructions is an empirical guess-and-test loop (Schoenfeld 1986).

ii. ***New thinking trends concerning the goals of teaching proofs***: For mathematicians, proofs play an essential role in establishing the validity of a statement and in enlightening its meaning. In the last decades more and more scholars claim that the situation in school is different: Hanna (1990) suggested distinguishing, in school geometry, between (a) proofs that only show that the theorem is true and (b) proofs that in addition explain and convince *why* the theorem is true. Dreyfus and Hadas (1996) showed that when a learning situation is provided, in which *students feel the need in proof* in order to be convinced and convince others (e.g., the need to show the existence of hypothesis), students search for a proper proof and then a proof becomes a meaningful mathematical tool for checking hypothesis. In these ways the importance of proof is focused in the level of its justifications and understanding and less in the formal classic way in which it presented.

iii. ***Dynamic geometry environments and proving***: The design of dynamic geometry learning environments raised a question about the place of the classical proof in the curriculum, since conviction can be obtained quickly and relatively easily: The dragging operation on a geometrical object enables students to apprehend a whole class of objects in which the conjectured attribute is invariant and hence to be convinced of its truth. The role of proof is then to provide the means to state the conjecture as a theorem. Dreyfus and Hadas (1996) argue that students' appreciation of the roles of proof can be achieved by activities in which the empirical investigations lead to unexpected, surprising situations. This surprise is the trigger for the question *why* and for the proof as an answer to this question.

## Cross-References


- ▶ Argumentation in Mathematics Education
- ▶ Concept Development in Mathematics Education
- ▶ Deductive Reasoning in Mathematics Education
- ▶ Logic in Mathematics Education
- ▶ Mathematical Proof, Argumentation, and Reasoning
- ▶ Realistic Mathematics Education
- ▶ The van Hiele Theory
- ▶ Visualization and Learning in Mathematics Education


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## Single-Sex Mathematics Classrooms

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### Keywords

Single-sex classroom; Single-sex school; Single-gender classroom; Single-gender school

### Introduction

In recent years it has become popular to replace the word "sex" in single-sex classrooms and single-sex schools with the word "gender." This substitution warrants some attention for "(p)recision is essential in scientific writing" (American Psychological Association [APA] 2010, p. 71).

The APA (2010) advocates the use of the term "*Gender* . . .when referring to women and men as social groups. *Sex* is biological; use it when the biological distinction is predominant" (p. 71). A similar distinction is made by the World Health Organization (WHO) (2012): "'Sex' refers to the biological and physiological characteristics that define men and women. 'Gender' . . . to the socially constructed roles, behaviours, activities, and attributes that a given society considers appropriate for men and women." Given that the division of students into same-sex classrooms is invariably based on biological characteristics, it is appropriate to retain the term sex in the heading of this contribution. However, explanations for differences associated with single-sex groupings are commonly linked to social expectations, perceptions, and conventions, that is, on gender-linked differences, as defined by APA (2010) and WHO (2012). For a more detailed discussion, see Leder (1992).

Historically, more emphasis has been placed on the education of boys than of girls and originally all-boys schools predominated. Over time, with increased expectations and demands for mass education, coeducational schools were added in many countries. In some countries, religious convictions have been, and are still, responsible for sex-segregated education – for example, in the predominantly Muslim countries such as Bahrain, Iran, and Saudi Arabia. In many others, economic and political considerations, as well as the increased importance attached to the education of girls, have led to the growth and ultimately dominance of coeducational schools. In contrast, in the United States of America with its strong history of coeducational schools, there appears to have been a revival of single-sex schooling, fuelled by legislative changes. This development is hotly and continuously deplored and contested by many inside and outside educational circles (see, e.g., Brown 2011; Halpern et al. (2011, 2012)).

## Characteristics

### Mathematics Classes in Single-Sex Schools

Unlike the often short life span of single-sex mathematics classes in coeducational schools, the single-sex grouping is maintained throughout the school life of students in single-sex schools. The mathematics performance and participation rates of boys attending single-sex schools have attracted some attention, but, as noted in a report by the US Department of Education (2005, p. xv), "males continue to be underrepresented in this realm of research." Issues related to girls' learning of mathematics have been a major focus of research comparing benefits of single-sex and coeducational schools. Findings reported some two decades ago by Leder (1992) that, when differences are found, they most frequently favor those in the single-sex school setting continue to be replicated. Referring to England and Wales, Thompson and Ungerleider (2004, p. 4) pointed to the increased publicity given to school examination results which publicize the consistent and superior achievements of students graduating from single-sex private and independent schools, with many of the highest scores coming from all-girls schools. In addition to a possible solution for the achievement gap, single-sex schooling is viewed in some jurisdictions as a means of balancing enrolments in subject areas within the coeducational public system in which there have been extreme imbalances between boys and girls.

Interpreting the finding of girls' better performance in mathematics in single-sex schools is problematic, however. When other factors are taken into account and in particular the fact that single-sex schools are often independent, private schools which attract students from higher socioeconomic families, it is clear that any advantages noted cannot be attributed simplistically to the single-sex composition of the school. Data from large-scale, international mathematics tests such as TIMSS and PISA illustrate unambiguously that students' socioeconomic background is an important variable influencing their performance in mathematics: in general, the higher the level of the socioeconomic background of students, the higher their performance on the mathematics component of these tests. Factors beyond students' background and system-related differences in human and physical resources have also been shown to contribute to different achievement outcomes. In summary, any apparent achievement advantages found in mathematics learning for girls attending a single-sex school cannot be attributed simplistically to one particular school characteristic, that is, the single-sex setting per se.

Longitudinal studies of the long-term impact of single-sex schooling are rare. Data from the National Child Development Study "a longitudinal study of a single cohort born in a particular week in 1958 in Britain" (p. 314) offers one such source (Sullivan et al. 2011). This group attended secondary school in the 1970s, at a time when about one-quarter of the cohort went to single-sex schools – a much higher proportion than is now the case. Sullivan et al. (2011, p. 311) report:

We find no net impact of single-sex schooling on the chances of being employed in 2000, nor on the horizontal or social class segregation of mid-life occupations. But we do find a positive premium (5 %) on the wages of women (but not men), of having attended a single-sex school. This was accounted for by the relatively good performance of girls-only school students in post-16 qualifications (including mathematics).

### Mathematics Classes in Sex-Segregated Classes in Coeducational Schools

In many countries, systematic documentation of differences in the mathematics achievement of boys and girls began in the early 1970s. Given the important gatekeeping role or critical filter played by mathematics into further educational and career opportunities, differences between the two groups, in favor of boys, in continued participation in advanced and post-compulsory mathematics courses were also noted with concern. The introduction of single-sex classes in coeducational schools, mostly aimed at secondary school students and not necessarily exclusively in mathematics, was among the initiatives mounted to redress the demonstrated achievement discrepancies. The move was considered to be consistent with the tenets of liberal feminism, that is, helping females attain achievements equal to those of males, and the apparent advantages for girls associated with the learning of mathematics in single-sex schools.

The findings reported from single-sex mathematics classes in formally coeducational schools are largely similar to those described for other subject areas. Girls typically liked the single-sex setting and performed somewhat better academically than in coeducational classes. In a number of the studies surveyed, boys were more ambivalent than girls about the single-sex setting with some indicating a firm preference for coeducational classes. These differences, however, could often be attributed to differences in student background factors or school organizational structures rather than the sex-segregated setting per se (Forgasz and Leder 2011).

In most of the studies located, the focus was on the shorter-term effects of the single-sex grouping. In the few studies in which longer-term effects were examined, earlier advantages attributed to the single-sex grouping appeared to dissipate: "The generally accepted view has been that for females, single-sex schooling is more advantageous" (OECD 2009, p. 44). Yet nuanced explorations of PISA data do "not uniformly support the notion that females tend to do better in a single-sex environment" (OECD 2009, p. 45).

Consistent explanations for the equivocal findings permeate the relevant scholarly literature: certain groups of students (e.g., those being harassed in a coeducational setting) were found to benefit from a single-sex environment, while for other groups it made no difference. Teacher strategies, instructional materials, and the prevailing school climate, rather than the sex grouping in the mathematics class, were more often found to be critical to students' success and perceptions of the class environment. Simplified and at times biased versions of these findings are regularly reported in the popular media and play a part in shaping the perceptions of the public and of stakeholders about the respective benefits of single-sex and coeducation schooling (Forgasz and Leder 2011).

To conclude, many complex and interacting factors influence the school learning environment – with a single-sex classroom setting per se unlikely to be the most influential. Some contexts, including the primary years of schooling and the longer-term effect of learning mathematics in a single-sex rather than a coeducational setting, have not yet received sufficient attention. For the present, proponents of single-sex education will focus on its putative benefits and critics on its disadvantages.

## Cross-References

▶ Cultural Influences in Mathematics Education
▶ Equity and Access in Mathematics Education

▶ Gender in Mathematics Education
▶ Mathematical Ability
▶ Socioeconomic Class in Mathematics Education

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# Situated Cognition in Mathematics Education

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## Keywords

Context; Knowing/knowledge; Learning; Participation; Situation; Transfer

## Introduction

"Situated cognition" is a loose term for a variety of approaches, in education and in other fields of inquiry, that value context. Its advocates claim that how one thinks is tied to a situation. "Situation" is another loose term; it may refer to a place (a classroom or a laboratory), but a situation may also reside in relationships with people and/or artifacts, e.g., "I am with friends" and "I am at my computer." This entry briefly considers the history of situated approaches before looking at the development of situated schools of thought in mathematics education. It then considers "knowing" and, briefly, research methodologies, implication for teaching, and critiques of situated cognition.

## Characteristics

### History

Marx's 11th thesis on Feuerbach, "Social life is essentially *practical*. All mysteries . . . find their rational solution in human practice and in the comprehension of this practice." (Marx 1845/1968, p. 30), remains a statement that few, if any, situated cognitivists would disagree with. Activity theory is an explicitly Marxist approach used by some mathematics education researchers which could be called "situated"; there are similarities and differences between these two approaches, e.g., "mediation" is central to activity theory but it is not a well-developed construct in current approaches labelled as "situated cognition" (see Kanes and Lerman 2007).

Cole (1996) claims that psychology once had two parts, one that could and one that could not be studied in laboratory experiments. He argues that the second part was lost in most of the twentieth-century psychology. One reason for this loss is that Western social sciences in the first half of the twentieth century were dominated by various forms of positivism, such as behaviorism in psychology (with a knock-on effect in education). Positivism is a form of empiricism which posits that we can obtain objective knowledge, a claim that is anathema to situated cognitivists. To a behaviorist, learning concerns conditioning, responses to stimuli, and attaching responses to environmental stimuli. From the 1950s onwards JJ and EJ Gibson, in the psychology of perception, argued differently that perceptual learning was a part of an agent's interaction with the environment; environments afford animals some actions/activities and constrain others – a chalkboard affords the construction of static geometric figures but an electronic whiteboard may afford the construction of dynamic geometric figures. This can be viewed as a form of situated cognition (where the situation is the environment) which has influenced some research in mathematics education; see Greeno (1994) for a consideration of affordances with reference to "situation theory" and mathematical reasoning. The waning of behaviorism as an academic paradigm in the West, circa 1970, however, did not immediately usher Cole's second psychology. In the place of behaviorism, mathematics education researchers largely embraced cognitive models of learning such as Piaget's genetic epistemology and information processing, both of which were content to capture data in laboratory conditions; note that this comment is not necessarily a criticism of these models per se but, rather, from a situated viewpoint, a comment on keeping research on learning within Cole's "first psychology."

## Mathematics Education

In mathematics education, "situated cognition" is often associated with studies of out-of-school mathematics towards the end of the twentieth century, Lave (1988) and Nunes et al. (1993) being early and influential examples of such studies. These studies presented data that people could do mathematics "better" in supermarkets or on the streets and argued that the mathematical processes carried out in out-of-school activities were radically different from those of school mathematics. These studies directly challenged the rationalist hegemony of academic (Western) mathematics and argued that a strong discontinuity exists between school and out-of-school mathematical practices.

According to Lave (1988), this discontinuity is a consequence of the fact that learning in and learning out of school are different social practices. School mathematics is, indeed, often ill-suited to out-of-school practices; in some cases the problems which arise in out-of-school mathematics are only apparently similar to school mathematics problems, but in reality there is a range of explicit and implicit restrictions which makes school methods unsuitable, and thus other methods are used (Masingila et al. 1996). Despite the evident discontinuity, some authors who do value context (situation) have observed an interplay between school and out-of-school mathematics: Saxe (1991) found evidence that school mathematics and the mathematics of street children's candy-selling practice in Brazil influence each other; Magajna and Monaghan (2003) found evidence that, in making sense of their practice, CAD-CAM technicians resorted to a form of school mathematics.

## Knowing

There are many constructs associated with situated cognition: community of practice (CoP), (legitimate peripheral) participation, boundaries, reification, and identity. This entry does not have space to consider these separately but they are all tied up with a central theme of knowing.

The verb "knowing" rather than the noun "knowledge" is the subheading of this section because situated cognitivists view this thing (knowledge/knowing) as something which results from doing (participation) rather than a passively acquired entity. Rather like the Gibsons' affordances, knowing is not an absolute attribute but a relative product of animal-environment interaction. The "person" in these interactions is equally not a laboratory subject but a whole person with goals and views of themselves in relation to the CoP in which they participate (hence the relevance of "identity"); such views have obvious relevance for mathematics education studies of classroom behaviors in terms of students' self-conceptions as "a good student," "cool," etc. But, as Kanes and Lerman (2007) point out, there are different nuances on "learning" within the situated cognition camp: a view that learning is a process that may or may not result from being a member of a CoP and a view that learning is subordinate to social processes, "learning is an integral part of generative social practice in the lived-in world" (Lave and Wenger 1991, p. 35).

Situated cognitivists views on knowing emerged partly in exasperation with dominant cognitive (non-situated) positions on knowledge:

> the effect on cognitive research of "locating" problems in "knowledge domains" has been to separate the study of problem solving from analysis of the situations in which it occurs ... "knowledge domain" is a socially constructed *exoticum*, that is, it lies at the intersection of the myth of decontextualized understanding and professional/academic specializations. (Lave 1988, p. 42)

Contrasts between situated and cognitive views on knowing/knowledge have important implication for the construct "transfer of knowledge" (or "transfer" for short), which is arguably the *philosopher's stone* of mathematics education research. To be fair to all, there are few serious researchers around of any persuasion who do not regard "transfer" as a highly problematic construct. Nevertheless, "transfer" (under the right conditions, which usually means "knowledge required in a new task is basically the same as knowledge acquired in a previous task") is a legitimate object of study for purely cognitive psychologists. It is a myth to radical expositions of situated cognition such as Lave (1988). Viewing transfer as a myth can be quite upsetting for practical mathematics educators who might well turn to their academic-situated colleagues and say "what, then, is the point in teaching?" Engle (2006), however, presents a situated view of transfer as "framing" – a means of interpreting phenomena. Engle (2006) examines a long-term science learning and teaching sequence with regard to learner construction of content and teacher framing of the contexts of learning in terms of time, "making references to both past contexts and imagined future ones . . .[to] make it clear to students that they are not just getting current tasks done, but are preparing for future learning" (456), and forms of learner participation. This view of transfer is far removed from the (non-situated) cognitivist view of "transfer of knowledge" and has potential for mathematics education, e.g., the framing of tool use in mathematics learning to promote intercontexuality.

## Research Methodologies

There is no research method specifically associated with situated cognition although methods used will be primarily qualitative and, possibly, mixed methods; it is hard to imagine how one might research being and knowing in mathematicized situations using only quantitative methods. Qualitative methods used in "situated research" hopefully suit the focus of the research. For example, it was noted above that Kanes and Lerman (2007) point out different nuances on "learning" within situated research and a focus on learning from being a member of a CoP may call for discourse analysis, and a focus on learning as an integral part of generative social practice in the lived-in world may call for ethnographic approaches.

## Implication for Teaching

Situated cognition is an approach to understanding knowing and does not prescribe a teaching approach. That said, reflection on situated cognition can be useful for teachers and teacher educators to critique their thinking about learning and teaching, as Winbourne and Watson (1998) do. They recognize the problems of students' school mathematical experience en bloc of providing a site for students to participate in a community of mathematicians but provide examples of lessons which could (and could not) be termed "local communities of (mathematical) practice" (p. 95), where the teachers "orchestrated" student participation so that student and teacher engagement with mathematics, rather than simple student behavioral compliance, was essential for the activity in the lessons.

### Critiques of Situated Cognition

There is no shortage of critiques since situated cognition has courted controversy since the publication of Lave (1988). These include "situated friendly" critiques such as Walkerdine (1997) which suggests that the regulation of individuals in discursive practice is not developed in Lave's work; attacks on the basic claims of situated cognition, such as Anderson et al. (1996); and questioning the existence of claims that a strong discontinuity exists between school and out-of-school mathematical practice (Greiffenhagen and Sharrock 2008).

## Cross-References


- ▶ Activity Theory in Mathematics Education
- ▶ Communities of Practice in Mathematics Education
- ▶ Discursive Approaches to Learning Mathematics
- ▶ Mathematization as Social Process
- ▶ Theories of Learning Mathematics


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# Socioeconomic Class in Mathematics Education


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## Keywords

Social class; Achievement; Equity; Assessment

## Definition

Research on the relationship between social class and socioeconomic status and achievement in mathematics.

## Overview

In many countries around the world, a correlation is found between working class or low socioeconomic status (SES) and achievement in mathematics. Secada (1992) traces recognition of the connection between social class, race, ethnicity, and other characteristics, with achievement in education to the United States Supreme Court case of Brown versus Board of Education in 1954. The major work on achievement in mathematics and social characteristics was begun in the mid-to late 1980s and has been a focus of a growing body of work ever since, with new theoretical perspectives developing (Valero and Zevenbergen 2004) and with new forums for dissemination and publication (e.g., the Political Dimensions of Mathematics Education (PDME) conferences of the early 1990s, followed by the conferences of the Mathematics Education and Society (MES) group). Here, the focus will be on social class and socioeconomic status and achievement in mathematics. Ethnicity and race and gender in mathematics education are separate entries in the encyclopedia.

Research in this area will be addressed in the following sections: statistical evidence, sociopolitical analyses, explanatory and analytical frameworks, and research on action or intervention.

### Statistical Evidence

Secada (1992) provides a thorough and structured analysis of data on achievement in the North American context. While it is clear that such analyses must be localized to be of use to researchers, educators, and policy makers, some of his concluding remarks are as relevant more than 20 years on. He argues that too often such analyses are carried out by researchers who are not in the mainstream of mathematics education research and thus do not impact sufficiently on mathematics education researchers, that social categories of students are unquestioned, and that the labelling masks the fact that the populations with lowest achievement are the poor and ethnic minorities.

Given the timing of the handbook in which Secada's chapter appears, it is to be expected that he would have high hopes for the reform agenda in the USA in terms of equity. Lubienski (2000) found, however, that there is evidence that disadvantaged students taught through the reform pedagogy are still underachieving in national tests. She draws on sociological theory to explain why the middle classes succeed whatever reforms take place, though that work has been challenged (Boaler 2003) on the basis of a study of forms of pedagogy that can claim to have been successful in equity terms. Nevertheless, as a general trend, such reproduction of advantage and disadvantage needs explanation; this is addressed below.

### Sociopolitical Analyses

Freire's Marxist approach can perhaps be seen as the earliest inspiration to researchers in mathematics education in relation to raising awareness of the idea that education is never neutral. His *Pedagogy of the Oppressed* (1970) contrasted a banking concept of education, identified with the oppressor, as against a critical pedagogy, with the goal of empowerment and emancipation of the oppressed. His work was based around several themes that have since been developed in the field. Freire, taking up the notion that knowledge is a social construct, raised the question of whose knowledge is to be valued; he argued for a constructivist view of learning, not one of "banking" inert knowledge; and that teaching is a political process and the teacher should work dialogically, learning from their students what matters to them in their lives.

Research on the everyday mathematics of indigenous, rural, and oppressed groups (Knijnik 2000) led to and was inspired by ethnomathematics (D'Ambrosio 1985), a sociopolitical theory which sees academic mathematics as just one of a range of mathematical systems used by people. D'Ambrosio (2010) argues that we remain unconscious of how academic mathematics, so dominant throughout the world, can be used for the good of society or for domination by the powerful of the powerless, the latter being the most common. Valuing academic, and therefore school mathematics, above all, marginalizes the lives and values of dominated groups.

Freire's constructivist view of learning, with elements from Vygotskian approaches too in his insistence on the unity of cognition and affect, emphasizes his rejection of the banking concept, in which knowledge does not relate to what matters to the lives of underprivileged students and is inert for all students. It particularly disadvantages the underprivileged. His critical pedagogic position has been taken up by many, including the *criticalmathematics* group (Frankenstein 1983). The concatenation of the two words signals the particular focus of researchers in that group, emphasizing the development of appropriate materials that challenge hegemonic views of the neutrality of mathematical knowledge and research studies of teaching and learning from this position. Activists such as Gutstein (2009) have also found Freire's work inspiring.

The dialogic view of teaching has been developed by Skovsmose (1994) in particular. Emphasizing democracy, a critique of the way that traditional/academic mathematics formats a view of the world, and the potential for equality that comes with a dialogic learning process, Skovsmose and his collaborators (e.g., Alrø and Skovsmose 2002) address the potential power of learning and teaching that engages with what matters in children's lives and with how they can change the world. Skovsmose's approach is often referred to as a critical mathematics position, with the two words separated.

Differences between the ethnomathematics and critical mathematics education positions were discussed by Vithal and Skovsmose (1997). Taking the case of South Africa as a context, though emphasizing the international implications, their detailed analysis of the potentialities of the two perspectives includes a concern for the empowerment of students when an ethnomathematics approach is taken, especially those disadvantaged by apartheid.

Critiques of the work described here come from the poststructuralist critique of critical theory (e.g., Ellsworth 1989; Walshaw 2004) which argues that empowerment is an enlightenment, universalist concept with no foundation other than ideology and from arguments that there is a confusion when attempting to harness everyday practices for the purposes of teaching mathematics in school, what Dowling (2001) calls the *myth of emancipation* (p. 32). This latter point is developed in the following section.

## Explanatory and Analytical Frameworks

While the statistical evidence confirms the correlation between low SES and low achievement in mathematics, and the sociopolitical perspectives argue forcefully for change, researchers need explanatory frameworks for why the correlation exists. It can be argued that without such analyses, any changes being made in pedagogy may come from principles and values but may not make any fundamental difference.

The sociological theories of Basil Bernstein and Pierre Bourdieu in particular, both Marxist sociologists, have been taken up by researchers in mathematics education to understand the causes of the correlation. The ideas of these sociologists of education have similarities and differences. From their Marxist origins, they both focus on consciousness as a product of social relations and in particular relations to the means of symbolic production.

Bourdieu (1977) introduced the notions of *habitus*, *cultural capital*, and *field*. In brief, the field provides the structuring practices which convey power and status. At the subjective level, habitus is the embodiment of culture, providing the lens through which the world is interpreted. The habitus of children from the middle classes may bring with it opportunity for power if it aligns with the expectations of the school. Thus, certain forms of culture endow the "possessor" with cultural capital that can be exchanged for gains that are valued, such as success in school mathematics. Zevenbergen (2001) provides clear description of the analytical tools Bourdieu's framework provides and uses it to analyze classroom interactions in a year-long ethnographic study. Gates (2004) uses Bourdieu's theories to examine teachers' beliefs from a sociological rather than cognitive perspective. In an analysis of mathematics achievement in the context of reforms and counterreforms of the curriculum in Victoria, Australia, Teese (2000) also employs Bourdieu's theories.

For Bernstein (e.g., 2000), language is an indicator of different relations to the means of symbolic production, children from working-class backgrounds exhibiting a restricted code and those from middle-class backgrounds exhibiting an elaborated code. Given that schools work in an elaborated code from the first day of children's participation in school, the manner in which schools reproduce advantage and disadvantage, the differential distribution of knowledge across social backgrounds becomes obvious. Key sociological concepts of those researching who succeeds and who fails in school mathematics, and why, include the nature of knowledge discourses, the distinction between the everyday and the "esoteric," and its effect on students (Dowling 1998; Cooper and Dunne 2000); the official and unofficial fields of pedagogic knowledge and how they are taken up and by whom (Morgan et al. 2002); the distinction between strong grammars, such as mathematical discourse, and weak grammars, such as education, set within notions of vertical and horizontal knowledge structures (Lerman 2010); and how forms of pedagogy can be modified to improve the achievement of disadvantaged students (Knipping et al. 2008).

Bernstein also shows how curriculum choices, the recontextualization of knowledge from one place, academic mathematics in our case, to another, school mathematics, is determined by ideology; what is deemed important for students to acquire is governed by beliefs and values, though usually implicitly. Researchers have taken up the issue of values in addressing how what currently manifests as mathematics in schools affects students.

Examining gender effects of forms of assessment in mathematics, Wiliam suggests:

> We are led to the conclusion that it is a third source of difference—the definition of mathematics employed in the construction of the test—that is the most important determinant of the size (and even the direction) of any sex differences. (Wiliam 2003, p. 194)

As Lawler says, in a reexamination of one of the earliest texts addressing disadvantage in mathematics (Reyes and Stanic 1988):

> Mathematics education does not work to realize the living of the child, but to enact in the child particular, culturally-defined, ways of operating and interacting that are deemed to be mathematical. We treat the content of mathematics as stable structures of conventional ideas, "inert, unchanging, and unambiguous 'things' that children learn" (Popkewitz 2004, p. 18). And although these things appear to make the learner more of an active participant by expanding the child's role in solving problems and applying their own thinking, we simultaneously make them less active in defining the possibilities and boundaries for their engagement. (Lawler 2005, p. 33)

Lawler argues that changes over some decades have not made a difference to who succeeds and who fails. Perhaps, the challenge not addressed so far, informed by postmodern thinking, concerns the mathematical content, not only in thinking about what to teach but why, whether mathematics should be taught to everyone, and why the field is so implicated in maintaining the high status of a mathematical qualification.

## Research on Action or Intervention

The literature on interventions and radical action is very broad and, for the most part, does not distinguish between the various social characteristics of disadvantaged groups, race, gender, social class, disability, or others. Examples can be found in the literature mentioned above, such as that of the critical mathematics group, the literature of the ethnomathematics group, or the proceedings of the Mathematics Education and Society conferences.

## Concluding Remarks

Localization of statistical evidence has been mentioned above but could be seen to be vital in all aspects of the issue of social class and socioeconomic status in mathematics education. Who is disadvantaged, what the causes might be, what status mathematics has, whether mathematics for all is part of the values, and what kinds of interventions might be effective are all informed by the theoretical and empirical studies described here but are different across the world. The research field lacks such analyses from many parts of the world, and the complicity of mainstream researchers in the status of mathematics in society may be the cause of the major focus being on other aspects of teaching and learning.

## Cross-References

► Cultural Diversity in Mathematics Education
► Equity and Access in Mathematics Education
► Indigenous Students in Mathematics Education
► Sociological Approaches in Mathematics Education

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# Sociological Approaches in Mathematics Education

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## Keywords

Agency; Critical analysis; Control; Curriculum; Discourse; Discrimination; Empirical investigation; Everyday knowledge; Feminist perspectives; Identity; Ideology; Institution; Instructional mechanism; Interaction; Knowledge; Linguistic habitus; Mathematics textbooks; Parental participation; Phenomenology; Social practice; Social process; Parental participation; Social semiotics; Stratification; Structuralist perspectives

## Topic and History

Sociological approaches in mathematics education are those where sociological theory guides and directs research. In research on mathematics education, they have a rather short history. They are offering vigorous and fresh perspectives, and they have received increasing attention during the last 25 years. By using methods of empirical investigation and critical analysis, they engage with the complex relationships between individuals, groups, knowledge, discourse, and social practice, aiming at a theoretical understanding of social processes in mathematics education. These relationships are often conceived as tensions between the micro level of individual agency and interaction and the macro level of the social structure of society. The institutions of mathematics education and their functioning, often in terms of social reproduction, are of crucial concern.

Sociological approaches in mathematics education refer to a field of study and a body of knowledge that are not defined by clear-cut boundaries. They use, recontextualize, and refine concepts and methods from the various branches of sociology and their neighboring disciplines. Naturally, sociology of education serves as the most convenient reservoir of reference for the sociological study of mathematics education (e.g., Bernstein, Bourdieu). However, studies from interpretive (interactionist (e.g., Mead) and ethnomethodological (e.g., Garfinkel)), phenomenological (e.g., Berger and Luckmann), critical (e.g., Adorno), structuralist (e.g., Althusser), poststructuralist (e.g., Foucault) and psychoanalytical (e.g., Lacan), political (e.g., Apple), feminist (e.g., Walkerdine), social semiotics (e.g., Halliday), and *discourse analytical* (e.g., Fairclough) perspectives have substantially contributed to our sociological understanding of mathematics education.

While only a few sociological studies of mathematics education had been published before the mid-1980s, a Fifth Day Special Program titled *Mathematics, Education, and Society* (Keitel et al. 1989) of the 6th International Congress on Mathematical Education (ICME-6) in 1988 achieved a breakthrough, quantitatively and in terms of its recognition, of research on *society and institutionalized mathematics education*, conceived as *the political* *dimensions of mathematics education*. This was the start of a series of international conferences, initially called Political Dimensions of Mathematics Education (PDME), then Mathematics Education and Society (MES), which served, and continues to serve, as the major forum for presenting and discussing research based on sociological approaches in mathematics education (Matos et al. 2008; Gellert et al. 2010).

## Issues of Research

Sociological approaches contribute in a particular way to what Lerman (2000) has called the "social turn in mathematics education research." While much research included in this "social turn" aims at conceiving mathematical learning as more social in character and as a result of action and interaction, sociological approaches in mathematics education investigate how mathematical knowledge is produced, distributed, recontextualized, reproduced, and evaluated by institutional practices. They particularly focus on how these practices shape identities and (re-)produce social stratifications. Another concern is the relationships between different contexts in which mathematical knowledge is transmitted, acquired, and assessed. As Ensor and Galant (2005) claim, many sociological studies of mathematics education are, at least implicitly, interested in the pedagogic forms and the mathematical knowledge supportive for social justice or try to state more precisely the pathologies that impede such a development. Equity and access are issues that motivate some sociological research in mathematics education. Jablonka (2009) holds that a prevalent ingredient of sociological approaches is critique, aiming at uncovering ideologies, making the invisible mechanisms of social functioning visible, thus making the unconscious conscious. Sociological approaches to research in mathematics education usually draw on qualitative research methods – exceptions prove the rule – which is much in accord with the skepticism of the "new sociology of education" of the 1970s in respect of a political arithmetic tradition.

While most of the presentations of sociological research at ICME-6 still had been of descriptive character and not systematically and explicitly based on sociological theories, they successfully kicked off substantial advances in the theoretical foundation of sociological approaches in mathematics education. Dowling's (1998) analysis of mathematical myths and pedagogic texts marks a milestone in the subsequent development of sociological theorizing in mathematics education. It examines and coordinates a wide range of theoretical positions, constructing a systematic and theoretically rooted language of description for analyzing mathematics textbooks sociologically. By providing mathematical activities that establish positions and messages differentially, mathematics textbooks construct a hierarchy of student voices through the distribution of the "myth of participation" (mathematics is a reservoir of use-values) and the "myth of reference" (mathematics offers a gaze on something other than itself). Mathematical texts for high-achieving students use abstraction and strategies of expansion to consistently foreground generalized academic mathematical messages. In contrast, texts for low-achieving students use localizing strategies to identify the students' voice with a public domain setting which is insulated from abstract mathematics. The curriculum mirrors the division of intellectual and manual labor, of class distinctions, and of code orientations. However, the ideological roots of mathematics curricula are far more hidden than overt and Dowling (1998) can be credited for contributing to their exposure. For many researchers, it provided an inspiring interpretation of the late work of the British sociologist of education Basil Bernstein (2000). In fact, Bernstein (2000) seems to have become the most common reference in studies of mathematics education that take sociological approaches. It provides an ample theoretical framework with strong internal coherence and explicit organizing principles – what Bernstein calls a strong grammar – that systematically links social structure with human agency, in particular for the context of pedagogic discourse. The widespread use of the concepts of the *pedagogic device*, *classification and framing values*, *recognition and realization rules*, *horizontal and vertical discourse*, and *recontextualizing fields* indicates a common focus and a coherent growing of sociological research in mathematics education. Studies of mathematics curriculum, of assessment of mathematical knowledge, of ability grouping, of pedagogic identities, and of classroom instruction practices, which will be exemplified in the next passage, are all central themes in sociological research in mathematics education. Most of the research examples link these central themes to each other.

Mathematics curricula can be usefully described and compared in terms of the strength of the boundaries established between everyday knowledge and academic mathematical knowledge, as well as between the areas that constitute the school subjects. Mathematics curricula for primary and for lower level secondary schools usually intend to connect school mathematical knowledge to the local and particular of everyday knowledge. This aim is reflected in the high proportion of word problems contained in the curriculum materials for the early grades. Gellert and Jablonka (2009) discuss how students face substantial intricacies of producing legitimate text in the classroom, if and because the recontextualization principle of the curriculum is generally not made sufficiently explicit in classroom practices. Cooper and Dunne (2000) investigate how students with different socioeconomic class backgrounds react to word and context problems. They analyzed large sets of data from the Key Stage 2 Tests for 10–11-year-old students in England. The study documents that students of families where the parents do manual work have significantly lower achievement when mathematics is interwoven with context. Cooper and Dunne find that these students tend to misinterpret the problems and to solve them with their everyday knowledge, which means that their mathematical competence is systematically underestimated in the tests. Wiliam et al. (2004) argue that for becoming successful in school mathematics, students need to develop a particular identity, in fact that of a young mathematics scholar, and that any other position towards mathematics, for instance, a more critical view of the nature of mathematics, is strictly discouraged by apparently neutral assessment practices that maximize differences between individuals and thus construct disparities in mathematics achievement. Ability grouping (streaming, setting, etc.) reinforces exclusion from the subject by constructing different mathematical *habitus* for different groups of learners (Zevenbergen 2005). Morgan et al. (2002) report that teachers' expectations and their subject position in the education discourse are heavily influential on their assessment practices. Consequently, teachers who teach in schools located in different social contexts emphasize different local assessment criteria, thus providing differential orientation towards mathematical knowledge, resulting in an unequal "preparation" of students for standardized mathematics achievement tests. In contexts of severe social discrimination and inferiority, what is transmitted and to be acquired is often emptied of any mathematical content. The tasks to be executed by students reflect a very weak classification between everyday and school knowledge, and consequently, the evaluative criteria appear to be weak or absent. It appears as a perverted form of recontextualization, when in socially discriminated contexts, the specialized knowledge of mathematics is subordinated to everyday knowledge and practices.

Sociological approaches to research on instructional practices have highlighted that reform agendas often overlook the different code orientations of groups of students. Lubienski (2000) argues that some instructional strategies that are highly valued in current mathematics education reforms disadvantage students who are characterized as of low socioeconomic status. She demonstrates how socioeconomically advantaged students tend to profit from intensive guided discussions in the classroom while more socioeconomically disadvantaged students become rather confused by conflicting mathematical ideas, suggesting that some characteristics of discussion-intensive mathematics classrooms might be more aligned with middle-class codes.

Apparently, the linguistic habitus of socioeconomically advantaged students work as cultural capital as in school – at least at the discursive level – the discursive practices are close to practices that are common in middle-class families. A similar effect has been observed by Brown (2000) who investigates parental participation in school mathematics. He reports that middle-class parents, when working together with their children on mathematics tasks, tend to emphasize the context-independent and general aspects of the tasks, while working-class parents focus strictly on the local and context-bound. Working-class children profit less than their middle-class peers from parental involvement in school mathematics.

Inside the mathematics classroom, various instructional mechanisms produce a stratification of achievement and success in mathematics that is not strictly based on the mathematical competence of the students. These mechanisms draw on students' unequal competences in recognizing the rules and reading the code of mathematics instruction. For instance, instructional strategies of embedding mathematics in mundane context and leaving implicit the relevance of that context in terms of the criteria for producing legitimate text separate the students along their code orientation. Teachers often show a well-distinguishable ability to maintain two different discourses at the same time, engaging some students in analytical mathematical arguments and others in substantial everyday reasoning. This observation is sociologically relevant since on the long run, the mathematical argument is institutionally more highly valued. Sociological approaches emphasize that the diversity, or heterogeneity, of groups of students is less a topic of concern than their positioning in hierarchies of social status.

Finally, research and reflection that explicitly call for more attention to sociopolitical dimensions are of fundamental importance for the sociological study of mathematics education. Here, the concept of power and its social, political, and educational ramifications is fundamental. Skovsmose (1994) introduces the notion of the formatting power of mathematics to indicate that mathematics colonizes large parts of reality and rearranges it. The transmission and acquisition of mathematical knowledge appear of direct social importance when concepts of critique, democracy, and *Mündigkeit* are brought together. For Valero (2009), power in mathematics education can be conceived in terms of the structural imbalance of knowledge control and of distributed positioning. The former view, which reflects a conflict theoretical stance, points to a constant struggle between structurally excluded and structurally included groups, in which the powerful tends to win and to succeed in cushioning the resistance on the side of the excluded. The latter view takes power as a relational capacity of social actors to draw on resources for self-positioning in situations. This definition does not only facilitate analyses of how mathematics and mathematics education is used in discourses affecting people's lives but also opens for a self-reflective perspective on how research in mathematics education is entangled in the distribution of power. Vithal (2003) investigates the role and potential power of mathematics education in postapartheid South Africa. By coining five pairs of concepts that work antagonistically and yet in cooperation with each other – freedom/structure, democracy/authority, context/mathematics, equity/differentiation, and, pulling these four together, potentiality/actuality – the fundament for a pedagogy of conflict and dialogue is laid out. Conceiving actuality as intrinsically conflicting, dialogue of various forms and at many levels is suggested to inspire and develop potentiality. Gates and Vistro-Yu (2003), taking on the distributions of mathematical knowledge and revisiting the program of *Mathematics for All*, describe mathematics as a gatekeeper to social progress and as a filtering device. They argue for a strong role of the mathematics education community to avoid and counteract the marginalization of some social groups. Gender, socioeconomic class, and ethnicity are discussed as examples of marginalized voices (and at the other side of the coin, there are dominant voices); in mathematics classrooms characterized by multiple discriminations, contradictions, and clashes in pedagogical practice, the marginalization tends to be reproduced and exacerbated. In essence (Skovsmose and Greer 2012), research on the sociopolitical dimensions of mathematics education, characterized by awareness of the inherently political nature of mathematics education and by acceptance of social responsibility, is based on, and continually develops, the critical agency of mathematics education researchers.

## Cross-References

- ▶ Competency Frameworks in Mathematics Education
- ▶ Critical Mathematics Education
- ▶ Discourse Analytic Approaches in Mathematics Education
- ▶ Equity and Access in Mathematics Education
- ▶ Gender in Mathematics Education
- ▶ Interactionist and Ethnomethodological Approaches in Mathematics Education
- ▶ Political Perspectives in Mathematics Education
- ▶ Poststructuralist and Psychoanalytic Approaches in Mathematics Education
- ▶ Recontextualization in Mathematics Education
- ▶ Socioeconomic Class in Mathematics Education

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# Sociomathematical Norms in Mathematics Education

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## Keywords

Sociomath norms; Social norms; Emergent perspective; Intellectual autonomy

## Definition

Sociomathematical norms are the normative criteria by which students within classroom communities create and justify their mathematical work. Examples include negotiating the criteria for what counts as a different, efficient, or sophisticated mathematical solution and the criteria for what counts as an acceptable mathematical explanation.

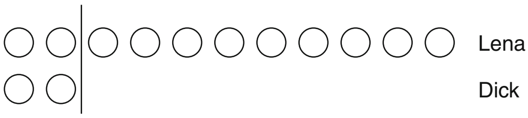
## Characteristics

### Social Norms

Social norms refer to the expectations that the teacher and students have for one another during academic discussions. Social norms are present in any classroom, including science and, English, for example. However, the social norms that are established within a *student*-centered classroom look very different from those in a traditional environment. Yackel and Cobb (1996) have documented at least four social norms that support student-centered instruction: Students are expected to (1) explain and justify their solutions and methods, (2) attempt to make sense of others' explanations, (3) indicate agreement or disagreement, and (4) ask clarifying questions when the need arises. The social norms for more traditional mathematics classes that are teacher-centered might involve expectations that the teacher explain one or more solution processes and that the students attempt to understand and repeat her reasoning on other problems. Social norms involve participants' expectations of each other during discussions and can be found in classrooms in any domain. For example, a *student*-centered science or literacy classroom might have similar social norms above such as explaining and justifying and understanding students' explanations.

### Sociomathematical Norms

While social norms focus on normative aspects of participation in any academic area, sociomathematical norms, on the other hand, are norms that are specific to *mathematical* activity. Similar to social norms, they can be found in any mathematics classroom, but they would look different depending on the goals and philosophy of instruction. They involve the teacher and students negotiating the *criteria* for what counts as an acceptable mathematical explanation, a different solution, an efficient solution, and a sophisticated solution in their classroom. For example, a social norm for a *student*-centered classroom might be that students are expected to explain their thinking, but what counts as an acceptable mathematical explanation must be determined among the teacher and students. For example, Stephan and Whitenack (2003) found that the *criteria* for what counts as an acceptable mathematical explanation in one first-grade class involved stating not only the procedures for finding an answer but also the reasons for the calculations as well as what these calculations and their results mean in terms of the problem. The criterion necessarily changes over time as the students and teachers give and take during their discussions. For instance, at the beginning of the year when a first-grade teacher asked her students to solve the problem, *Lena has 11 hearts, Dick has 2 hearts, how many more hearts does Lena have than Dick*, some students gave the answer 9 while others said 11. Students felt obliged to explain their thinking (social norm), but their discussion simply focused on their calculations, e.g., "I counted up 9 more to get to 11." Students who thought the answer was 11 argued that Lena




**Sociomathematical Norms in Mathematics Education, Fig. 1**

has 11 more than Dick so the answer is 11. Since students' explanations only drew on their calculations, the teacher attempted to initiate a discussion about *why* someone might count up to get the answer. Because the criterion for what counts as acceptable explanation in math class involved sharing their calculations, students did not know how to explain *why* they counted up. The teacher drew circles on the board to support students as they tried to explain why counting up was legitimate (Fig. 1).

A student came to the board, drew a vertical line after the second "heart," and counted by ones up from Dick's two hearts to "make them have the same amount." In this way, the teacher, through the use of diagrams, helped the students begin to learn that an acceptable explanation must involve their reasons for their procedures.

The criterion for what counts as an acceptable mathematical explanation might be different depending on the teaching approach used. For example, in more traditional classrooms, what counts as an acceptable mathematical explanation might involve describing only the calculations that one used in their procedure. In the Lena and Dick problem above, a traditional setting might find the following explanation acceptable, "I counted up two more from 9 on my fingers," without any reference to why that method has meaning and leads to a correct answer.

## Origin of the Term

The term sociomathematical norms was first coined by Cobb and Yackel (1996) as they built a framework for analyzing *student*-centered, or what they called, inquiry-based classrooms. They drew on the emergent perspective, a theory that says learning occurs both cognitively as well as in social interaction. Using the emergent perspective, they created the following framework to help themselves and others interpret how the teacher and students are interacting and learning in a classroom:

| Social | Individual |
|---|---|
| Classroom social norms | Beliefs about own role, others' roles, and the general nature of mathematical activity in school |
| Sociomathematical norms | Mathematical beliefs and values |
| Classroom mathematical practices | Mathematical conceptions |

As the framework shows, an individual forms his beliefs about his role in the class, his mathematical beliefs, and his mathematical learning as he participates in and contributes to the social and sociomathematical norms and classroom mathematical practices of his classroom community. Cobb and Yackel (1996) stress that learning is both an individual and social process with neither taking primacy over the other.

## Growth of the Concept

Many mathematics education researchers have acknowledged the importance of paying close attention to the establishment of certain sociomathematical norms in a variety of classroom settings. In fact, some argue that while inquiry social norms are mandatory for creating student-centered mathematics classrooms, they are insufficient for supporting mathematical growth (Pang 2001). Pang found that teachers, who established both strong social and sociomathematical norms for inquiry instruction, saw more mathematical growth in their students than those who had only established strong inquiry social norms. Given that sociomathematical norms focus more on the quality of the *mathematical* contributions in class, Pang's finding makes sense.

Mathematics education researchers have extended Yackel and Cobb's sociomathematical norms research by analyzing the development of these norms at the elementary (Pang 2001; Stephan and Whitenack 2003; Levenson et al. 2006), middle (Akyuz 2012), high (Kaldrimidou et al. 2008), and college level (Rasmussen et al. 2003). Findings indicate that negotiating the criterion for what counts as different, efficient, sophisticated, and an acceptable explanation in inquiry settings are an important focus of the teachers' practice.

At the elementary level, Levenson et al. (2006) extended the work on what counts as an acceptable explanation when they found that one teacher's criterion for what counts as acceptable involved practically based explanations (those ground in realistic contexts), even though she knows that some of her students are capable of giving more mathematically based explanations (those that are devoid of pictures and are more abstract). Additionally, Pang found that teachers are excellent at establishing social norms that are consistent with inquiry-based instruction, but not as much with sociomathematical norms. This is a concern since Pang argues that mathematical discussions arise out of sociomathematical norms, not social norms. Therefore, teachers must reconceptualize mathematics in their classrooms going beyond just expecting students to explain. Stephan and Whitenack (2003) identified a fifth sociomathematical norm, the criteria for what counts as an adequate mathematical diagram.

Of the research conducted at the middle and high school levels, most focus on documenting the sociomathematical norms that are established in higher level mathematics. Kaldrimidou et al. (2008) found that the criteria for what counts as an acceptable mathematical explanation was very procedural in a high school mathematics class they observed while Akyuz (2012) found that it was more conceptual (or meaning based) in one middle school class founded on inquiry-based instruction. Hershkowitz and Schwarz (1999) documented two new sociomathematical norms as they studied students who used a computer program to aid in their instruction. These two norms involved the criteria for what counts as mathematical evidence and what counts as a good hypothesis.

Keen attention to sociomathematical norms is even important within college level mathematics classrooms. Rasmussen et al. (2003) elaborate the criteria for what counts as a different, elegant, and efficient solution as well as acceptable explanation in an inquiry-based differential equations class. They also argue that the criteria for what counts as acceptable should often involve more than the procedures for solving the problem.

Other researchers have attempted to teach sociomathematical norm development in their professional development workshops (Shriki and Lavy 2005) as well as with preservice teacher instruction (Dixon et al. 2009). Some articles detail the role of the teacher in establishing these norms (McClain 2005). Additionally, sociomathematical norms have gained attention in other research fields as well with Johnson (2000) coining the term "sociophysics norms" to refer to the criteria for what counts as inquiry-based physics discourse.

## Common Issues

The research based upon sociomathematical norms is growing both within the field of mathematics education as well as other disciplines. When an idea like this takes root and begins to grow, oftentimes, it can change from its original meaning. The most common way sociomathematical norms are misinterpreted in the literature today involves losing the fact that they deal with the *criterion* for what counts as good mathematical discourse. The fact that students are expected to give different ideas in class can be cast as a social norm, but the *criterion* for what counts as different is negotiated within the realm of *mathematics*. It is the role of the teacher to lead the negotiation of these criteria and, therefore, the criterion for what counts as an acceptable mathematical explanation depends upon the teacher's own criterion, often influenced by the mathematics community.

Other research sometimes conflates sociomathematical norms with students' beliefs. For example, the fact that Marcos always gives procedural explanations and believes that math involves calculating answers is not a sociomathematical norm. Rather, that is his belief about what mathematics is (the individual side of Cobb and Yackel's framework).

In summary, sociomathematical norms refer to the criteria by which solutions are determined as different, efficient, and sophisticated and explanations are deemed mathematical acceptable in a classroom. The teacher and her students create these criteria together as they solve problems and engage is discourse with one another. Sociomathematical norms are present in any mathematics classroom; however, the criteria for what counts as mathematical solutions and explanations would probably look different from classroom to classroom, depending on how teacher- or student-centered the instruction is. Sociomathematical norms are different from social norms in that the former are specific to *mathematics* talk. Additionally, social norms are easier for teachers to establish in their classrooms, but mathematics grows out of sociomathematical norms, making it extremely important for teachers to make them a clear focus of their teaching practice. This is one area that deserves more attention and research.

## Cross-References


- ▶ Argumentation in Mathematics
- ▶ Learner-Centered Teaching in Mathematics Education
- ▶ Manipulatives in Mathematics Education
- ▶ Mathematical Proof, Argumentation, and Reasoning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# Stoffdidaktik in Mathematics Education


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## Keywords


Stoffdidaktik; Gymnasium; Ingenierie didactique; Elementarization; Grundvorstellungen; Pedagogical content knowledge

## Definition

An approach to mathematics education and research on teaching and learning mathematics (i.e., didactics of mathematics), which concentrates on the mathematical contents of the subject matter to be taught, attempting to be as close as possible to disciplinary mathematics. A major aim is to make mathematics accessible and understandable to the learner.

## History

Stoffdidaktik or "subject matter didactics" (translation suggested by the entry author) has been a prominent approach to mathematics education and research into teaching and learning mathematics (i.e., didactics of mathematics) in German-speaking countries (e.g., Austria, Germany, and parts of Switzerland). It grew out of one of the two main strands of German-speaking didactics of mathematics in the first half of the twentieth century, namely, university studies that focused on the teaching of mathematics in "gymnasium," the most demanding type of school at that time in Germany. This strand was different from the strand that focused on teacher training for primary and the majority of lower secondary schools. With professors of mathematics at university interested in mathematics education (like Felix Klein and Heinrich Behnke), it had authors basically coming from university institutions and teachers of gymnasium, who published in well-established journals mainly read by mathematics teachers of gymnasium (like "Zeitschrift für den mathematischen und naturwissenschaftlichen Unterricht ZMNU," later "MNU" or "Unterrichtsblätter für Mathematik und Naturwissenschaften UMN"). With the widening of research approaches in didactics of mathematics during the second half of the twentieth century, Stoffdidaktik somehow widened its perspective to the teaching of mathematics in all types of schools, but lost its position as one of two major approaches in German-speaking didactics of mathematics. The title of Reichel's (1995) plenary talk at a conference of German-speaking didacticians is quite revealing: "Is there a future for subject specific didactics?" (for detailed description of this development cf. Steinbring 2011, pp 44–46). Nowadays, Stoffdidaktik is mainly published in journals aiming at practicing teachers of all levels of schooling in German-speaking countries (in journals such as "Der Mathematikunterricht MU"; URL: http://www.friedrich-verlag.de/go/Sekundarstufe/Mathematik/Zeitschriften/Der+Mathematikunterricht).

## Characteristics

According to Steinbring (2011, p. 45), Stoffdidaktik is characterized by the assumption that mathematical knowledge – researched and developed in the academic discipline – is essentially unchanged and absolute. ". .. it specifically proceeds to prepare the pre-given mathematical disciplinary knowledge for instruction as a mathematical content, to elementarise it and to arrange it methodically." As protagonist of subject matter didactics, Griesel (1974, p. 118) has identified the following features of "didactically oriented content analysis" as he prefers to name the approach: "The research methods of this area are identical to those of mathematics, so that outsiders have sometimes gained the impression that, here, mathematics (particularly elementary mathematics) and not mathematics education is being conducted." In terms of research methodology, this is a very clear and somehow very restricted preference, which – at least in terms of research methods – makes it difficult to distinguish Stoffdidaktik from mathematics.

Furthermore, Griesel continues: "The goal of 'didactically oriented content analysis' which essentially follows mathematical methods is to give a better foundation for the formulation of content-related learning goals and for the development, definition and use of a differentiated methodical set of instruments" (Griesel, 1974, p. 118, both translations by Heinz Steinbring 2011, p. 45). The practice of "content-oriented analysis" up to the 1960s suggests that implicitly

Stoffdidaktik starts from the assumption that after a decent mathematical analysis, one will find one and only one best way to teach a certain content matter, which then should be incorporated into mathematics textbooks (for a critical description of this feature of Stoffdidaktik see Jahnke 1998, p. 68).

In the preface of a book series, which Griesel himself identifies as a prototypical example of Stoffdidaktik, Griesel (1971, p. 7) identifies six areas, which are important for the progress of didactics of mathematics. The first two are of utmost importance, especially the first one: research into the content, the methods, and the application of mathematics; and didactical ideas and insights, "which make it possible to attend better, or at all, to a subject area within instruction." For him, the first area was most successful at that time. The other four influential factors are general experience, statistically based evidence about instruction, insights into the mathematical learning process, and the development-psychological and sociological conditions (translations from Steinbring 2011, p. 45). With these statements, Griesel identified some limitations of "content-oriented analysis" using mathematical methods. He even went as far as calling them meaningless if the necessary follow-up empirical investigations show that the results of Stoffdidaktik are meaningless for the learning process of mathematics.

From an international perspective, the approach closest to Stoffdidaktik is the French approach of "ingenierie didactique" – didactical engineering – especially its "a priori analysis" part. Stoffdidaktik shares with didactical engineering a focus on disciplinary mathematics, its history, and its epistemology. Especially for the "a priori" part, didactical engineering as well as Stoffdidaktik in its entirety heavily depend on a detailed analysis of the content, history, and epistemology of the mathematical content matter under analysis. If taken as a preparation to a teaching experiment, the a priori part of didactical engineering tends to enact the very same activities and methods, as Stoffdidaktik would apply. A difference between these two approaches appears when the actual practice is taken into consideration: From the very beginning, didactical engineering is also interested in the teacher and learner of the subject matter under consideration, their preknowledge before a teaching (experiment), and the consequences after a teaching experiment. Traditional Stoffdidaktik was not interested in the human side of the teaching-learning process nor did it traditionally look into the consequences of a certain setup of the teaching-learning process (for a detailed comparison, see Sträßer 1996). The reason for the relative negligence of these aspects may be the idea of the one and only best way to teach a certain subject matter, which allows to forget about alternatives.

The notion of "pedagogical content knowledge" ("PCK"), which was introduced into the debate on the professional knowledge of teachers by Shulman (1987), is also close to Stoffdidaktik. With PCK as "understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (Shulman, 1987, p. 8), PCK shares a close link to subject matter knowledge with Stoffdidaktik. In contrast, Stoffdidaktik tends to be more authoritarian, looking for the best one and only mathematical solution, but cares less for the personal aspects of the teaching and learning process – with Shulman's concept of "content knowledge" confirming the importance of disciplinary mathematics for the teaching and learning of the subject.

## Some Examples

A rather comprehensive exemplar of Stoffdidaktik is the book entitled "*Mathematik wirklich verstehen*" ("*Really understanding mathematics*") by Kirsch (1987), which covers a major part of lower secondary mathematics (especially numbers and functions with a foundation in set theory). In some sense, the title marks an important difference: While mathematics tends to prove a statement, Stoffdidaktik aims at understanding the statement. Vollrath (1974/2006) can be taken as the complement on equations and elementary (school) algebra. Holland (1996/2007) covers geometry in lower secondary mathematics teaching. Danckwerts and Vogel (2006) with a book on teaching calculus entitled "Analysis verständlich unterrichten" (How to teach calculus understandably) confirm the effort of Stoffdidaktik to teach mathematics in an accessible, understandable manner.

The internationally best known example of Stoffdidaktik is a plenary by Kirsch at the ICME congress in Karlsruhe (Kirsch, 1977) entitled "Aspects of Simplification in Mathematics Teaching." The title mirrors the utmost importance of disciplinary mathematics, which Stoffdidaktik prepares for teaching this subject. In order to make mathematics accessible in teaching, Kirsch suggests four activities:

- Concentration on the mathematical heart of the matter
- Including the "surroundings" of mathematics
- Recognizing and activating preexisting knowledge
- Changing the mode of representation

which are often summarized under the concept "elementarization" – with a long tradition in Germany (see the famous series of books by Felix Klein *Mathematics from an Advanced Standpoint* – original title: *Elementarmathematik vom höheren Standpunkt aus*).

For Reichel (1995), "the so-called Stoffdidaktik was the most important part" of German didactics of Mathematics. In his "perhaps amplified understanding of that term," he adds a list of 15 research areas to traditional Stoffdidaktik. Besides other areas mentioned, Stoffdidaktik should play a major role when analyzing the image of mathematics, in assessment questions, in research on using computers, and on language and (teaching) mathematics – to cite but a few from Reichel's list. This already shows that Reichel has a concept of Stoffdidaktik which clearly goes further than the traditional epistemology of school mathematics, content analysis, elementarization, and teaching methods with Stoffdidaktik as a major part of research work in didactics of mathematics.

## Recent Development

Reichel's (1995) text indicates a development with Stoffdidaktik in the German-speaking didactics of mathematics: In the last quarter of the twentieth century, Stoffdidaktik has lost its importance as one of the most important and widespread research approaches in the German-speaking community. Young researchers widened the narrow perspective of traditional Stoffdidaktik by taking into account more aspects than disciplinary mathematics, its history, and epistemology. In this respect, a major move was the suggestion of taking into account the beliefs, ideas, and knowledge of the learner of mathematics. Vom Hofe (1995) was the most prominent advocate of this opening up of Stoffdidaktik to the learner by suggesting to care for the "Grundvorstellungen" (i.e., basic beliefs and ideas) of the learner to link mathematics, the individual (especially: learner), and reality. Grundvorstellungen are seen as a way to better understand sense making of an individual, ways of representation that an individual develops, and her/his way of using ideas and concepts with respect to reality. In doing so, the concept of Grundvorstellungen is not only meant as a normative idea to inform curriculum construction but also as a way to describe the strategies and mindsets of a (potential or actual) learner. Four concepts structure this approach to didactics of mathematics, namely, the individual, the context, the Grundvorstellung, and mathematics.

## Cross-References

▶ Cultural Influences in Mathematics Education
▶ Curriculum Resources and Textbooks in Mathematics Education
▶ Design Research in Mathematics Education
▶ Didactic Transposition in Mathematics Education
▶ History of Research in Mathematics Education
▶ International Comparative Studies in Mathematics: An Overview
▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes

▶ Mathematics Teachers and Curricula
▶ Pedagogical Content Knowledge in Mathematics Education
▶ Teacher as Researcher in Mathematics Education

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# Structure of the Observed Learning Outcome (SOLO) Model

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## Keywords

Cognitive development; neo –Piagetian model; Nature of learning; Assessment framework

## Definition

Biggs and Collis (1982) described the **S**tructure of the **O**bserved **L**earning **O**utcome (SOLO) Model (commonly referred to as the SOLO Taxonomy) as a general model of intellectual development. SOLO had its origins in the stage development ideas of Piaget and the information processing concepts of the 1970s. It can be considered within the broad research framework referred to as neo-Piagetian. As such, SOLO has much in common with the writings of Case (1992), Halford (1993), and Fischer and Knight (1990) to name a few.

## Characteristics

Central to SOLO is the view that there are "natural" stages in the growth of learning any complex material or skill. Also, these stages "are similar to, but not identical with, the developmental stages in thinking described by Piaget and his co-workers" (Biggs and Collis 1982, p. 15).

The SOLO Model has its roots in the analysis of responses to questions posed in a variety of subject/topic areas. The focus of analysis was on specifying "how well" something was learned, as a balance to the more traditional approach of "how much" has been learned. The insight of Biggs and Collis was that the structural organization of knowledge was the issue that discriminated well-learned from poorly learned material.

For SOLO, learners actively construct their understandings by building upon earlier experiences and understandings. In doing this, learners pass through sequential qualitatively different "stages" that represent a coherent view of their world. This development is a result of processes of interaction between the learner and his or her social and physical environment.

Hence, understanding is viewed as an individual characteristic that is both content and context specific (Biggs and Collis 1991). SOLO emerges as a means of describing the underlying structure of an individual's performance at a specific time that is determined purely from a response. Describing the structure of a response is seen as a phenomenon in its own right, without necessarily representing a particular stage of intellectual development of the learner (Biggs and Collis 1982).

The progressive structural complexity in responses, i.e., cognitive development, is described in two ways. First is based upon the nature or abstractness of the task/response and is referred to as the *mode*. The second is based on a person's ability to handle, with increased sophistication, relevant cues within a mode and is referred to as the *level* of response.

## SOLO Modes

SOLO postulates that all learning occurs in one of five *modes of functioning*, and these are referred to as Sensorimotor, Ikonic, Concrete Symbolic, Formal, and Post-formal. The five modes of thinking are described (briefly) below.

| | |
|---|---|
| Sensorimotor (soon after birth) | A person reacts to the physical environment. For the very young child, it is the mode in which motor skills are acquired. These play an important part in later life as skills associated with various sports evolve |
| Ikonic (from 2 years) | A person internalizes actions in the form of images. It is in this mode that the young child develops words and images that can stand for objects and events. For the adult this mode of functioning assists in the appreciation of art and music and leads to a form of knowledge referred to as intuitive |
| Concrete Symbolic (from 6 or 7 years) | A person thinks through use of a symbol system such as written language and number systems. This is the most common mode addressed in learning in the upper primary and secondary school |
| Formal (from 15 or 16 years) | A person considers more abstract concepts. This can be described as working in terms of "principles" and "theories." Students are no longer restricted to a concrete referent. In its more advanced form, it involves the development of disciplines. |
| Post-formal (possibly at around 22 years) | A person is able to question or challenge the fundamental structure of theories or disciplines |

It is important to note that the ages provided above are approximate indications of when a mode becomes available and is context dependent. There is no implication that a person who is able to respond in the concrete symbolic mode in one context is able or would wish to respond in the same mode in other contexts.

Nevertheless, an implication of this description is that most students in primary and secondary school are capable of operating within the Concrete Symbolic mode. Because of this, the Concrete Symbolic mode is considered the *target* mode for instruction at school, and teaching techniques need to be adopted generally to suit learners working in this mode. In the case of a secondary student in certain topics, some may still respond to stimuli in the Ikonic mode, while others may respond with Formal reasoning.

Each mode has its own identity and its own specific idiosyncratic character. While earlier acquired modes are needed to move to new modes of abstraction, these earlier modes remain available to the individual. Within each mode, responses become increasingly complex as the cycle of learning develops. This growth is described in terms of *levels* using the same generic terms for each mode. A level refers to a pattern of thought revealed in what a learner says, writes, and/or does.

### SOLO Levels

Three *levels* form a *cycle* of development. These descriptions of levels indicate an increasing sophistication in a learner's ability to handle tasks associated with a mode.

Unistructural: The student focuses on the domain/problem but uses only one piece of relevant data and so may be inconsistent.

Multistructural: Two or more pieces of data are used without any relationships perceived between them. No integration occurs. Some inconsistency may be apparent.

Relational: All data are now available, with each piece woven into an overall mosaic of relationships. The whole has become a coherent structure. No inconsistency is present within the known system.

Each level integrates the level before it thus logically acquiring the elements of the prior level. At the same time, each level forms a logical and empirically consistent structured whole. An important consequence is that all learner responses should be able to be allocated to a particular level, a mixture of levels, or a mixture of adjoining levels (referred to as transitional responses).

Research into SOLO levels since the 1990s (Campbell et al. 1992; Pegg 1992) when students' responses were analyzed over a greater range of learning situations than had been undertaken in earlier research identified more than one cycle of levels within each mode. In the case of two cycles of growth identified within a mode, the cycles describe a continuous pattern of development with Relational level of the first cycle linking to the Unistructural level in the second cycle. This work has resulted in a greater understanding of cognitive development (Pegg 2003).

### Conclusion

SOLO is a general framework for systematically assessing quality in terms of both structural and hierarchical characteristics. The strength of SOLO is the linking of the hierarchical nature of cognitive development (modes) and the cyclical nature of learning (levels). Each mode/level of functioning has its own integrity and structure and its own idiosyncratic selection and use of data. The main strength of the framework is in its ability to offer systematic and objective qualitative assessments of learning.

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# Students' Attitude in Mathematics Education

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## Keywords

Affect; Beliefs; Emotions; Interpretative approach; Students' failure in mathematics

## Definition

The construct of attitude has its roots in the context of social psychology in the early part of the twentieth century. In this context, attitude is considered as a state of readiness that exerts a dynamic influence upon an individual's response (Allport 1935).

In the field of mathematics education, early studies about attitude towards mathematics already appeared in 1950, but in many of these studies the construct is used without a proper definition.

In 1992, McLeod includes attitude among the three factors that identify affect (the others are emotions and beliefs), describing it as characterized by moderate intensity and reasonable stability. But the definition of the construct remains one of the major issues in the recent research on attitude: as a matter of fact, there is no general agreement among scholars about the very nature of attitude.

Therefore, in this entry, the issue of the definition of attitude towards mathematics (and also of the consequent characterization of positive and negative attitude) is developed in all its complexity.

## The Origin of the Construct

Since the early studies, research into attitude has been focused much more on the development of measuring instruments than towards the theoretical definition of the construct, producing methodological contributions of great importance, such as those of Thurstone and Likert.

As far as mathematics education is concerned, early studies about attitude towards mathematics already appeared in 1950: Dutton uses Thurstone scales to measure pupils' and teachers' attitudes towards arithmetic (Dutton 1951). The interest in the construct is justified by the vague belief that "*something called* '*attitude*' *plays a crucial role in learning mathematics*" (Neale 1969, p. 631).

In these studies, both the definition of the construct and the methodological tools of investigation are inherited from those used in social psychology: in particular, attitude is seen as "*a learned predisposition or tendency on the part of an individual to respond positively or negatively to some object*, *situation*, *concept*, *or another person*" (Aiken 1970, p. 551). Recourse to the adverbs "*positively or negatively*" is very evident: indeed a lot of attention by researchers is focused on the correlation between positive/negative attitude and high/low achievement. Aiken and Dreger (1961), regarding this alleged correlation between attitude and achievement, even speak of a *hypothesis of the etiology of attitudes towards maths*. Aiken (1970, p. 558) claims: "*obviously*, *the assessment of attitudes toward mathematics would be of less concern if attitudes were not thought to affect performance in some way*."

## The Problematic Relationship Between Attitude and Achievement

Until the early nineties, research into attitude within the field of mathematics education focuses much more on developing instruments to measure attitude (in order to prove a causal correlation between *positive* attitude towards maths and achievement in mathematics) rather than on clarifying the object of the research.

But the correlation between attitude and achievement that emerges from the results of these studies is far from clear. Underlining the need for research into attitude, Aiken (1970) refers to the need of clarifying the nature of the influence of attitude on achievement: he reports the results of many studies in which the correlation between attitude and achievement is not evident. Several years later, Ma and Kishor (1997), analyzing 113 studies about attitude towards mathematics, confirmed that the correlation between positive attitude and achievement is not statistically significant.

In order to explain this "failure" in *proving* a causal correlation between positive attitude and achievement, several causes have been identified: some related to the inappropriateness of the instruments that had been used to assess attitude (Leder, 1985) and also achievement (Middleton and Spanias 1999), others that underline the lack of theoretical clarity regarding the nature itself of the construct *attitude* (Di Martino and Zan 2001).

In particular, until the early nineties, most studies did not explicitly provide a theoretical definition of attitude and settled for operational definitions implied by the instruments used to measure attitude (in other words, they implicitly define positive and negative attitude rather than giving a characterization of attitude). Up until that time, in mathematics education, the assessment of attitude in mathematics is carried out almost exclusively through the use of self-report scales, generally Likert scales. Leder (1985) claims that these early attempts to measure attitudes are *exceptionally primitive*. These scales generally are designed to assess factors such as perspective towards liking, usefulness, and confidence. In mathematics education a number of similar scales have been developed and used in research studies, provoking the critical comment by Kulm (1980, p. 365): "*researchers should not believe that scales with proper names attached to them are the only acceptable way to measure attitudes*."

Other studies have provided a definition of the construct that usually can be classified according to one of the following two typologies:

1. A "simple" definition of attitude which describes it as the positive or negative degree of affect associated to a certain subject.
2. A "multidimensional" definition which recognizes three components of the attitude: affective, cognitive, and behavioral.

Both the definitions appear to be problematic: first of all a gap emerges between the assumed definitions and the instruments used for measuring attitude (Leder 1985). Moreover, the characterizations of *positive* attitude that follow the definitions are problematic (Di Martino and Zan 2001).

In the case of the simple definition, it is quite clear that "positive attitude" means "positive" emotional disposition. But even if a positive emotional disposition can be related to individual choices (e.g., which and how many mathematics courses to take), there are many doubts about the correlation between emotional disposition and achievement (McLeod 1992, refers to data from the Second International Mathematics Study that indicates that Japanese students had a greater dislike for mathematics than students in other countries, even though Japanese achievement was very high). Moreover, a positive emotional disposition towards mathematics is important, but not a value per se: it should be linked with an epistemologically correct view of the discipline.

In terms of multidimensional definition, it is more problematic to characterize the positive/negative dichotomy: it is different if the adjective "positive" refers to emotions, beliefs, or behaviors (Zan and Di Martino 2007). The assessment tools used in many studies try to *overcome* this difficulty returning a single score (the sum of the scores assigned to each item) to describe attitude, but this is inconsistent with the assumed multidimensional characterization of the construct. Moreover, the inclusion of the behavioral dimension in the definition of attitude exposes research to the risk of circularity (using observed behavior to infer attitude and thereafter interpreting students' behavior referring to the inferred attitudes). In order to avoid such a risk, Daskalogianni and Simpson (2000) introduce a bidimensional definition of attitude that does not include the behavioral component.

An interesting perspective is that identified by Kulm who moves to a more general level. He considers the attitude construct functional to the researcher's self-posed problems and for these reason he suggests (Kulm 1980, p. 358) that "*it is probably not possible to offer a definition of attitude toward mathematics that would be suitable for all situations, and even if one were agreed on, it would probably be too general to be useful*."

This claim is linked to an important evolution in research about attitude, bringing us to see attitude as "*a construct of an observer's desire to formulate a story to account for observations*," rather than "*a quality of an individual*" (Ruffell et al. 1998, p. 1).

## Changes of Perspective in Research into Attitude in Mathematics Education

In the late 80s, two important and intertwined trends strongly influenced research about attitude in mathematics education.

In the light of the high complexity of human behavior, there is the gradual affirmation of the interpretative paradigm in the social sciences: it leads researchers to abandon the attempt of explaining behavior through measurements or general rules based on a cause-effect scheme and to search for interpretative tools. Research on attitudes towards mathematics developed, in the last 20 years, through this paradigm shift from a normative-positivistic one to an interpretative one (Zan et al. 2006). In line with this, the theoretical construct of "attitude towards mathematics" is no longer a predictive variable for specific behaviors, but a flexible and multidimensional interpretative tool, aimed at describing the interactions between affective and cognitive aspects in mathematical activity. It is useful in supporting researchers as well as teachers in interpreting teaching/learning processes and in designing didactical interventions.

Furthermore, the academic community of mathematics educators recognized the need for going beyond purely cognitive interpretations of failure in mathematics achievement. Schoenfeld (1987) underlines that lack at a metacognitive level may lead students to a bad management of their cognitive resources and eventually to failure, even if there is no lack of knowledge. The book "Affect and mathematical problem solving" (Adams and McLeod 1989) features contributions by different scholars regarding the influence of affective factors in mathematical problem solving.

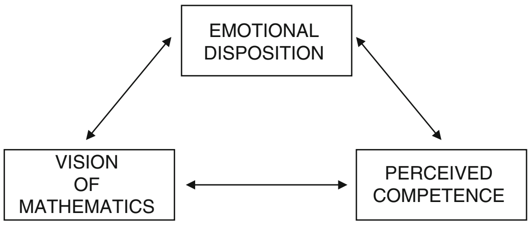
This gives a new impulse to research on affect, and therefore on attitude, in mathematics, with a particular interest on the characterization of the constructs. There is the need for a theoretical systematization and a first important attempt in this direction is done by McLeod (1992). He describes the results obtained by research about attitude, in particular underlining the significant results concerning the interpretation of gender differences in mathematics (Sherman and Fennema 1977); but he also points out the problems that emerged in the research about attitude (and more general affective construct), underlining the need for theoretical studies to better clarify the mutual relationship between affective constructs (emotions, beliefs, and attitudes): "*research in mathematics education needs to develop a more coherent framework for research on beliefs, their relationship to attitudes and emotions, and their interaction with cognitive factors in mathematics learning and instruction*" (McLeod 1992, p. 581).

Moreover, McLeod highlights the need to develop new observational tools and he also emphasizes the need for more qualitative research. Following this, narrative tools began to assume a great relevance in characterizing the construct (Zan and Di Martino 2007), in observing changes in individual's attitude (Hannula 2002), in assessing influence of cultural and environmental factors on attitude (Pepin 2011), and in establishing the relationship between attitudes and beliefs (Di Martino and Zan 2011).

## The TMA Model: A Definition of Attitude Grounded on Students' Narratives

In the framework described, following an interpretative approach based on the collection of autobiographical narratives of students (more than 1800 essays with the title "Maths and me" written by students of all grade levels), Di Martino and Zan (2010) try to identify how students describe their relationship with mathematics. This investigation leads to a theoretical characterization of the construct of attitude that takes into account students' viewpoints about their own experiences with mathematics, i.e., a definition of attitude closely related to practice. From this study it emerges that when students describe their own relationship to mathematics, nearly all of them refer to one or more of these three dimensions:

- Emotions
- Vision of mathematics
- Perceived competence




**Students' Attitude in Mathematics Education, Fig. 1** The TMA model for attitude (Di Martino and Zan 2010)

These dimensions and their mutual relationships therefore characterize students' relationship with mathematics, suggesting a three-dimensional model for attitude (TMA) (Fig. 1):

The multidimensionality highlighted in the model suggests the inadequacy of the positive/negative dichotomy for attitude which referred only to the emotional dimension. In particular the model suggests considering an attitude as negative when at least one of the three dimensions is negative. In this way, it is possible to outline different profiles of negative attitude towards mathematics.

Moreover, in the study a number of profiles characterized by failure and unease emerge. A recurrent element is a low perceived competence even reinforced by repeated school experience perceived as failures, often joint with an instrumental vision of mathematics.

As Polo and Zan (2006) claim, often in teachers' practice the diagnosis of students' negative attitude is a sort of black box, a claim of surrender by the teacher rather than an accurate interpretation of the student's behavior capable of steering future didactical action. The identification of different profiles of attitude towards mathematics can help teachers to overcome the "black box approach" through the construction of an accurate diagnosis of negative attitude, structured in the observation of the three identified dimensions, and aimed at identifying carefully the student's attitude profile.

## Cross-References

▶ Affect in Mathematics Education
▶ Gender in Mathematics Education
▶ Metacognition
▶ Teacher Beliefs, Attitudes, and Self-Efficacy in Mathematics Education

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## Task-Based Interviews in Mathematics Education

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### Keywords

Clinical interview; Teaching experiment; Problem solving; Task design

### Definition

Interviews in which a subject or group of subjects talk while working on a mathematical task or set of tasks.

#### The Clinical Interview

Task-based interviews have their origin in clinical interviews that date back to the time of Piaget, who is credited with pioneering the clinical interview. In the early 1960s, the clinical interview was used in order to gain a deeper understanding of children's cognitive development (e.g., Piaget 1965, 1975). Task-based interviews have been used by researchers in qualitative research in mathematics education to gain knowledge about an individual or group of students' existing and developing mathematical knowledge and problem-solving behaviors.

#### Task-Based Interview

The task-based interview, a particular form of clinical interview, is designed so that interviewees interact not only with the interviewer and sometimes a small group but also with a *task environment* that is carefully designed for purposes of the interview (Goldin 2000). Hence, a carefully constructed task is a key component of the task-based interview in mathematics education (Maher et al. 2011). It is intended to elicit in subjects estimates of their existing knowledge, growth in knowledge, and also their representations of particular mathematical ideas, structures, and ways of reasoning.

In preparing a clinical task-based interview, certain methodological considerations warrant attention and need to be considered in protocol design. These require attention to issues of reliability, replicability, task design, and generalizability (Goldin 2000). Some interviews are structured, with detailed protocols determining, in advance, the interviewer's interaction and questions. Other protocols are semi-structured, allowing for modifications depending on the judgment of the researcher. In situations where the research is exploratory, data from the interviews provide a foundation for a more detailed protocol design. In other, more open-ended

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situations, a task is presented and there is minimal interaction of the researcher, except, perhaps, for clarification of responses or ensuring that the subjects understand the nature of the task.

## Methodology

As subjects are engaged in a mathematical activity, researchers can observe their actions and record them with audio and/or videotapes for later, more detailed, analyses. The recordings, accompanied by transcripts, observers' notes, subjects' work, or other related metadata, provide the data for analyses and further protocol design. Data from the interviews are then coded, analyzed, and reported according to the research questions initially posed.

## Techniques and Resources

A variety of techniques are used in task-based interviews, such as thinking aloud and open-ended prompting (Clement 2000). These can be modified and adjusted, according to the judgment of the researcher.

Task-based interviews are used to investigate subjects' existing and developing mathematical knowledge and ways of reasoning, how ideas are represented and elaborated, and how connections are built to other ideas as they extend their knowledge (Maher 1998; Maher et al. 2011). Episodes of clinical, task-based interviews can be viewed by accessing the Video Mosaic Collaborative, VMC, website (http://www.videomosaic.org) or Private Universe Project in Mathematics (http://www.learner.org/workshops/pupmath). An example of a task-based interview in which the interviewee is engaged with the interviewer as well as the task environment that was designed by the researchers, see http://hdl.rutgers.edu/1782.1/rucore00000001201.Video.000062046. The episode shows nine-year-old Brandon, explaining the notation he used to explain his reasoning. It also shows how the interviewer's intervention, asking Brandon if the solution reminded him of any other problem, prompted him, spontaneously, to provide a convincing solution for an isomorphic problem (Maher and Martino 1998). A second example from the content strand of algebra is a task-based interview of Stephanie, an 8th grade girl who has been asked to build a model for $(a + b)^3$ with a set of algebra blocks. Stephanie, earlier in the interview, has successfully expanded $(a + b)^3$ algebraically to the expression $a^3 + 3a^2b + 3ab^2 + b^3$ and is challenged by the researcher in this clip to find each of the terms as it is modeled in the cube that she builds. In this example, the researcher is assessing Stephanie's ability to connect her symbolic and physical representations as well as observing how she navigates the transition from a two-dimensional model of $(a + b)^2$ to a model that involves three dimensions. All nine of the clips from this interview are available on the Video Mosaic Collaborative website and can be found by searching for the general title: *Early algebra ideas about binomial expansion, Stephanie's interview four of seven*. The full title of clip 5 is *Early algebra ideas about binomial expansion, Stephanie's interview four of seven, Clip 5 of 9: Building $(a + b)^3$ and identifying the pieces*. The link to this clip is http://hdl.rutgers.edu/1782.1/rucore00000001201.Video.000065479.

## Task-Based Interviews for Assessment

Paper and pencil tests are limited in that they do not address conceptual knowledge and the process by which a student does mathematics and reasons about mathematical ideas and situations. Adaptations of the clinical task-based interview have been useful in describing student knowledge and providing insight into how mathematical solutions to tasks are built by students. By providing a structured mathematical task, researchers can gain insight into students' mathematical thinking (Davis 1984). Also, teachers can use task-based interviews in their classrooms to study how young children think about and learn mathematics as well as to assess the mathematical knowledge of their students (Ginsburg 1977). Assessments of the mathematical understanding and ways of reasoning in problem-solving situations of small groups of students can also be made with open-ended task-based assessments (Maher and Martino 1996). See http://www.learner.org/workshops/pupmath/workshops/wk2trans.html.

An example of a group interview facilitated by researchers Carolyn Maher and Regine Kiczek with four 11th grade students who have been working on combinatorics problems as a part of a longitudinal study of children's mathematical reasoning since they were in elementary school (Alqahtani, 2011). In this session they were discussing the meaning of combinatorial notation and the addition of Pascal's identity in terms of that notation. They were asked to write the general form of Pascal's identity with reference to the coefficients of the binomial expansion. Their work during the session indicates their recognition of the isomorphism between the binomial expansion and the triangle and can be viewed at http://videomosaic.org/viewAnalytic?pid=rutgers-lib:35783.

### The Teaching Experiment

According to Steffe and Thompson (2000), a teaching experiment is an experimental tool that derives from Piaget's clinical interview. In this context, the interviewer and interviewee's actions are interdependent. However, it differs from the clinical interview in that the interviewer intervenes by experimenting with inputs that might influence the organizing or reorganizing of an individual's knowledge in that it traces growth over time. In a teaching experiment, researchers create situations and ways of interacting with students that promote modification of existing thinking, thereby creating a focus for observing the students' constructive process. There typically is continued interaction with the student (or students) by the researcher who is attentive to major restructuring of and scaffolding growth in the student's building of knowledge. In these ways, the teaching experiment makes use of and extends the idea of a clinical interview.

Yet a teaching experiment is similar to a task-based interview in several ways. First, a problematic situation is posed. Second, as the interviewer assesses the status of the student's reasoning in the process of interacting with the student, new situations are created in the attempt to better understand the student's thinking. Also, as in some task-based interviews, protocols may be modified as observation of critical moments suggests (Steffe and Thompson 2000).

### Significance

There is substantial and growing evidence that clinical task-based interviews and their variations provide important insight into subjects' existing and developing knowledge, problem-solving behaviors, and ways of reasoning (Newell and Simon 1972; Schoenfeld 1985, 2002; Ginsburg 1997; Goldin 2000; Koichu and Harel 2007; Steffe and Olive 2009; Maher et al. 2011). The interviews provide data for making students mathematical knowledge explicit. They offer insights into the creative activity of students in constructing new knowledge as they are engaged in independent and collaborative problem solving.

## Cross-References

▶ Inquiry-Based Mathematics Education
▶ Problem Solving in Mathematics Education
▶ Questioning in Mathematics Education

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## Teacher as Researcher in Mathematics Education

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### Keywords

Teacher as researcher; Teacher training; Teacher knowledge

The term "teacher as researcher" is usually used to indicate the involvement of teachers in educational research aiming at improving their own practice. The teachers-as-researchers movement emerged in England during the 1960s, in the context of curriculum reform and extended into the 1980s. Cochran-Smith and Lytle (1999) reviewed papers and books published in the United States and in England in the 1980s disseminating some experiences of teacher research. The main feature of the teacher research movement during this period seems to be an "explicit rejection of the authority of professional experts who produce and accumulate knowledge in "scientific" research settings for use by others in practical settings" (1999, p. 16). Within this movement, teachers are no longer considered as mere consumers of knowledge produced by experts, but as producers and mediators of knowledge, even if it is local knowledge, to be used in a specific school or classroom. This knowledge aims at improving teaching practice.

In mathematics education worldwide, the teachers-as-researchers movement has been the subject of debate within the mathematics educators' community and of several papers presenting results of these programs or discussing certain aspects of teacher research (see Huillet et al. 2011). In these debates, the contention pivoted around whether its outputs could be regarded as research. Many research endeavors conducted by teachers do not fill the requisites of formal research, such as systematic data collection and analysis, as well as dissemination of the research results. Some researchers distinguish two forms of teacher research in practice: formal research, aimed at contributing knowledge to the larger mathematics education community, and less formal research, also called practical inquiry or action research, which aims to suggest new ways of looking at the context and possibilities for changes in practice (Richardson 1994). A major aim of most action research projects is the generation of knowledge among people in organizational or institutional settings that is actionable – that is, research that can be used as a basis for conscious action (Crawford and Adler 1996).

The International Group for the Psychology of Mathematics Education (PME) started a working group called "teachers as researchers" in 1988. This group met annually for 9 years and published a book based on contributions from its members (Zack et al. 1997). The book comprised accounts of teachers' different experiences of enquiry in several countries and using several methods which basically aimed to improve teaching practice. In 2003 (PME27), members of a plenary panel intituled "Navigating between theory and practice. Teachers who navigate between their research and their practice" shared their experience on how they connect their role of teacher and researcher (Novotná et al. 2003). This panel was followed by a discussion group called "research by teachers, research with teachers" which met at PME in 2004 and 2005, and working sessions on "teachers researching with university academics" (2007–2009).

Some mathematics educators claimed that teachers as researchers typically focus on their pedagogical practice, rarely challenging the mathematical content of their teaching (Huillet et al. 2011). They support this claim in terms of a review of several papers of the teachers-as-researchers movement in education. In most of the papers reviewed, the focus is on teachers' classroom practices. They report on a study where teachers were not researching their own practice but the Mathematics for Teaching (MfT) limits of functions for secondary school level. They suggest that teachers get involved in research that puts mathematics at the core: research on Mathematics for Teaching, with attention to both mathematical and pedagogical issues and their intertwining in practice.

The idea of using research in teacher training arose long time ago. Yang (2009) contends that in China, a school-based teaching research system exists since 1952. In 1992, Clary claims that action research can become an efficient mean of training. In recent years, research conducted by teachers has become an important part of some teacher education programs (see Benke et al. 2008).

## Cross-References

- ▶ Communities of Inquiry in Mathematics Teacher Education
- ▶ Reflective Practitioner in Mathematics Education

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# Teacher Beliefs, Attitudes, and Self-Efficacy in Mathematics Education

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## Keywords

Beliefs; Attitudes; Self-efficacy; Affect; Teaching efficacy

Beliefs, attitudes, and self-efficacy are all aspects of the affective domain (McLeod 1992). The affective domain can be conceptualized as an internal representational system, comprising emotions, attitudes, beliefs, morals, values, and ethics (DeBellis and Goldin 2006). These are often placed on a continuum, with feelings and emotions at one end, characterized as short-lived and highly charged, and beliefs at the other end, typified as more cognitive and stable in nature (Philippou and Christou 2002). In the context of mathematics, the affective domain was introduced to explain why learners who possessed the cognitive resources to succeed at mathematical tasks still failed (Di Martino and Zan 2001; see also Affect in Mathematics Education). In the context of teachers of mathematics, over the last 30 years there has been a growing interest in how affective factors influence classroom practice, specifically with reference to beliefs (Thompson 1992; Philipp 2007), attitudes (Ernest 1989), and self-efficacy (Bandura 1997).

Philipp (2007) defines beliefs as "the lenses through which one looks when interpreting the world" (p. 258). There are many different types of beliefs that may influence teaching, including but not limited to beliefs about mathematics, beliefs about the teaching of mathematics, beliefs about the learning of mathematics, beliefs about students, beliefs about teachers' own ability to do mathematics, to teach mathematics, etc. Recognition of the power of beliefs to affect teaching has led to investigations into the beliefs of preservice teachers and the role that their experiences as mathematics students plays in their initial beliefs about what it means to teach mathematics (cf. Fosnot 1989; Skott 2001) and the role of teacher education programs to reshape these beliefs (Green 1971). Research on teachers' beliefs is complicated by a number of factors, including the often blurry boundary between beliefs and knowledge (Wilson and Cooney 2002) and beliefs and attitudes/emotions, as well as challenges in finding ways to measure beliefs and their impact. There is a substantial amount of literature on consistencies (e.g., Leatham 2006; Liljedahl 2008) and inconsistencies (e.g., Hoyles 1992; Speer 2005) between teachers' espoused beliefs, enacted beliefs, actual beliefs, and the attributed beliefs that the researchers assign to them.

Attitudes can be defined as "a disposition to respond favourably or unfavourably to an object, person, institution, or event" (Ajzen 1988, p. 4). Attitudes can be thought of as the responses that individuals have to their belief structures. That is, attitudes are the manifestations of beliefs (Liljedahl 2005). Negative attitudes towards mathematics can interfere with teacher learning. Unfortunately, these negative attitudes can be very difficult to change in adults (Evans 2000). Research on the relationship between teachers' attitudes and teacher practice is rare (Philipp 2007). In her cross-cultural study, Ma (1999) found that basic attitudes towards mathematics along with their lack of confidence in their own abilities affected teachers' willingness to engage in mathematical problem solving with their students. Ernest (1988) found some indications that attitudes towards teaching mathematics were more influential in teachers' practice than their attitudes towards mathematics. Other desirable attitudes of mathematics teachers that have been discussed in the literature are curiosity (Simmt et al. 2003), high motivation for success for themselves and their students (Rowan et al. 1997; Kukla-Acevedo 2009), as well as appreciation for the elegance of solutions and for a "good" problem (Ball 2002).

Teachers' self-efficacy sits on the boundary between beliefs and attitudes as it also incorporates emotional factors, i.e., confidence and anxiety. The research often distinguishes between, and sometimes conflates, personal teaching efficacy, teachers' beliefs in their own ability to teach effectively, and general teaching efficiency or outcome expectancy, which relates to teachers' beliefs that teaching can make a difference (Tschannen-Moran et al. 1998). Teacher self-efficacy has been found to influence teachers' attitudes and practice (Riggs and Enochs 1990), commitment to teaching (Coladarci 1992), and student achievement (Ashton and Webb 1986); however, research in this area is challenged by difficulties in clearly defining and measuring self-efficacy and its impact (Bandura 1993). There has also been considerable interest in the factors that influence self-efficacy (Bandura 1997), particularly in preservice teacher education, as it has been suggested that self-efficacy is most malleable early in teachers' careers (Hoy 2004). Interestingly, Swars et al. (2009) note that if teachers' efficacy beliefs are connected to the traditional teacher-centered teaching approaches, they will be in tension with the constructivist philosophies of current reform curricula in mathematics. So, if teacher efficacy matters at all, we need to ensure that it is associated with "appropriate" pedagogical beliefs.

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# Teacher Education Development Study-Mathematics (TEDS-M)

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## Keywords

Assessment; Knowledge; Mathematics; Pedagogy; Teacher education; Botswana; Canada; Chile; Chinese Taipei; Georgia; Germany; Malaysia; Norway; Oman; Philippines; Poland; Russian Federation; Singapore; Spain; Switzerland; Thailand; United States

## Definition

TEDS-M is the first empirical cross-national study of teacher preparation to collect data on the organization, curriculum, processes, and outcomes of teacher education from national probability samples of institutions, teaching staff, and students in 17 countries (Botswana, Canada, Chile, Chinese Taipei, Georgia, Germany, Malaysia, Norway, Oman, Philippines, Poland, Russian Federation, Singapore, Spain, Switzerland, Thailand, and the United States). TEDS-M was designed to focus on the outcomes of the mathematics preparation of teachers at the primary and lower secondary levels and to serve as a valuable tool to help inform and develop mathematics teacher preparation policy for future mathematics teachers.

The TEDS-M study was carried out under the aegis of the International Association for the Evaluation of Educational Achievement (IEA) and was made possible by a major grant from the US National Science Foundation. The College of Education at Michigan State University (MSU) and the Australian Council of Educational Research (ACER) were the joint international study centers (ISCs) for TEDS-M under the executive direction of Principal Investigator Maria Teresa Tatto of MSU. To design and carry out the study, the ISCs worked in collaboration with the International Association for the Evaluation of Educational Achievement (IEA) Data Processing and Research Center (DPC), the IEA Secretariat in Amsterdam, Statistics Canada, and the TEDS-M national research centers in the 17 participating countries. Together, these teams of researchers and institutions conceptualized the study, designed and administered the instruments, collected and analyzed the data, and reported the results.

The TEDS-M study findings in detail can be found in Tatto et al. 2012, and in the IEA website [http://www.iea.nl/?id=20] along with additional reports, and the publicly available data from the study. This entry focuses on the following persistent questions addressed by TEDS-M: What characterizes the institutions and the curriculum of teacher education programs? What are the

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characteristics of future primary and secondary teachers who are expected to teach mathematics? What are the outcomes of teacher education concerning professional teachers' knowledge and beliefs in mathematics?

### Institutions

The TEDS-M found that the nature of preservice teacher preparation institutions is diverse within and across countries. There is a wide array of programs residing in public and private institutions; some in universities and some in colleges outside universities. Some offer programs only in education, some are comprehensive with regard to the fields of study offered. Some offer university degrees, and some do not. Teacher education programs are typically categorized according to whether the opportunities to learn that they offer are directed at preparing future teachers for primary schools or for secondary schools. However, this categorization proved to be an oversimplification within the context of TEDS-M and likely within the larger international context. The terms *primary* and *secondary* do not mean the same thing from country to country. There is no universal agreement on when primary grades end and secondary grades begin. Therefore, programs were defined by *types*, according to their purposes using two organizational variables – grade span (the range of school grades for which teachers in a program were being prepared to teach) and teacher specialization (whether the program was preparing specialist mathematics teachers or generalist teachers). Primary program types were grouped according to whether they prepare specialist teachers of mathematics or generalist teachers and then subdivided into three groups according to the highest grade level for which they offer preparation: (1) program types that prepare teachers to teach no higher than grade 4, (2) program types that prepare teachers to teach no higher than grade 6, and (3) program types that prepare teachers to teach no higher than grade 10. The specialist teachers of mathematics constituted group (4). At lower secondary level, program types were placed in two groups, according to whether graduates from those program types would be eligible to teach (5) no higher than grade 10 or (6) up to the end of secondary schooling (Tatto et al. 2012).

### Curriculum

In the TEDS-M study participating institutions provided detailed information about the academic and professional content of their preservice teacher education programs. This included information about the number of subject areas graduates would be qualified to teach (i.e., specialists versus generalists) and the number of hours of instruction allocated to each area. Regarding specialization, one distinct pattern emerged. While most programs prepare future primary teachers to teach more than two subjects, those preparing future secondary teachers, for the most part, prepare them to teach one or two subjects. Regarding the relative emphasis given to specific areas of the teacher education program – as indicated by the number of hours allocated to each – the data revealed that teacher education programs generally offer courses in four areas: (a) liberal arts, (b) mathematics and related content (academic mathematics, school mathematics, and mathematics pedagogy), (c) educational foundations, and (d) pedagogy. Specifically regarding mathematics-related courses, TEDS-M found that in general, those programs that intend to prepare teachers to teach higher curricular levels such as lower and upper secondary provide, on average, opportunities to learn mathematics in more depth than those programs that prepare teachers for the primary level. Thus, on average, future lower and upper secondary teachers had greater opportunity to learn mathematics, both at the tertiary level as well as the school level, than future primary teachers. The exception to this pattern was found within the primary mathematics specialist group where higher opportunity to learn tertiary mathematics was reported more frequently than within any of the other program groups. Regarding school mathematics in particular, preservice teacher education programs in the countries participating in the study included all or a combination of some of the following topics: numbers; measurement; geometry; functions, relations, and equations; data representation, probability, and statistics; calculus; and validation, structuring, and abstracting. But these programs typically rationed the quantity and depth of future primary teachers' opportunities to learn school level mathematics (with primary teachers predominantly studying topics such as numbers, measurement, and geometry above any other topics). As programs prepare teachers for higher grades, the proportion of areas reported as having been studied increases. Importantly, TEDS-M found that the Asian countries and other countries whose future teachers did well on the TEDS-M assessments did offer algebra and calculus as part of future primary and lower secondary teacher education. And while the secondary curriculum across a large number of countries calls for instruction in basic statistics, the study found a general gap in this area in teacher education as reported by future teachers. This variability is mirrored in the opportunities to learn in the mathematics pedagogy domains between primary and lower secondary groups. In other areas TEDS-M found that opportunity to learn how to teach diverse students was highly variable with many countries reporting few or no opportunities to learn in this domain. Opportunity to learn general pedagogy was high among all primary programs and most secondary programs. Most programs preparing future primary teachers provide opportunities to make connections between what they learn in their programs and future teaching practice; but in the secondary program groups these opportunities were not as prevalent. The TEDS-M findings regarding overall opportunities to learn in mathematics teacher education reflect what seems to be a cultural norm in some countries, namely, that teachers who are expected to teach in primary – and especially the lower primary – grades need little in the way of mathematics content beyond that included in the school curriculum. The pattern among secondary future teachers is generally characterized by more and deeper coverage of mathematics content; however, there was more variability in opportunities to learn mathematics and mathematics pedagogy among those future teachers being prepared for lower secondary school (known in some countries as "middle school") than among those being prepared to teach Grade 11 and above. Not surprisingly, the countries with programs providing the most comprehensive opportunities to learn challenging mathematics had higher scores in the TEDS-M tests of knowledge. In TEDS-M, primary level and secondary level teachers in high-achieving countries such as Chinese Taipei, Singapore, and the Russian Federation had significantly more opportunities than their primary and secondary counterparts in the other participating countries to learn university and school level mathematics. This tendency seems to be closely related to the expectation that primary schools can be staffed with generalist teachers, defined in this study as teaching three or more subjects. Although this assertion may seem reasonable, the question of how much content knowledge teachers need to teach effectively is still an issue of much debate. The TEDS-M findings signal an opportunity to examine how these distinct approaches play out in practice. If relatively little content knowledge is needed for the lower grades, then a lesser emphasis on mathematics preparation and nonspecialization can be justified. The key question is whether teachers prepared in this fashion can teach mathematics as effectively as teachers with more extensive and deeper knowledge, such as that demonstrated by specialist teachers. Although TEDS-M has not provided definitive conclusions in this regard (this question necessitates studying beginning teachers and their influence on student learning), this question is currently under investigation by a study called FIRSTMATH, as a follow-up of TEDS-M, also funded by NSF and based at Michigan State University (Tatto 2010). What TEDS-M does show is that within countries, future teachers intending to be mathematics specialists in primary schools had higher knowledge scores on average than their generalist counterparts, and similarly, future teachers intending to teach upper secondary had higher scores on average than those intending to teach lower secondary grades (see Tatto et al. 2012).

### The Characteristics of Future Primary and Secondary Teachers Who Are Expected to Teach Mathematics

The TEDS-M study found that different countries' policies designed to shape teachers' career trajectories have a very important influence on who enters teacher education and who eventually becomes a teacher. These policies can be characterized as of two major types (with a number of variations in between): career-based systems where teachers are recruited at a relatively young age and remain in the public or civil service system throughout their working lives, and position-based systems where teachers are not hired into the civil or teacher service but rather are hired into specific teaching positions within an unpredictable career-long progression of assignments. In a career-based system, there is more investment in initial teacher preparation, knowing that the education system will likely realize the return on this investment throughout the teacher's working life. While career-based systems have been the norm in many countries, increasingly the tendency is toward position-based systems. In general, position-based systems, with teachers hired on fixed, limited-term contracts, are less expensive for governments to maintain. At the same time, one long-term policy evident in all TEDS-M countries is that of requiring teachers to have university degrees, thus, securing a teaching force where all its members have higher education degrees. These policy changes have increased the individual costs of becoming a teacher while also increasing the level of uncertainty of teaching as a career.

A major part of TEDS-M involved examining the participating countries' policies for assuring the quality of future teachers. The study found great variation in these policies, especially with respect to the quality of entrants to teacher education programs, the methods for assessing the quality of graduates before they can gain entry to the teaching profession (e.g., periodic formative and summative examinations both written and oral, a thesis requirement, and others), and at the organizational level, the accreditation of teacher education programs. The TEDS-M data indicated a positive relationship between the strength of quality assurance arrangements and country mean scores in the TEDS-M tests of mathematics content knowledge and mathematics pedagogy knowledge. Countries with strong quality assurance arrangements, such as Chinese Taipei and Singapore, scored highest on these measures. Countries with weaker arrangements, such as Georgia and Chile, tended to score lower on the two measures of future teacher knowledge. These findings have implications for policymakers concerned with promoting teacher quality. Quality assurance policies and arrangements can make an important difference to teacher education. These policies can be designed to cover the full spectrum, from polices designed to make teaching an attractive career to policies for assuring that entrants to the profession have attained high standards of performance. The TEDS-M findings point to the importance of ensuring that policies designed to promote teacher quality are coordinated and mutually supportive. The TEDS-M data shows that countries such as Chinese Taipei and Singapore that do well on international tests of student achievement, such as TIMSS, not only ensure the quality of entrants to teacher education but also have strong systems for reviewing, assessing, and accrediting teacher education providers. They also have strong mechanisms for ensuring that graduates meet high standards of performance before gaining certification and full entry to the profession.

Aside from qualifications, TEDS-M found that future teachers being prepared to teach at the primary and secondary school levels were predominantly female, although there were more males at the higher levels and in particular countries. They seemed to come from well-resourced homes, and many reported having access to such possessions as calculators, dictionaries, and DVD players, but not personal computers – now widely considered essential for professional use. The latter was especially the case among teachers living in less affluent countries such as Botswana, Georgia, the Philippines, and Thailand. The TEDS-M survey found that a relatively small proportion of the sample of future teachers who completed the survey did not speak the official language of their country (which was used in the TEDS-M surveys and tests) at home. Most future teachers described themselves as above average or near the top of their year in academic achievement by the end of their upper secondary schooling. Among the reasons the future teachers gave for deciding to become teachers, liking working with young people and wanting to influence the next generation were particularly prevalent. Many believed that despite teaching being a challenging job, they had an aptitude for it (see Tatto et al. 2012).

### The Outcomes of Teacher Education: Mathematics Professional Knowledge and Beliefs

#### Mathematics Content and Mathematics Pedagogy Content Knowledge for Teaching

Regarding the mathematics and mathematics pedagogy content knowledge of future teachers, the TEDS-M study provides the first solid evidence, based on national samples, of major differences across countries in the (measured) mathematics knowledge outcomes of teacher education. The answer to the TEDS-M research question about the teaching mathematics knowledge that the future primary and secondary teachers had acquired by the end of their teacher education was clear, for the most part, this knowledge varied considerably among individuals within every country and across countries. The difference in mean mathematics content knowledge (MCK) scores between the highest and lowest-achieving country in each primary and secondary program group was between 100 and 200 score points, or one and two standard deviations. This difference is a substantial one, comparable to the difference between the 50th and the 96th percentile in the whole TEDS-M future teacher sample. Differences in mean achievement across countries in the same program group on mathematics pedagogical content (MPCK) were somewhat smaller, ranging from about 100–150 score points. Therefore, within each program group (e.g., preparing teachers to teach *(1) no higher than grade 4, (2) no higher than grade 6, (3) no higher than grade 10, (4) as specialists, (5) no higher than grade 10, and (6) up to the end of secondary schooling*) and by the end of the teacher preparation programs, future teachers in some countries had substantially greater mathematics content knowledge and mathematics pedagogical content knowledge than others. On average, future primary teachers being prepared as mathematics specialists had higher MCK and MPCK scores than those being prepared to teach as primary generalists. Also, on average, future teachers being prepared as lower and upper secondary teachers (e.g., group 6) had higher MCK and MPCK scores than those being prepared to be only lower secondary teachers. In the top-scoring countries within each program group, the majority of future teachers had average scores on mathematics content knowledge and mathematics pedagogy content knowledge at or above the higher anchor points (see Tatto et al. 2012). In countries with more than one program type per education level, the relative performance on MCK and on MPCK of the future teachers with respect to their peers varied. For instance, the mean mathematics content knowledge score of future primary teachers in Poland ranked fourth among five countries preparing lower primary generalist teachers, but first among six countries preparing primary mathematics specialists. An important conclusion of the TEDS-M study is that the design of teacher education curricula can have substantial effects on the level of knowledge that future teachers are able to acquire via the opportunities to learn provided to them (see Tatto et al. 2012).

#### Beliefs

The TEDS-M study assessed beliefs about the nature of mathematics (e.g., mathematics is a set of rules and procedures, mathematics is a process of enquiry), beliefs about learning mathematics (e.g., through teacher direction, through student activity), and beliefs about mathematics achievement (e.g., mathematics is a fixed ability) (Philipp 2007; Staub and Stern 2002). We found that in general, educators and future teachers in all countries were more inclined to endorse the pattern of beliefs described as conceptual in orientation and less inclined to endorse the pattern of beliefs described as computational or direct-transmission. Several countries showed endorsement for the belief that mathematics is a set of rules and procedures. The view that mathematics is a fixed ability was a minority one in all countries surveyed, yet its existence is still a matter of concern because it implies a less inclusive approach to teaching mathematics to all learners. The TEDS-M data shows important cross-country differences in the extent to which such views are held. The program groups within countries endorsing beliefs consistent with a computational orientation were generally among those with lower mean scores on the knowledge tests. In some high-scoring countries on our knowledge tests, however, future teachers endorsed the beliefs that mathematics is a set of rules and procedures as well as a process of enquiry (see Tatto et al. 2012). The TEDS-M findings thus showed endorsement for both of these conceptions within mathematics teacher education. This finding suggests the importance for teacher education institutions to find an appropriate balance on these conceptions when designing and delivering the content of their programs (Tatto 1996, 1998, 1999).

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▶ Mathematics Teacher Education Organization, Curriculum, and Outcomes

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# Teacher Supply and Retention in Mathematics Education

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## Keywords

Recruitment; Supply; Attrition; Retention; Morale

## Definition

Issues and strategies concerning the supply and retention of high-quality mathematics teachers in primary and secondary classrooms.

## Characteristics

The supply and retention of high-quality mathematics teachers are crucial to the success of any education system. Faced with mounting evidence that the most important in-school influence on student achievement is teachers' knowledge and skill (Hattie 2009), policy makers are paying closer attention to strategies likely to recruit, prepare, and retain the best possible teachers. While policy decisions about pupil-teacher ratios, initial teacher education pathways, and teaching conditions influence countries' overall supply and demand balance, a universal and relatively unfulfilled demand for high-quality mathematics teachers prevails. For some developing countries, this demand is evident across all sectors. For countries that – in various ways – produce sufficient numbers of generalist teachers for primary schools, the focus is on a search for ways to ensure sufficient numbers of well-qualified mathematics specialist teachers for upper primary and/or secondary schools (Tatto et al. 2012).

In looking to address teacher quality, concerns about the sufficiency of mathematics and pedagogical content knowledge, both at the recruitment and graduate phase of teacher education, are central. A trend is for countries to require teacher graduates to meet additional criteria measured by tests of mathematics knowledge or periods of probationary teaching in schools before gaining professional certification. Efforts to increase recruitment of potential mathematics teachers have prompted the design of alternative teacher education pathways which provide additional mathematics content focus. An allied recruitment issue is the trend for an increasing proportion of career switchers to enter teaching. For many career switchers, their experiences of learning mathematics are distal and often confined to mathematics service courses. Findings from a large-scale survey study of elementary and middle school teachers in the USA (Boyd et al. 2011) suggesting that career switchers may be less effective at teaching math than other teachers during their first year of teaching warrant further investigation of how the use of mathematics in previous careers might impact on the quality of students' mathematical learning experiences and design of teacher education programs.

Echoing findings from the UK and South Africa, Ingersoll and Perda (2010) claim that the shortage of quality mathematics teachers in the USA is not just an issue of recruitment – but also an issue of retention. Common across many education systems, high teacher attrition rates are linked to inadequate degree of classroom autonomy, inadequate provision of professional development opportunities, unrealistic workloads, and pupil discipline and behavior problems. Collectively, these contribute to high levels of dissatisfaction, high stress levels, and low teacher morale. Importantly, in many countries retention impacts differentially – with high poverty, high minority, and urban public schools experiencing higher rates of turnover.

Efforts to increase retention rates focus on induction programs for newly qualified teachers. However, while access to induction programs is becoming commonplace, the effectiveness of induction varies across and within educational systems (Britton et al. 2012). Charged with enacting reforms in mathematics teaching, beginning teachers need opportunities to engage and experiment with ambitious mathematics teaching within a culture of expansive whole-school learning. Likewise, efforts to support experienced mathematics teachers' professional growth have highlighted the value of communities of practice that lead to increased investment by teachers in ways that develop long-term teaching trajectories while simultaneously strengthening their professional identities as mathematics teachers. To counter effects of low morale, professional learning experiences must involve deliberate acknowledgement of teachers' strengths of current practices (Graven 2012). Importantly, efforts focused on building teacher quality need also to be partnered with supportive teacher education contexts (Artzt and Curcio 2008), supportive school contexts (Johnson 2012), and informed by evidenced-based research (Alton-Lee 2011).

## Cross-References

▶ Communities of Practice in Mathematics Teacher Education
▶ Mathematical Knowledge for Teaching
▶ Mathematics Teacher Identity
▶ Models of Preservice Mathematics Teacher Education
▶ Teacher Education Development Study-Mathematics (TEDS-M)

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# Teacher-Centered Teaching in Mathematics Education

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## Keywords

Intellectual heteronomy; Direct instruction; Explicit instruction; Special education

## Definition

Teacher-centered teaching is an approach to teaching that places the teacher as the director of learning and is mainly accomplished by lecture, repetitive practice of basic skills, and constructive feedback.

### Intellectual Heteronomy

Many researchers have contended that one of the most important contributions that education can make in individuals' lives is to their development of autonomy (Piaget 1948/1973). Autonomy is defined as the determination to be self-governing to make rules oneself rather than rely on the rules of others to make one's decisions (heteronomy). Kamii (1982) suggests that autonomy is the ability to think for oneself and make decisions independently of the promise of rewards or punishments. In relation to education, Richards (1991) distinguishes between two types of traditions in the mathematics education of children, what he terms *school mathematics* and *inquiry mathematics*. School mathematics is what is typically thought of as a teacher-directed environment in which learning mathematics is a process of both memorizing teacher-modeled rules and procedures and solving routine problems that often have little significance to the real world until mastery of the teacher's solution methods is attained. Heteronomy is fostered here as students learn to replicate what the teacher has shown them.

*Teacher-centered instruction* has been around for years and generally refers to a complex pedagogy that places the teacher as the mathematical authority for learning. This approach to teaching and learning has enjoyed prominence for decades despite recent pushes towards student-centered teaching. Teacher-centered classrooms can best be described as environments in which the teacher emphasizes mastery of content and basic skills and transfers knowledge primarily by lecture and repetition. The students are viewed as recipients of information and can master the skills by repeated practice and memorization. The term *teacher-centered instruction* is also known as *direct instruction* and *explicit instruction* in educational circles.

### Contrast with Student-Centered Instruction

Recent research has suggested that teachers shift their practices towards more student-centered instruction (Yackel and Cobb 1994; Hiebert et al. 1997; Tarr et al. 2008) primarily to promote higher and deeper engagement of students with the mathematics. Additionally, *Mathematics Education in Europe* reports that many European countries have reconceptualized their mathematics instruction towards more student-centered teaching (http://eacea.ec.europa.eu/education/eurydice/documents/thematic_reports/132EN_HI.pdf).

While some researchers have shown that students perform better on standardized tests when taught using teacher-centered instruction, others have shown the opposite, leaving room for exploring which of the characteristics of both approaches can be used to maximize learning. The table below illustrates the major differences between teacher- and student-centered approaches and represents a merging of two tables found at the sites: www.nclrc.org/essentials/goalsmethods/learncentpop.html and http://assessment.uconn.edu/docs/TeacherCenteredVsLearnerCenteredParadigms.pdf.

| Teacher centered | Learner centered | Theme |
|---|---|---|
| Focus is on instructor | Focus is on both students and instructor | Role of instructor |
| Instructor talks; students listen passively | Students interact with instructor and one another, students are engaged | |
| Mathematical learning is transmitted from teacher to student | Students construct knowledge through gathering and synthesizing information and integrating it with the general skills of inquiry, communication, critical thinking, problem solving, and so on | Lesson design |
| Emphasis is on acquisition of knowledge outside of the context in which it will be used | Emphasis is on using and communicating knowledge effectively to address issues arising in real-life contexts | |

(*continued*)

| Teacher centered | Learner centered | Theme |
|---|---|---|
| Lessons are designed so that the mathematics can be broken into small manageable pieces | Lessons are designed around a problematic situation that students must solve without much pre-lecture. Students mathematize the situation | |
| Students work alone | Students work in pairs, in groups, or alone depending on the purpose of the activity | Role of students |
| Environment is viewed as competitive and individualistic | Culture is cooperative, collaborative, and supportive | |
| Instructor monitors and corrects every student utterance | Students talk without constant instructor monitoring; both instructor and students provide analyze solutions, particularly when questions arise | Assessment |
| Instructor answers students' questions | Students answer each other's questions using instructor as an information resource | |
| Assessment is used to monitor learning | Assessment is used to monitor learning and to inform instruction | |
| Classroom is quiet | Classroom is noisy and interactive | |

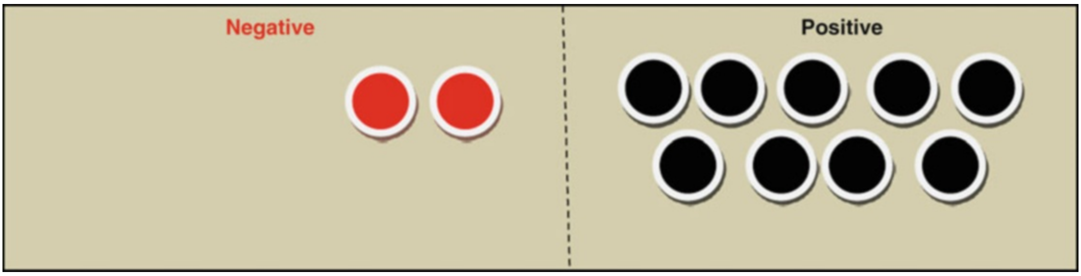
## Characteristics

### Role of the Instructor

The role of the instructor in a teacher-centered classroom is to impart knowledge onto the student through lecture and modeling of the mathematics concept(s). A typical lesson format might consist of reminding students of the work they did previously or eliciting prerequisite knowledge that is needed to begin a new concept. Once the groundwork has been laid by the teacher, she states the objectives for the class period and proceeds to lecture, drawing upon a variety of sources. Generally speaking, the teacher's goal is to illustrate how and why a basic skill or concept works by showing how to solve a diverse set of problems. She breaks the modeling down into chunks that will be more easily understood by the students. When the modeling is complete and student questions have been answered, the teacher will have them practice solving very similar problems either independently or with peers. At this time, she will walk around the room to monitor student behavior and provide positive and/or negative feedback. The guidance here is heavy with students practicing and making corrections to their errors until mastery is attained. Most of the talk is teacher directed with little student talk. There are obviously variations in this lesson design and often direct instructors attempt to make the lesson more engaging by relating some of the mathematics to real life and by using manipulative or notations. These manipulatives and diagrams/notations are controlled by the teacher and used as a modeling device.

### Direct Instruction Lesson Design

In a direct instruction approach, the teacher might begin with students working through a series of prerequisite skills, like whole number operations and inequalities (e.g., $3 < 8$). When the prerequisite skills are mastered, then the teacher explicitly states the objectives. To help students see the importance of integers, she might show some examples of real-world situations that illustrate integer concepts. Then, the teacher models for students how to solve problems. For example, it is common to show students how to order integers on a horizontal number line by starting with a zero marked near the middle of the line and counting the necessary spaces either left or right for each integer. In a direct instruction environment, the lesson is carefully structured and the teacher is the center of the activity, showing students how to place integers on the number line and order integers appropriately. During this part of instruction, the teacher asks questions as a way to monitor whether students are able to repeat the skills she has shown them. Next, the class enters a period of guided practice with the teacher monitoring student progress and giving immediate feedback. For students still struggling with the skill, she might give prompts or hints to help them along their




**Teacher-Centered Teaching in Mathematics Education, Fig. 1**

learning. When students demonstrate accuracy without teacher assistance, they are asked to work independently to reach mastery of the skill.

An instructional strategy that is becoming increasingly acknowledged as useful for lessons that follow direct instruction is called concrete-semiabstract-abstract (CSA) design. Take, for example, the teaching integer operations. A CSA approach would have students represent integers with two colored chips (see also Bennet and Musser 1976; Maccini and Hughes 2000; Flores 2008), black being positive and red representing negative integers. Basically, a black chip and a red chip cancel each other out and represent what is called a *zero pair*. What is crucial to understanding this scenario is that zero added to any amount will not change the original amount. For example, if a student has 5 black chips (a + 5 value) and adds 2 red and 2 black chips, she has to recognize that, although the total number of chips has increased to 9, the value of 5 remains the same since the 4 chips represent zero.

In the example below, students have already reached mastery of the integer concept of ordering positive and negative numbers and are being introduced to the operation of addition for the first time. Following the CSA design, the teacher first shows students a workmat separated into two areas, a negative and a positive area. At the concrete phase of instruction, students are taught how to model an integer word problem with chips. For example, consider the problem, "In State College, Pennsylvania, the temperature on a certain day was −2 °F. The temperature rose by 9 °F by the afternoon. What was the temperature that afternoon?" (Maccini and Hughes 2000). Teachers would model the problem by placing two red chips on the negative side of the mat and nine black chips on the positive side (Fig. 1).

A conversation might look like this:

T: What is the temperature at the beginning of the day? (T displays a mat with chips on the overhead projector while each student has the same at their desks.)

Ted: −2.

T: So since the temperature is 2° below zero, is negative, we would put two red chips on the negative side. Go ahead, get two red chips and put them on your own mat. Now, what does the next part of the word problem say? Maya?

Maya: The temperature rose by 9 °F by the afternoon.

T: The temperature rose by 9 °F by the afternoon. That means we should put nine black chips on the positive part of the mat. Please put those on there (Maya puts nine black chips on the positive side). Now, to find out the temperature at the end of the day, we need to take out zero pairs. A zero pair is one red chip and one black chip. They equal zero because one positive and one negative chip cancel each other out, make 0. So we can just take them away. I'll take away one set (physically removes one red and one black chip together). You do the next one Grace. (Grace comes to the overhead and takes a red and black chip off the mat). How many chips do we have left?

Gwen: 7.

T: Right, seven chips. What color are they? (Students say "black.") Right. Are black chips positive or negative?

Students: Positive.

T: OK, so it is a positive 7 °F at the end of the day. Let's try another one.

As students work out more and more examples, either with the teacher or in small groups, the teacher walks around and gives immediate feedback concerning the correctness of their methods and answers. In the next phase, the semi-concrete, students are given a worksheet that is structured so that students continue their previous activity but instead of using actual chips, they are to draw their chips on the work paper and solve the problem with their drawings. Again, the teacher provides positive and/or corrective feedback as the students solve these problems.

Finally, in the abstract phase, students are asked to write symbolic equations for integer word problems and use rules for addition/subtraction of integers to solve them. Mastery of the skills at each of the phases is required before moving on to the next phase. If a student produces an incorrect answer or method, the teacher reteaches by modeling the methods again. The students practice the modeled methods until mastery is attained.

***The Role of the Students***. As can be seen in the example above, the teacher does a great job of modeling and explaining to the students the steps behind integer addition. She has placed integers in a real-world context, using manipulatives to help students make sense of the concept. The students, for their part, are required to follow her steps and answer questions as best they can throughout the modeling. The teacher has broken down integer operations into one small chunk, working with addition first. Once students master addition problems, through repetition and feedback, the teacher will move to subtraction. The role of the student is to practice the skill enough to master the content. The classroom environment is fairly quiet with little interaction between students, unless the teacher allows them to practice with one another.

***Assessment***. Assessment is typically conducted as a way to monitor student success in performing the skills that have been taught. In this way, assessment occurs on a regular basis and immediate feedback is given to students. The goal is to reach mastery on basic skills and move on to more sophisticated ones.

### Further Areas of Research

There are a number of studies that show students who have received direct instruction outperform students who received student-centered instruction. Typically, these tests revolve around mathematical achievement on calculational proficiency. However, critics of the teacher-centered approach also cite studies showing that students who received student-centered instruction perform equally well on calculational problems and outperform their teacher-centered peers on critical thinking problems. It is clear that research shows disparate results and the mathematics education field must work towards reconciling these differences. One suggestion that seems to be popular in the special education field, and supported by statements from the National Mathematics Advisory Panel (2008), is to merge explicit and student-centered instruction together (Hudson et al. 2006; Scheuermann et al. 2009). Proponents of this approach are typically from special education and advocate making instruction more realistic and hands on (like the integer example), but simultaneously scaffolding students' learning by explicit and direct means.

## Cross-References

▶ Learner-Centered Teaching in Mathematics Education

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# Teaching Practices in Digital Environments

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## Keywords

Teaching practices; Representational infrastructures; Communication infrastructures; Graphical, numerical, symbolic, and geometric environments; Community of practice; Community of inquiry; TPACK theory; Instrumental approach; Documentational genesis; Instrumental orchestration

## Definition

The teacher's activities and methodologies with the use of digital technologies: changing uses of digital technologies in the last years, main and recent issues, theoretical perspectives, and considerations for the future.

## Characteristics

Much of the research related to the use of digital technologies in mathematics education has focused on learners and on the particular effects that a given technology might have on the nature and quality of student learning. Over the past decade, in a shift that has occurred more generally in mathematics education research, researchers have begun to pay more attention to the existing practices of teachers, as they relate to the use of technology, and the changes that such practices might or must undergo in order to more effectively make use of available technologies. This change of focus is driven in part by the fact that despite the availability of and institutional support for digital technologies, the everyday practice of most teachers has changed little with respect to the use of technology (Laborde 2008). This entry focuses on the changing uses of digital technologies over the past 30 years and provides an overview of the theoretical perspectives that have been developed over the past decade to study ways of understanding and supporting changing teaching practices.

### Teachers' Changing Uses of Digital Technologies

Since its introduction in schools – in the 1980s – the use of ICT (information and communication technology) in teaching mathematics has had two main functions: (a) as a support for the organization of the teacher's work (producing work sheets, keeping grades) and (b) as a support for new ways of doing and representing mathematics. The past decade has seen an evolution of technology itself with the introduction of new communication and representational infrastructures (Hegedus and Moreno-Armella 2009). The representational infrastructures used in mathematics education can involve specific software for teaching topics such as statistics, algebra, and modelling as well as graphical, numerical, symbolic, and geometric environments that are used to represent mathematical objects. Over time, teachers have moved from content-specific graphical and mathematical programs toward more generic and multi-representational environments (Thomas 2006).

The communication infrastructures (such as electronic mail, web platforms, and social networks) have become useful both for teacher professional development and for teaching practice in the classroom. In the first case, teachers can become members of communities of colleagues in the same school, in a network of schools or in a teacher education program (as community of practice, in the sense of Wenger 1998), or in a research program (as community of inquiry, Jaworski 2006). They can participate in these communities in synchronous and asynchronous activities aimed at sharing materials, designing curricular plans, doing teaching experiments, collecting data for assessment, and discussing results. In the second case, they can organize their classroom activities in ways that combine face-to-face interactions with distance ones mediated by these infrastructures.

The use of digital environments in classroom in recent years has changed from a more "private" to a "public" use that integrates the private use (Hegedus and Moreno-Armella 2009; Robutti 2010), as predicted in Sinclair and Jackiw (2005). This shift, which echoes the historical shift from the use of individual handheld slate to blackboards, can be described again in terms of the technology available. White the computer laboratory and handheld technology (calculators) settings featured individual or small group interactions with the technology that could not easily be shared with the whole class and with new infrastructures combine the public-private uses or reverse the dominant interaction. In the former case, handheld devices can be connected to the teacher's computer, which projects student-generated work to a large public screen or to an interactive whiteboard. Further, Robutti (2010) documents that "blended" approach, in which the public screen not only displays the student work in real time, providing immediate feedback, it enables individual students to compare and connect their own work with that of others. In the latter case, teachers can use projectors or interactive whiteboards to enable whole classroom sharing of digital representations, thus retaining control of the use of the technology and reducing the need for student instrumentation – such a modality has become increasingly frequent in both primary and secondary school classrooms.

## Teacher Practice and Technology: Theoretical Perspectives

Over the past decade, there has also been a shift in focus from the learner to the teacher, echoing the broader increase of attention in mathematics education research. Early research involved studying the variables, such as attitudes and levels of proficiency, which affect teachers' use of a given technology (Thomas 2006). Subsequent attention was placed on the interaction that might occur between teachers' proficiency with and attitude toward technology use and their proficiency with and attitude toward mathematics. For example, in the case of DGS (and other dynamic mathematical environments), the dynamic/visual conception of a given mathematical object or relationship that the technology offers might not accord with the static/algebraic conception that a given teacher has developed – or that the textbook and assessment items assume. The resulting mismatch will have an important effect on the way a given technology is used (see Laborde 2001; Sinclair and Robutti 2012) and on the related learning process.

By extending the well-known PCK framework to TPACK (Koehler and Mishra 2009), researchers have also drawn attention to the way new technology resources interact with teacher's pedagogical and content knowledge. This framework highlights the fact that technology use cannot change (or be changed) in isolation of other aspects of teacher practice. This echoes the extensive research that documents the way in which the use of technology changes the learner and the learner's understanding. With a dual focus on the teacher and the learner, Borba and Villarreal (2005) have coined the phrase "humans-with-media," a term that emphasizes the way in which the technology is considered part of these communities and can influence teaching and learning processes.

More recently, however, researchers have sought to better theorize their understanding of teaching practices using technology in such a way to move beyond the logical demarcation of types of teacher knowledge perpetuated by TPACK. Two main approaches have emerged. Ruthven's (2009) Structuring Features of Classroom Practice framework identifies five structuring features of classroom practice that shape the choices that teachers make when integrating new technologies: working environment, resource system, activity structure, curriculum script, and time economy. So, for example, in terms of *resource system*, teachers must decide how they will build a coherent set of elements that function in a complementary manner in the classroom – this might involve choosing a digital tool that uses the same kind of notation that is used in the textbook or encouraging students to take notes on their laptops, where their technology-based explorations are taking place, instead of in their notebooks. Ruthven et al. (2009) used this framework to identify the various adaptations of teaching practices in their study of teachers' use of graphing software at lower secondary level.

The second approach draws on the notion of "instrumental genesis," which has been extensively used to study the way in which tool and person coevolve and which has focused on the ways in which learners go from being untutored operators of a given tool to being proficient users. Guin and Trouche (2002) extend this notion to "instrumental orchestration," which focuses more specifically on technology integration in teaching and learning. In particular, instrumental orchestration involves practices that take into account both the constraints involved in using a tool and the way in which students' use of the tool develops. Orchestration is described in terms of two variables: (1) "didactical configuration" is the arrangement of artifacts in the environment, and (2) "exploitation mode" is the way the teacher decides to exploit a didactical configuration for the benefit of her didactical intentions. Drijvers et al. (2010) introduce also the "didactical performance," which involves the ad hoc decisions taken while teaching about how to perform in the chosen didactic configuration and exploitation mode.

Gueudet and Trouche (2009) call "documentational genesis" the way teachers go from being untutored operators of materials (any kind of teaching resource, including digital technologies) to being proficient users of them. As teachers develop ways of using these materials, they turn into documents that have stable usage schemes. This approach enables researchers to attend to the broad range of materials involved in a particular lesson, as well as the relationship between a teacher's preparation of it and its implementation in the classroom.

### Central Considerations for the Future

As Stacey (2002) argues, "new technology renders some traditional examination questions obsolete and others problematic" (p. 11). As such, even in situations where there is a high adoption of technology in teaching (e.g., the use of CAS in university-level courses in Canada) (Buteau et al. 2009), assessment continues to be pencil and paper driven. However, many have argued that until teachers develop practices in which technology is used both in formative and summative assessment, the putative effects of these technologies (increasing the focus on conceptual understanding, enabling broader forms of mathematical expression, empowering student agency and creativity, etc.) will be greatly compromised.

## Cross-References


▶ Information and Communication Technology (ICT) Affordances in Mathematics Education
▶ Instrumentation in Mathematics Education
▶ Learning Practices in Digital Environments
▶ Technology and Curricula in Mathematics Education

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# Technology and Curricula in Mathematics Education

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## Keywords

Technology; Mathematics; Curriculum

## Definition

The relationship between technology and mathematics curriculum from the perspective of research, mathematical practices in the classroom, and recent learning theories with mathematical digital technologies.

## Characteristics

Mathematics was one of the earlier subjects to make use of the computer in the classroom, and the first digital computers were primarily developed to solve differential equations, having evolved from Babbage's automatic calculating machine. In the 1970s computer programming began to be taught in schools in some countries around the world, although this was not explicitly linked to the mathematics curriculum, despite the fact that programming is strongly related to the idea of variable and algorithm. At a similar time the Logo programming language was developed by Seymour Papert and colleagues with its best known feature being an on-screen turtle which could be controlled by programming commands (Papert 1980). During the 1980s Logo began to be used in schools, and evidence from empirical studies suggested that Logo could engage young students in exploring mathematical ideas such as ratio and proportion, geometry, variables, functional variation, recursive processes, mathematical generalization, and its symbolization (Hoyles and Noss 1992). Interestingly different perspectives began to emerge in terms of the relationship of technology use and the mathematics curriculum, with some people arguing that technology use in the mathematics classroom should fit with the existing curriculum and others arguing that technology should be used to introduce students to complex mathematical ideas that had previously been inaccessible to them, for example, introducing the underlying ideas of calculus to primary school students (Kaput 1994). The Logo programming language had been developed as a means of transforming school mathematics and the curriculum. By contrast in the 1980s, dynamic geometry environments were developed (e.g., The Geometer's Sketchpad and Cabri) to support the learning of geometry within the curriculum, although dynamic geometry environments have evolved so that they can be used within different curriculum areas (e.g., functions and trigonometry).

By the early 1990s a wide range of technologies were available to be used within school mathematics, including graph plotting packages, spreadsheets, and computer algebra packages which have been developed for university and professional mathematicians (e.g., Mathematica). Sometimes these technologies have been designed to fit with particular aspects of the school mathematics curriculum (e.g., the graph plotting package Autograph or the statistics education package Fathom), sometimes they have been designed to make mathematics accessible to new groups of students (e.g., SimCalc which was designed to democratize the learning of calculus), and sometimes they have been adapted from technologies that had not been designed for educational purposes (e.g., spreadsheets, see, e.g., Sutherland and Rojano 1993).

Nowadays technologies for learning mathematics are increasingly available on mobile devices, which include calculators and tablet computers, and such devices linked to the Internet can provide students with access to a wide range of mathematical digital technologies. However, research clearly shows that whatever the designer's intentions students can use technologies developed for learning mathematics for nonmathematical purposes (Bartolini Bussi and Mariotti 2008). For example, students might use dynamic geometry tools to draw shapes or pictures on the screen instead of constructing mathematical objects using geometrical properties. Theories of learning with mathematical digital technologies provide explanations for why this is the case and at the same time offer a framework for developing classroom practices that exploit the potential of technology for mathematical learning. For example, the theory of instrumental genesis (Artigue 2002) distinguishes between the technology (artifact) and the instrument, separating what relates to the intention of the designer (the technology) and what is constructed by the user and relates to the context of use (the instrument). This theory has been used to explain the discrepancy between the students' behavior and the teacher's intentions and points to the importance of the design of mathematical activities and the role of the teacher within technology-enhanced learning environments. Other theories such as the theory of semiotic mediation and the theory of constructionism also provide frameworks for designing technology-enhanced learning environments for mathematics (Drijvers et al. 2010).

The extended presence of computers, calculators, and mobile devices in schools, as well as three decades of using these technologies in education at an experimental level, has resulted in an increasing interest in the relationship between mathematics curriculum and technology development among researchers, teachers, parents, educational authorities, and curriculum designers and developers. Nevertheless, the potential impact that such technologies may have in the official curriculum has been and still is a controversial issue in these communities. Despite this controversy, research experiences with a variety of computer programs and tools have already influenced curriculum changes in many countries, and this has happened in different ways, such as (1) connecting different mathematics curricular areas, both at the same and at different school levels, due to the possibility to work with multiple digital representations (which are dynamically linked to each other) of one concept or situation (e.g., the concept of function); (2) giving students an early access to powerful mathematical ideas (e.g., mathematics of variation); (3) incorporating new topics in the curriculum (e.g., 3D geometry); (4) making it possible for students to analyze large authentic data sets in statistics; and (5) removing classic topics. With regard to this last point, in the early 1990s, manipulative aspects of algebra were substantially reduced in the UK national curriculum at secondary school level (Sutherland 2007) but have since been reintroduced and are increasingly emphasized due to an appreciation of the importance of symbolic manipulation with paper and pencil for developing symbol sense. In this respect there is a continuing debate about the relative importance of paper-and-pencil mathematics versus computer-based mathematics in terms of developing mathematical knowledge and understanding, with many people arguing that digital technologies for mathematics do not replace paper-and-pencil technologies.

The relationship between technology and the mathematics curriculum is constantly in flux, changing over time and varying between countries, from countries like the USSR in the 1980s, which considered informatics as "a new mathematics" and introduced meta-content, such as discovery, collaboration, generalization, transfer, and mathematics across different subject areas (Julie et al. 2010), to countries that explicitly introduce in the mathematics curriculum the use of software such as dynamic geometry, Logo, spreadsheets, graphing calculators, computer algebra systems (CAS), and applets for the teaching of specific mathematical domains either in a compulsory way (e.g., Hong Kong, Russia, France) or in an optional way (e.g., South Africa, Mexico, Brazil, and Central American countries). A common denominator in many of these examples is a disparity between implementation of the use of such technology in the mathematics classroom (which tends to be teacher-centered) and the pedagogical strategies suggested in the curriculum documents (such as learner-centered and exploratory or experimental approaches). Overall it is widely recognized that at the level of the classroom, mathematics teachers are not exploiting the potential of technologies for learning mathematics despite what might be specified in the curriculum and despite the research evidence that indicates the ways in which technologies could be used in mathematics education (Assude et al. 2010).

As part of the non-static relationship of technology with curriculum, technological evolution and innovation are also potential factors of mathematics curricular changes. The progress made in improving dynamic geometry programs offers students access to advanced geometric ideas in three dimensions, mainly changing the point of view of a 3D scene and in this way, visually obtaining full information of a 3D object. In computer algebra systems (CAS), it is possible to either de-emphasize manipulative skills and focus students' work on conceptual tasks (Kieran 2007) or promote conceptual and technical aspects of mathematics (Lagrange 2003). Applets and other programs run on digital tablets which allow students to physically touch and manipulate representations of mathematical objects. Recent versions of spreadsheets offer a friendly environment for mathematical modeling tasks using (hot-linked) graphical, symbolic, and numeric representations of phenomena of the physical world, which opens up the possibility of promoting mathematical modeling approaches more generally in the curriculum. For example, the open-source software GeoGebra favors the connection between Euclidean, Cartesian, and analytic geometry.

Beyond the potential or real influence of the use of digital technologies in the official mathematics curriculum, the actual implementation of effective modes of technology use in the mathematics classroom is still a big challenge, a situation that represents an opportunity for promising future research. In this regard, the experience from longitudinal studies dealing with alternative technology-enhanced mathematics curricula provides an important antecedent. One of the innovations in some of these experiences is the use of technology for curriculum design and development with a functional approach to algebra at secondary, tertiary, and university levels. For example, in the pioneering technology project Computer-Intensive Algebra (Penn State University), beginning algebra concepts were introduced in mathematical modeling contexts, and students used specialized software to work with numerical, graphical, and symbolic representations of functions of one and two variables (Fey, et al. 1991). In a similar way, in the VisualMath curriculum project (Yerushalmy and Shternberg 2001), specialized software with multiple nonsymbolic representations of functions was used, in a functional approach in which letters represent quantities that vary, and solving equations consists of identifying a particular case of the comparison of two functions. It is worth mentioning that in such experimental studies, technology is one of the main factors of curriculum change, demonstrating the feasibility of its implementation in an educational system.

Nowadays Internet connectivity potentially changes the ways in which digital technologies can be integrated into the mathematics curriculum, both in terms of applications that are available in "the cloud" and because teachers and students can work collaboratively within virtual communities. Teachers are increasingly using the Internet to access teaching resources and organize their work. Internet connectivity is also changing the way in which digital technologies are being designed, with, for example, the dynamic geometry software GeoGebra developed as a free software within the open-source software movement. A promising area of future research is to investigate the ways in which connectivity can transform mathematical practices in schools and in particular whether students can use social media to create collaborative mathematical communities. Another important area of research relates to the ways in which digital technologies can be used to assess the learning of mathematics, and many people consider that assessment practices have to change before digital technologies become fully integrated into the mathematics curriculum.

## Cross-References


- ▶ Information and Communication Technology (ICT) Affordances in Mathematics Education
- ▶ Technology Design in Mathematics Education
- ▶ Types of Technology in Mathematics Education


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# Technology Design in Mathematics Education

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## Keywords

Technology design; Microworlds; Virtual communities; Web design

## Introduction and Historical Background

The role of the teacher, educational context, and design are three key factors, called by Drjvers (2012) decisive and crucial to promote or hinder the successful integration of digital technology in mathematics education. By using the term “design,” the author means not only the design of digital technology involved but also the design of corresponding tasks and activities and the design of lessons and teaching, in general.

An appropriate design, according to the author, refers explicitly to the instrumental genesis model which considers co-emergence of technical mastery to use technology for solving mathematical problems and the genesis of mental schemes leading to conceptual understanding (Drjvers 2012). As such, the model seeks a match between didactical and pedagogical functionality in which digital tool is incorporated with the tool’s characteristics and affordances. It also emphasizes a priority of pedagogical and didactical considerations as main guidelines and design heuristics over technology’s limitations and properties related to its affordances and constrains (Drjvers 2012).

This global definition of “design” related to the technology use in mathematics education was given by Drijvers during his plenary talk at the ICME Congress in Seoul, Korea, in 2012, which reflects 40 years of history after S. Papert’s talk, also during the ICME Congress in Exeter, Great Britain, in 1972, expressing ideas of the *micro-world vision* which set up a long-term guidelines in research and development of the technology design principles (Healy and Kynigos 2010). The turtle geometry microworld (and related programming language LOGO specifically designed for learning) grounded in the theory of constructionism is one of the first examples of technology design which – instead of being the aid to teach school mathematics – provides an opportunity to make mathematics “more learnable” where computers are used “as mathematically expressive media with which to design an appropriate mathematics fitted to the learner” (Healy and Kynigos 2010, p. 63; see also Papert 1980 and Kynigos 2012).

In their analysis of technology development in mathematics education based on 25 years of publication in the JRME (Journal for Research in Mathematics Education), Kaput and Thompson (1994) argue that “technology can reinforce any bias the user or designer brings to it" (p. 678) and do it by changing fundamentally the experience of doing and learning mathematics and this in three main aspects, interactivity, control, and connectivity. At the early stages, computer environments were reproducing a "human-human" *interactivity* via so-called CAI (computer-assisted instruction) by putting computers in the role of teacher presenting standard skill-based materials. Designers had a full *control* to engineer constrains and supports, create agents to perform actions for the learner (resources, aid, feedback, representation systems), and thus could influence students' mathematical experiences (Kaput and Thompson 1994). Finally, *connectivity* was seen as linking teachers to teachers, students to students, students to teachers, and, in a more general sense, the world of education to wider worlds of home and work (Kaput and Thompson 1994). At the more subtle level, the technology development involved "a gradual reshaping or expansion of human experience – from direct experience in physical space to experience mediated by the computational medium" (Kaput and Thompson 1994, p. 679).

This vision was directing mainstream of research and practice of designing technology-enhanced learning and teaching environments over the past 20 years that resulted in building of interactive microworlds that foster modelling and collaboration by "layering of mathematical and scientific principles and abstraction and embedding increasing problem-solving complexity into the software" (Confrey et al. 2009, p. 20). Such environments need to be engaging for the students so to help them to achieve goals they find compelling by making, at the same time, mathematics "visible to students and expressed in a language with which they can connect" (Confrey et al. 2009, p. 20). Kaput et al. (2007) use the term "infrastructure" which implies not only material support for activity but also social systems at different size scales, like communities of practice (in the sense of Lave and Wenger 1991). Related to the users of mathematics and mathematics education software, this implies active participation in a practice as an intrinsic property of membership, "whether one uses the technology as interactive tool or as a medium in which one designs and builds interactive artifacts (technology as 'tutee')" (Kaput et al. 2007, pp. 177–178).

## Example of a Microworld

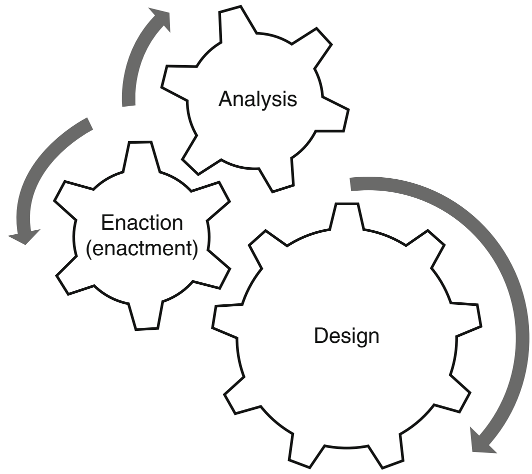
Design of a dynamic visualization software environment 3DMath (Christou et al. 2006) aimed at enabling learners to construct, observe, and manipulate geometrical figures in a 3D-like space. General principles of the design meeting these purposes are based on three major fields of educational theory: *constructivist perspective* about learning as personally constructed and achieved by designing and making artifacts that are personally meaningful, *semiotic perspective* about mathematics as a meaning-making endeavor that encourages multiple representations of knowledge, and *fallibilist nature of mathematics* where knowledge is a construction of human beings and is subject to revision (Christou et al. 2006). Also, according to the authors, core visual abilities must be taken into account: perceptual constancy, mental rotation, perception of spatial positions, perception of spatial relationships, and visual discrimination – accumulation of representations makes possible the creation of mental images.

Related to these perspectives, design principles for the 3DMath would (a) allow students to see a geometric solid presented in several possible ways; (b) introduce software-controlled speed and directions of rotations that can enable students to devise strategies of movement and anticipated their results; (c) integrate intuitive interface allowing the learner to make and design personally meaningful artifacts by means of rich semiotic resources enabling multiple perspectives and representations; (d) help students to focus on mental images; (e) be rich in the ability to manipulate and transform solids; (f) focus on observation, construction, and exploration; (g) contribute to the development of visual abilities (dragging, tracing, measuring, adding text, and diagrams); and (h) add export of construction and control of available (hidden) options (Christou et al. 2006).

## Virtual Learning Communities in Mathematics Education

Virtual communities emerge in early 2000s expanding affordances of the Internet technology while allowing for designing collaborative learning environments in mathematics. For example, a *Math Forum* community brings together teachers, students, parents, software developers, mathematicians, math educators, professionals, and tradespeople. While having different experience, expertise, and interest in mathematics by playing different roles, they all contribute in building sustainable learning space with a variety of educational resources that helps to scaffold each other's understanding of mathematics (Renninger and Shumar 2004). From the point of view of the design, we note two key features, namely, the *content* with extensive archives and links to information and the *interactive tools* that promote information exchange, discussion, and community building (Renninger and Shumar 2004). This type of the design lets participants to try out and select different ways of working with the content and thus facilitate learning driven by their personal questions and interests (Renninger and Shumar 2004). The website provides them with services that support learning, such as *Problem of the Week* section with five interactive, nonroutine, challenging problems posted weekly accompanied further with solutions and explanations; *Ask Dr. Math* service allowing posing and answering frequently asked questions from the members; and *Teacher2Teacher* discussion forum, examples of lessons, projects, games, and a newsletter (Renninger and Shumar 2004).

The design of virtual communities is a cyclic process which reflects a design-based research (DBR) model of the CASMI (Communauté d'Apprentissages Scientifiques et Mathématiques Interactifs, www.umoncton.ca/cami) community (Fig. 1). The model illustrates an innovative research approach suitable for studying complex problems in real, authentic contexts in collaboration with practitioners. Research and development happens through continuous cycles of design, enactment, analysis, and redesign which would lead to sharable theories that help communicate relevant implications to practitioners and to other educational designers (Design-Based Research Collective 2003).




**Technology Design in Mathematics Education, Fig. 1** A DBR cycle for the CASMI community (Freiman and Lirette-Pitre 2009, p. 248)

The DBR model allowed for implementing five techno-pedagogical principles in the CASMI: *friendly welcome* allowing everyone (students, teachers, parents) to join the community at any time; *math challenge* using authentic, complex, and contextualized problems to which every member can submit a solution via an e-form on the website; *formative individual feedback* provided by mentors (mostly university students) aiming to encourage each participant to be persistent and continue to participate; *acceptance of variety* of styles and strategies valuing different ways of thinking as rich and valuable contribution to the community; and open communication as a vehicle of the community to promote knowledge sharing and knowledge building through collaboration and discussion (Freiman and Lirette-Pitre 2009).

A newest development of collaborative models of technology-enhanced learning environments is grounded in what Gadanidis and Namukasa (2012), referring to the works of Levy (1998) and Borba and Villarreal (2005), call "integral component of a *cognitive ecology* of the human-with-technology *thinking collectives*" (p. 164). As new media affordances, Gadanidis and Namukasa (2012) mention democratization, as new media learning resources are available from any place with the Internet access; multimodality that connects physical, linguistic, cognitive, and symbolic experiences; collaboration that allows for new ways of thinking collaborative, participatory, and distributed; and performance metaphor with multimedia authoring tools used to create online content which are orchestrated (programmed) as "stage" "scenes," "actors" making of the Web a "performative medium" (p. 167).

## Applications and Task Design with Technology

Designing tasks with interactive technology is yet another direction of research and practice of mathematics education over the past decades. According to the epistemic model developed by Leung (2011), the task design "focuses on pedagogical processes in which learners are empowered with amplified abilities to explore, re-construct (or re-invent) and explain mathematical concepts using tools embedded in a technology-rich environment" (p. 327). Sinclair (2006) brought attention to the issues related to the use of interactive web-based applets (web-based sketches) whose design principles are under-researched. A design of the computer-based tasks is grounded in a complex combination of a variety of theories in mathematics education on the use of manipulatives, teaching approaches for some specific topics, and structure of classroom discussions, which can be borrowed and adapted for the use in the technological environment often in the constructivist perspective which seeks in helping students to build their own understanding by connecting new ideas and prior understanding; the activity theory is also used to explain learning as being dependent on personal experience and can be mediated by the tools. In its turn, the activity theory can be linked to the affordance theory in a way that Martinovic et al. (2012) conceptualize as a "handshake" which is prerequisite for an action by a subject, for example, a student, during the explorative activities on mathematics software, needs to be able to use features of the software and to consider the objects constructed in/by the software as material/real.

Interactive geometry sketches based on two reflection tasks were designed in Sinclair's study using the Geometer's Sketchpad and saved as JavaSketches (Jackiw 2002, cited in Sinclair 2006, p. 32). The findings from the experimentation with teachers and students reveal several issues related to some technical problems with the sketches, student difficulties with the wording of questions and instructions, as well as interpretation of mathematical concepts embedded in the applets; therefore more research is needed in order to develop strategies to gather information about the needs and the abilities of end users during the design process (Sinclair 2006, pp. 34–35).

## New Paths in Research and Practice with Technology Design

A recent development in the design of technology applications to support mathematics learning is related to the *mobile technology*, to constructing *complex integrated systems* by combining microworlds and virtual community features, and *use of games*.

### Mobile Learning Design

First reports are coming from pilot studies about design for mobile devices, such as cellular phones with the use of *Sketch2Go* and *Graph2Go* (Botzer and Yerushalmy 2007). The first enables students to sketch graphs (constant, increasing, and decreasing functions) and get an immediate feedback of the drawn graph and present a graph of the rate of change thus reinforcing visual exploration of (physical temporal) phenomena and providing with qualitative indication of the ways in which sketch drawn by the user changes; this motivates students to experiment with a given situation, analyze it, and reflect upon it. The second application (*Graph2Go*) is a graphing calculator which operates for given sets of function expressions and enables the dynamic transformation of functions (including changing parameters of algebraic expression, Botzer and Yerushalmy 2007, p. 314).

### Combining Multiple Platforms

The design of a virtual learning environment that integrates synchronous and asynchronous media with an innovative multiuser version of a dynamic math visualization and exploration toolbox is discussed by Stahl (2012) using the example of the VMT with GeoGebra. The combination of features of the computer-supported collaborative learning (CSCL) software (VMT – Virtual Mathematics Team platform that engages learners in significant discourse and practicing teamwork) and dynamic mathematics software (such as Geometer's Sketchpad, Mathematica, Cabri, or GeoGebra allowing users to manipulate geometric diagrams and equations). The combination of both environments, VMT and GeoGebra, helped to overcome issues related to multi-user collaboration by means of a client–server architecture. This allows "multiple distributed users to manipulate constructions and to observe everyone else's actions in real time (through immediate broadcast by the server and further logged in detail for replay and research)" (Stahl et al. 2012, p. 5).

### Design of Games for Teaching and Learning Mathematics

While exploring educational potential of computer games in mathematics, Hui (2009) mentioned several general categories of games, such as action, adventure/quest, fighting, puzzle, role-play, simulations, sports, and strategy games. For mathematics education, problem solving and deductive reasoning as well as skills like numerical calculation and monetary skills are mentioned by the author as the most viable avenues for acquisition and application of mathematics in computer games (Hui 2009).

Kafai (2006) discussed two different perspectives on design of games for learning: making games for learning instead of playing games for learning. The instructionist perspective builds on a vision that making a game for practicing the multiplication tables can make the learning of academic matters more fun, if not easier by embedding school-like exercises in a computer game (Kafai 2006). For example, a game called *How the West Was Won* offered the players to throw a dice than perform various arithmetic operations on the numbers to determine how far to advance a token on the board (Kafai 2006). By mentioning *Math Blaster* as another example of thousands of instructional games on the market, the author mentions that little is known about which features make an educational game good for learning and few studies are available on what are successful design features for good educational games (Kafai 2006). As about constructionist perspective, the main idea expressed by Kafai (2006) is that rather than embedding "lessons" directly in games, the goal should be directed to providing students with opportunities to construct their own games and thus new relationships with knowledge in the process, as shows the study of primary children and preservice teachers designing games with representing fractions in different ways. Not only this opportunity makes possible game design environment in which the user can load fraction design tools with a set of objects and graphic tools for creating, representing, and operating on fractions (like splitting, fair sharing) and fraction objects. Moreover, there were also tools allowing students and preservice teachers – designers to share, annotate, and modify their designs using electronic discussion forum. Researchers found that conversation and discussion among participants were essential in helping the designers build more sophisticated representations (Kafai 2006).

## Cross-References

▶ Learning Environments in Mathematics Education
▶ Learning Practices in Digital Environments
▶ Technology and Curricula in Mathematics Education
▶ Types of technology in Mathematics Education

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# The Learning Framework in Number

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## Keywords

Early arithmetic; Learning trajectories; Counting; Number word sequences; Numerals

## Description

The Learning Framework in Number (LFIN, Fig. 1) consists of a set of progressions of student learning related to early arithmetic. Each progression relates to a specific domain of mathematics learning and, taken together, the domains are

**Stages: Early Arithmetical Learning**
- 0 - Emergent Counting
- 1 - Perceptual Counting
- 2 - Figurative Counting
- 3 - Initial Number Sequence
- 4 - Intermediate Number Sequence
- 5 - Facile Number Sequence

**Levels: Forward Number Word Sequences (FNWS) & Number Word After**
- 0 - Emergent FNWS.
- 1 - Initial FNWS up to 'ten'.
- 2 - Intermediate FNWS up to 'ten'.
- 3 - Facile with FNWSs up to 'ten'.
- 4 - Facile with FNWSs up to 'thirty'.
- 5 - Facile with FNWSs up to 'one hundred'.

**Levels: Numeral Identification**
- 0 - Emergent Numeral Identification.
- 1 - Numerals to '10'
- 2 - Numerals to '20'
- 3 - Numerals to '100'
- 4 - Numerals to '1000'

**Levels: Backward Number Word Sequences (BNWS) & Number Word Before**
- 0 - Emergent BNWS.
- 1 - Initial BNWS up to 'ten'.
- 2 - Intermediate BNWS up to 'ten'.
- 3 - Facile with BNWSs up to 'ten'.
- 4 - Facile with BNWSs up to 'thirty'.
- 5 - Facile with BNWSs up to 'one hundred'.

**The Learning Framework in Number, Fig. 1** The Learning Framework in Number (LFIN) (Adapted with permission from Wright et al. (2006), p. 20)

interrelated. Each progression takes a summary form referred to as a model and consisting of a table, setting out progressive levels of knowledge of the domain. The LFIN relates to young children's early arithmetical learning and was the first example of such a framework (Wright 1986, 1991, 1994, 1998). Figure 1 shows LFIN in summary form and includes models for four domains: Early Arithmetical Learning; Forward Number Word Sequences; Backward Number Word Sequences; and Numeral Identification (Steffe 1992; Wright et al. 2006; Wright 2008). The origin of LFIN is independent of that of learning trajectories (Simon 1995) and instructional progressions (Gravemeijer 2004). Nevertheless, LFIN has been described (Clements and Sarama 2009) and can be regarded as a set of interrelated learning/instructional trajectories.

## Origin

LFIN was initially developed as part of a research study of the knowledge progression across a school year, of children in the first and second years of school (Wright 1991, 1994). In this study, use of a process of videotaped, interview-based assessment enabled the profiling of the knowledge of 45 children – 15 drawn from each of three classrooms – on LFIN at the beginning, middle, and end of the school year. Table 1 shows the progress of 15 students from a class in the Kindergarten year. This study not only showed the kinds of knowledge progressions typical of students in the first and second years of school but also highlighted the relatively wide range of knowledge within a given classroom.

## Applications

LFIN has been used in several research studies to chart the progress of very large cohorts of students (Thomas and Ward 2002; Wright and Gould 2002). These studies were undertaken in conjunction with large-scale, systemic implementations of new initiatives in early arithmetic instruction, which adopted or drew on LFIN as a guiding pedagogical model (Bobis et al. 2005). Table 2 is drawn from a study in which 23,121 students with ages ranging from 4.5 to 9.9 were assessed to determine their

**The Learning Framework in Number, Table 1** Five-year olds – Kindergarten Year – School B, March August November (Adapted from Wright (1994), p. 34)

| | Boy/girl | Counting stage 0–5 | | | Forward N.W.S. 0–5 | | | Backward N.W.S. 0–5 | | | Numeral identification 0–4 | | | Spatial patterns 0–3 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | G | 1 | 1 | 1 | 3 | 3 | 3 | 1 | 0 | 1–3 | 1 | 1 | 1 | 0 | 0 | 0 |
| 2 | B | 1 | 2 | 2 | 3 | 2 | 4 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| 3 | B | 2 | 2 | 2 | 5 | 5 | 5 | 3 | 5 | 5 | 2 | 3 | 4 | 1 | 1 | 1 |
| 4 | G | 2 | 2–3 | 3 | 3 | 4 | 4 | 1–3 | 3 | 3 | 1 | 1 | 2 | 1 | 2 | 1 |
| 5 | B | 2 | 3 | 3 | 3 | 3 | 5 | 2 | 3 | 3 | 1 | 2 | 2 | 1 | 2 | 2 |
| 5 | G | 3 | 3 | 3 | 4 | 5 | 5 | 3 | 3 | 3 | 1 | 2 | 2 | 1 | 2 | 2 |
| 7 | G | 2–3 | 3 | 3 | 3–4 | 5 | 5 | 3 | 3 | 3 | 1 | 3 | 3 | 2 | 2 | 2 |
| 7 | B | 2 | 3 | 3 | 3 | 5 | 5 | 3 | 3 | 3 | 1 | 2 | 3 | 1 | 2 | 2 |
| 7 | B | 2 | 3 | 3 | 3 | 3–4 | 5 | 0 | 1–3 | 3 | 1 | 2 | 3 | 0 | 2 | 2 |
| 10 | G | 3 | 3 | 3 | 4 | 5 | 5 | 4 | 3 | 5 | 3 | 3 | 3 | 1 | 2 | 2 |
| 11 | G | 3 | 3 | 4–5 | 3 | 4 | 5 | 1–3 | 3 | 3 | 1 | 2 | 3 | 1 | 2 | 2 |
| 12 | G | 3 | 3 | 4–5 | 5 | 5 | 5 | 3–4 | 5 | 5 | 1 | 2 | 3 | 2 | 1 | 2 |
| 13 | G | 3 | 5 | 5 | 4 | 5 | 5 | 3 | 5 | 5 | 1 | 3 | 3 | 2 | 3 | 2 |

Notes: In column one, the order has been determined by considering the data from the November interviews (the value appearing on the right in each cell) as follows: The counting stage is considered first, then the levels are considered, in order from left to right
The Counting Stage corresponds to the Stage of Early Arithmetical Learning
Table entries in the form of a range, e.g., two to three, rather than a single level or stage, indicate that the precise level or stage could not be determined

**The Learning Framework in Number, Table 2** Number of children at each stage, for each age group (n = 23,121) (Adapted with permission from Wright and Gould (2002))

| | 4.5–4.9 years | 5.0–5.9 years | 6.0–6.9 years | 7.0–7.9 years | 8.0–8.9 years | 9.0–9.9 years |
|---|---|---|---|---|---|---|
| Emergent | 552 | 1,406 | 388 | 113 | 39 | 18 |
| Perceptual | 555 | 3,254 | 2,032 | 825 | 262 | 97 |
| Figurative | 56 | 887 | 1,749 | 1,321 | 441 | 182 |
| Counting on | 13 | 322 | 1,445 | 2,413 | 1,563 | 772 |
| Facile | 0 | 29 | 259 | 682 | 765 | 681 |
| Totals | 1,176 | 5,898 | 5,873 | 5,354 | 3,070 | 1,750 |

stage on the domain of Early Arithmetical Learning.

Finally, LFIN has been used extensively in professional practice in at least eight countries, as a guiding model for both classroom instruction and intensive intervention with low-attaining students (Wright et al. 2006).

## Cross-References

- ▶ Hypothetical Learning Trajectories in Mathematics Education
- ▶ Number Teaching and Learning

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# The van Hiele Theory

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## Keywords

van Hiele theory; van Hiele levels; van Hiele phases; Geometry; Cognitive development

## Definition

The van Hiele Theory had its beginnings in the 1950s in the companion doctoral work of husband and wife team Pierre van Hiele and Dina van Hiele-Geldof. Dina died in 1959 and Pierre continued to develop and refine the Theory that is explored thoroughly in his 1986 book, *Structure and Insight*.

Much of the resurgence of interest in teaching of Geometry that began in the 1980s and 1990s can be traced to the ideas developed in the van Hiele Theory. Detailed accounts and summaries of this early, but still highly relevant, work can be found in the following: Clements and Battista (1992), Fuys et al. (1988), Burger and Shaughnessy (1986), Hoffer (1981), Lesh and Mierkiewicz (1978), Mayberry (1981), and Usiskin (1982).

## Characteristics

The Theory has two main aspects that combine to provide a philosophy of mathematics education (not only of Geometry). The two key aspects to the theory are:

1. Levels that students grow through in acquiring competence and understanding
2. Teaching phases that assist students to move through the levels

Van Hiele's ideas have much in common with those of Piaget (Piaget et al. 1960) in that they ascribe student understanding to a series of levels or stages. However, there are important differences between the two theories. For example, the van Hiele Theory:

- Places explicit importance on the role of language in moving through the levels.
- Concentrates on learning rather than development; hence the focus is on how to help develop student understanding.
- Postulates that ideas at a higher level result from the study of the structure at the lower level.

Most of the research effort has been directed at the van Hiele levels of thinking – a hierarchical series of categories that describe cognitive growth in students. The second and equally important aspect that has not received the same degree of scrutiny or acknowledgement is the notion of five teaching phases to help guide activities that lead students from one level to the next.

### Van Hiele Levels

Van Hiele envisaged five levels, and these are described below within the context of two-dimensional geometrical figures:

Level 1. Students recognize a figure by its appearance (i.e., its form or shape). Properties of a figure play no explicit role in its identification.

Level 2. Students identify a figure by its properties, which are seen as independent of one another.

Level 3. Students no longer see the properties of figures as independent. They recognize that a property precedes or follows from other properties. Students also understand relationships between different figures.

Level 4. Students understand the place of deduction. They use the concept of necessary and sufficient conditions and can develop proofs rather than learn them by rote. They can devise definitions.

Level 5. Students can make comparisons of various deductive systems and explore different geometries based upon various systems of postulates.

Although these descriptions are content specific, van Hiele's levels are actually stages of cognitive development: "the levels are situated not in the subject matter but in the thinking of man" (van Hiele 1986, p. 41). Progression from one level to the next is not the result of maturation or natural development. It is the nature and quality of the experience in the teaching/learning program that influences a genuine advancement from a lower to a higher level, as opposed to the learning of routines as a substitute for understanding.

It is this focus on teaching that pervades the ideas inherent in the van Hiele's writings – so much so that the "theory" is perhaps better described as pedagogical rather than psychological, as many (or most) of the problems identified in students' learning have their basis in teaching practices rather than in the cognitive processes that may underlie performance.

It is important to state that the van Hiele levels are not without controversy. Some of these issues are discussed in Pegg and Davey (1998).

### Van Hiele Phases

In terms of the van Hiele phases, the initial work in this area appeared through the doctoral thesis of Dina van Hiele-Geldof. Her thesis was translated from Dutch into English as part of the investigation led by Geddes (see Fuys et al. 1984) and provides a valuable insight into how the phase concept emerged.

The purpose of Dina's study was to detail her experiences and teaching procedures with 2 Year 7 (12 years old) Geometry classes over a year. The students were studying Geometry for the first time, and the main question posed in the study was to see if it was possible to follow a teaching approach that allowed students to develop from one level to the next in a continuous process.

As a result of this work, five phases were identified that allowed students to move from one level to the next. The descriptions of the phases (see Pegg 2002) given below are adapted from Dina's last paper that was written just before her death and also translated into English by the Geddes team.

Phase 1: Information (Inquiry). This part of the process allows students to discuss what the area to be investigated is about.

Phase 2: Directed Orientation. Out of the first phase and the resulting discussion, students begin to look at the area to be studied in a certain way. This part of the process involves the teacher in directing the class to explore the object of study by means of a number of simple tasks.

Phase 3: Explicitation. As a result of the manipulation of materials and the completion of simple tasks set by the teacher, the need to talk and to converse about the subject matter becomes important. During the early part of the process, the students are encouraged to use their own language. However, over time the teacher gradually incorporates, where appropriate, correct technical terms.

Phase 4: Free Orientation. Here students are given a variety of activities and are expected to find their own way to a solution. The teacher's role is to encourage different solutions to the problems as well as the inventiveness of the students.

Phase 5: Integration. The students achieve an overview of the area of study by themselves. They are now clear of the purpose of the study and have reached the next level.

As with the van Hiele levels, there is an intuitive appeal about the learning phases outlined above.

### Summary

The van Hiele theory is directed at improving teaching by organizing instruction to take into account students' thinking, which is described by a hierarchical series of levels. According to the theory, if students' levels of thinking are addressed in the teaching process, students have ownership of the encountered material. As a result, they have the potential to develop *insight* (the ability to act *adequately* with intention in a *new situation*). For the van Hieles, the main purpose of instruction is the development of such insight.

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# Theories of Learning Mathematics

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## Keywords

Complexity; Learning theories; Models and modeling; Models versus theories; Theories of mathematics education

## Definition

According to Karl Popper, widely regarded as one of the greatest philosophers of science in the twentieth century, *falsifiability* is the primary characteristic that distinguishes scientific theories from ideologies – or dogma. For example, for people who argue that schools should treat creationism as a scientific theory, comparable to modern theories of evolution, advocates of creationism would need to become engaged in the generation of falsifiable hypothesis and would need to abandon the practice of discouraging questioning and inquiry. Ironically, scientific theories themselves are accepted or rejected based on a principle that might be called *survival of the fittest*. So, for healthy theories on development to occur, four Darwinian functions should function: (a) variation, avoid orthodoxy and encourage divergent thinking; (b) selection, submit all assumptions and innovations to rigorous testing; (c) diffusion, encourage the shareability of new and/or viable ways of thinking; and (d) accumulation, encourage the reusability of viable aspects of productive innovations.

## Characteristics

### The History and Nature of Theory Development

To describe the nature of theories and theory development in mathematics education, it is useful to keep in mind the preceding four functions and to focus on two books that have been produced as key points during the development of mathematics education as a research community: *Critical Variables in Mathematics Education* (Begle 1979) and *Theories of Mathematics Education* (Sriraman and English 2010).

Begle was one of the foremost founding fathers of mathematics education as a field of scientific inquiry; and his book reviews the literature and characterizes the field when it was in its infancy. For example, before 1978, the USA's National Science Foundation had funding programs to support curriculum development, teacher development, and student development; but, it had no comparable program to support knowledge development (i.e., research). Similarly, before 1970, there was no professional organization focusing on mathematics education research or theory development; there was no journal for mathematics education research; and in the USA, just as in most other countries, there existed no commonly recognized *curriculum standards* for school mathematics. Furthermore, most mathematics educators thought of themselves as being curriculum developers, program developers, teacher developers, or student developers (i.e., teachers) – and only secondarily as researchers. And, if any theories were invoked to guide their research or development activities, these theories were mainly borrowed from educational psychology such as Bloom's *taxonomy of* educational objectives, Gagne's *behavioral objectives* and learning *hierarchies*, Piaget's *stage theory*, Ausabel's *advanced organizers and* meaningful verbal learning, and later Vygotsky's *socially mediated learning*, and Simon's *artificial intelligence* models for cognition. However, the practitioners' side of these mathematics education researchers made it difficult for them to ignore the fact that very few of their most important day-to-day decision-making issues were informed in any way by these borrowed theories.

In contrast to the preceding state of affairs, Sriraman and English's (2010) book clearly documents a shift beyond theory borrowing toward theory building in mathematics education; the relevant theories draw on far more than psychology, and the mathematics education research community has become far more international – and far more multidisciplinary in its membership. Furthermore, the field changed significantly after the National Council of Teachers of Mathematics (NCTM) published its nationally endorsed *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989), commonly referred to as the Standards in the USA. Since then, similar documents were produced in many other countries throughout the world. But, to what extent have these documents been products of empirical research and theory development instead of dogma? The NCTM Standards themselves were not based on any research per se, but simply an envisioning of what mathematics education in classrooms, i.e., in practice might look like and what the appropriate content might look like, keeping the learner in mind.

### Curricular Standards and Mathematics Education Research

Two decades later, consider the USA's newest *Common Core State Curriculum Standards* (CCSC 2012). In this case, there clearly exist some instances where these *CCSC Standards* were informed by the work of a few researchers.

But, it is equally clear that this document was produced using a process of political consensus building in which the views of some stakeholders were given great attention (e.g., university-based mathematicians and teacher educators), whereas others were ignored almost completely (e.g., engineers, social scientists, and other heavy users of mathematics outside of schools). Consequently, the *CCSC Standards* exhibit little recognition of the fact that, outside of school in the twenty-first century, many new kinds of problem-solving situations abound in which new types of mathematical thinking are needed. In fact, little is said in the *CCSC Standards* that could not have been said when Begle was in his prime. For example:

- The mathematics education community still does not know how to operationally define measurable conceptions of almost any of the higher-level understandings or abilities that the *CCSC Standards* refers to as "mathematical practices." So, the only goals that are stated in ways that can be documented and assessed tend to be the CCSC's long lists of "things students should know and be able to do" (i.e., declarative statements {facts} or condition-action rules {skills}).
- In spite of the CCSC's claim of being based on *research-based learning progressions*, it still is unclear how the mastery of the *CCSC*'s lists of "things students should know and be able to do" interacts with the development of higher-order "conceptual understandings of the type which are needed to conceptualize (i.e., mathematize by quantifying, dimensionalizing, coordinatizing, systematizing) situations that do not occur in a pre-mathematized form. In particular, it is unclear how (or whether) the CCSC's lists of "things students should know and be able to do" should be treated as "prerequisites" which must be "mastered" before students should be introduced to deeper and higher-order conceptual understandings and abilities. Furthermore, modern research in the learning sciences clearly has shown that (a) students' and teachers' conceptual understandings of most "big ideas" in the K-12 curriculum develop (in parallel and interactively) over time periods of many years and (b) students' conceptual understandings of these "big ideas" are a great deal more *situated* and *socially mediated* than theories of 30 years ago led educators to believe.
- Modeling continues to be characterized as the application of concepts (traditionally) taught in school. Yet, research in the learning sciences clearly is showing that, in modern societies, in students' everyday lives outside of schools and departments of mathematics, many of the situations that students need to mathematize involve (a) integrating ideas and procedures drawn from more than a single textbook topic area and (b) using more than a single, solvable, and differentiable function. For example, in problem-solving situations that involve data analysis and statistics, Bayesian and Fisherian computational models tend to be far more accessible and powerful than traditional methods that depend on Calculus and the use of traditional analytic methods. And, in situations that involve several interacting agents, issues often arise that involve feedback loops, second-order effects, and issues such as maximization, minimization, or stabilization. And again, graphics-oriented computational models make it possible for quite young children to deal effectively with situations that no longer need to be postponed until after courses in Calculus.

Perhaps the most important general theme that cuts across many of the chapters in Sriraman and English's book is that, from early number concepts through proportional reasoning and Calculus, the mathematics education community in general has been quite naïve about: (a) what it means to "understand" nearly every "big idea" in the K-12 curriculum; (b) how these understandings develop along dimensions such as concrete-abstract, intuition-formalization, or situated-decontextualized; (c) what it means for one concept or ability to be prerequisite to another; and (d) how understandings of both "big ideas" and basic "facts and skills" evolve as interconnections and distinctions develop.

Begle's powerfully influential *School Mathematics Study Group (SMSG) projects* provide clear instances of a curriculum development project that attempted to make research and theory development important parts of their collective agenda. For example, the *National Longitudinal Study of Mathematics Abilities* (*NLSMA*) was an important part of SMSG initiatives. Nonetheless, in their introduction to Begle's book, Wilson and Kilpatrick reported that "*(Begle) tried to persuade the SMSG advisory board to sponsor research as well as curriculum development, but he was not successful (p. x.).*" Similarly, in his keynote address at the *First International Congress of Mathematics Education*, Begle stated:

> I see little hope for any further substantial improvements in mathematics education until we turn mathematics education into an experimental science – until we abandon our reliance on philosophical discussions based on dubious assumptions, and instead follow a carefully constructed pattern of observation and *speculation*, the pattern so successfully employed by physical and natural scientists.
>
> Much of what goes on in mathematics education is based on opinions that are so firmly held that the thought of doubting them crosses very few minds. Yet, most of these opinions have no empirical substantiation, and in fact many of them are, if not wrong, at least in need of serious qualifications (p. xvi).

In other words, Begle believed that a large share of what paraded as theory in mathematics education was (and continues to be) dogma. For example, in his reviews of the literature in topic areas ranging from problem solving to teacher development, Begle identified many examples of dubious opinions which continue to go unquestioned.

- Concerning Teacher-Level Knowledge: *Despite all of our efforts, we still have no way of deciding, in advance, which teachers will be effective and which will not. Nor do we know which training programs will turn out effective teachers and which ones will not (p. 29) .... The outcomes of teaching does not depend just on the teacher (or the program used) but rather is the result of complex interactions among teachers, students, the subject matter, the instructional materials available, the instructional procedure used, the school and community, and who knows what other variables (p. 32) .... Many of our common beliefs about teachers are false, or at the very best rest on shaky foundations. For example, the effects of a teacher's subject matter knowledge and attitudes on students learning seem to be far less powerful than most of us had realized (p. 54).*

  Most mathematics educators surely believe that teacher-level understandings of topics to be taught should involve understanding both more and also differently than students. But, we still know little about the nature of these teacher-level understandings.
- Concerning Problem Solving: *A substantial amount of effort has gone into attempts to find out what strategies students use in attempting to solve mathematical problems .... But no clear-cut directions for mathematics education are provided by the findings of these studies. In fact, there are enough indications that problem-solving strategies are both problem- and student-specific to suggest that hopes of finding one (or a few) strategies which should be taught to all (or most) students are far too simplistic (p. 145).*

In the NCTM's most recent *Handbook of Research in Mathematics Education* (Lester 2007), the chapter on problem solving (Lesh and Zawojewski 2007) concludes that very little has changed since Begle's time. New words (such as metacognition, or habits of mind) have been introduced to replace previously discredited constructs (such as those reviewed by Begle), but the following fundamental issues remain. (a) Strategies, heuristics, or other meta-level procedures which seem to provide useful after-the-fact descriptions of what successful problem solvers' behaviors seem to have done do not necessarily provide prescriptions of what novice problem solvers should do next during ongoing problem-solving activities, and (b) if attention focuses on a small number of larger or more general rules of behavior, then these general rules tend to lack prescriptive power. But, if attention focuses on a larger number of smaller or more specific rules of behavior, then knowing when to use such behaviors is a large part of what it means to understand them. And transfer of learning that was expected to occur in such studies has been unimpressive.

## Theories Versus Models

Sriraman and English's book identifies a trend in which theory development shifts toward model development; and modeling perspectives are being used to provide alternatives to traditional theories related to topic areas ranging from teacher development to problem solving; and accompanying design research methodologies are being used to supplement what can be investigated using more traditional methods.

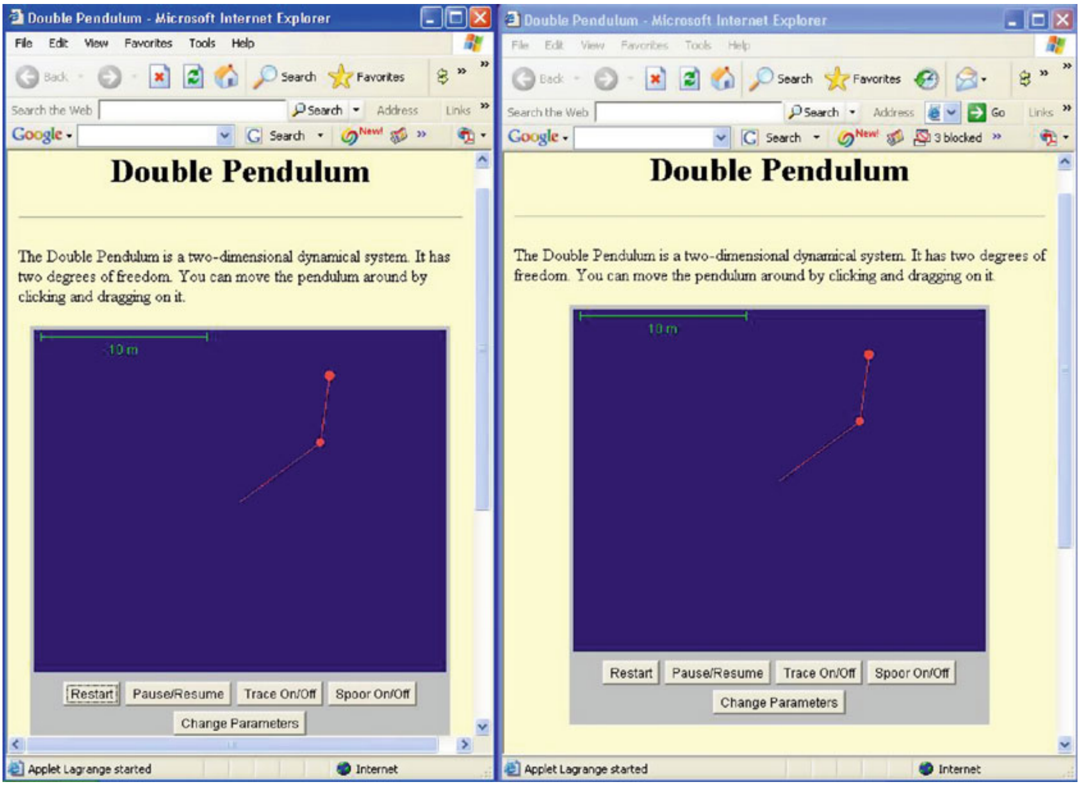
The key assumption that underlies a model and modeling perspective is that all relevant "subjects" – including not only students and teachers but also researchers themselves – are model developers. Students develop models to make sense of mathematical problem-solving situations that do not occur in a pre-mathematized form. Teachers develop models to make sense of students' model development activities. And researchers develop models of interactions among students, teachers, and learning environments. For example, in the case of both teaching and problem solving, it is widely recognized that highly effective people not only do things differently than their less experienced or less effective counterparts, but they also see (or interpret) things differently. Furthermore, the interpretation systems that they develop are both learnable and assessable – as well as being powerful, sharable, and reusable (i.e., transferrable).

Similarly, according to MMP, students' conceptual understandings of "big ideas" are expected to involve conceptualizing (mathematizing or mathematically interpreting) situations; relevant models are expected to involve the gradual integration, differentiation, reorganization, and adaptation of existing models. In other words, for a given "big idea" in the K-12 curriculum, a large part of "conceptual understanding is expected to involve the development of powerful, sharable, and reusable models.

To highlight some other important ways that model development is expected to contribute to theory development, while at the same time being different than theory development, it is useful to shift attention to curriculum development and program development. Critics often accuse the mathematics education research community of failing to provide "scientifically sound" empirical evidence about curriculum materials that "work" (ref needed). But, most mathematics education researchers are also practitioners – e.g., teachers, teacher educators, or developers of curriculum materials. And it is precisely their practitioner side that makes them aware of the uselessly simplistic nature of most studies claiming to show that some curriculum innovation "works" – using standardized and randomly assigned "treatment groups" and "control groups" in situations where (a) the criteria for "working" tend to be poorly aligned with the most important goals of the curriculum that is used, (b) it is well known that "working" depends on far more than the curriculum materials themselves, and (c) curriculum innovations don't simply act on students and teachers – students and teachers also react (or act back)! So, successful curriculum innovations usually involve continual adaptations – based on the strengths and weaknesses of individual students and teachers and based on their reactions at various stages of implementation.

## The Complexity of Models in Mathematics Education

To see why no two situations are never exactly alike and why the same thing never happens twice, consider the following. During the 1980s and 1990s, a number of learning theorists who wanted to apply their learning theories to mathematics education developed a methodology called *aptitude-treatment-interaction studies* (ATI). These ATI studies recognized that, even in very simple learning situations (e.g., one student and one teacher), different students reacted differently to a given treatment. So, attempts were made to identify profiles of student attributes (A1, A2, . . ., An) which could be matched with alternative preplanned treatment attributes (T1, T2, . . . Tm). But, the results of these ATI studies showed that progressively finer-grained student and treatment profiles not only led to unworkable combinatorial nightmares, but they also involved feedback loops in which students

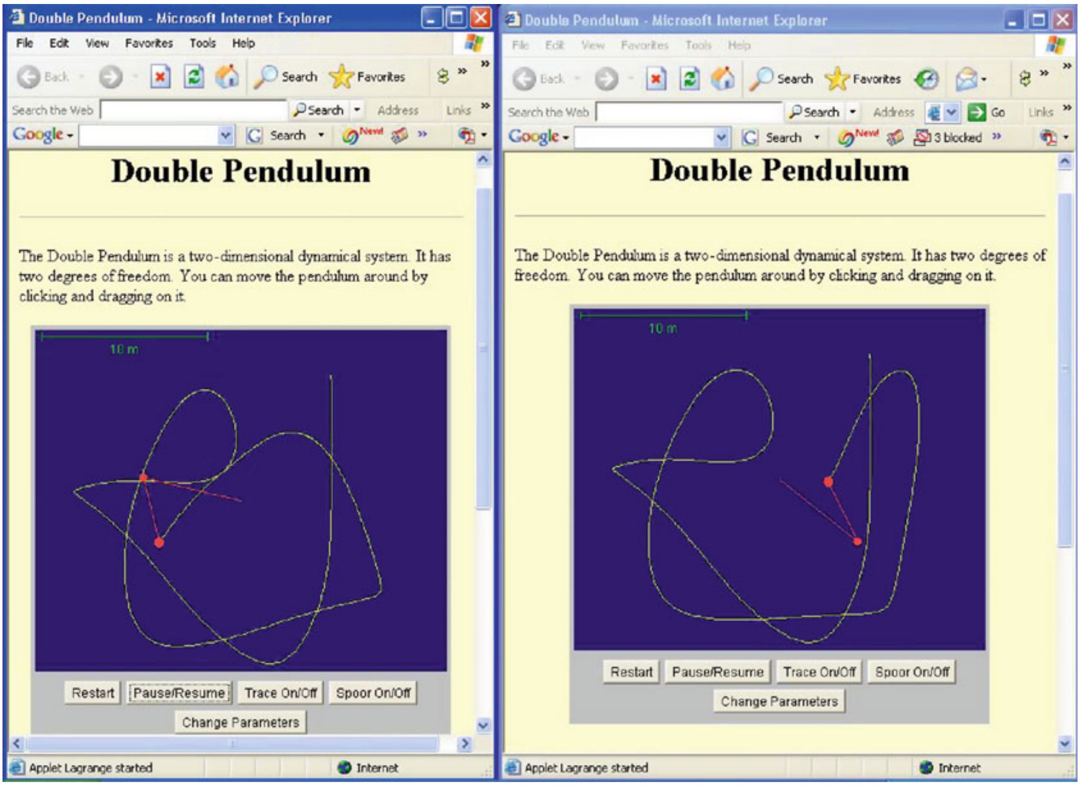



**Theories of Learning Mathematics, Fig. 1** Two identical starting points for a double pendulum system

acted on treatments as much as treatments acted on students. So, what emerged in such situations is similar to what happens when two identically configured double pendulums are set in motion at exactly the same time. Within a few cycles, identical systems will function in ways that are quite different – and unpredictably so (as shown in the Figs. 1 and 2 below).

These kinds of systems are studied in a branch of mathematics known as *complexity theory*. And one thing complexity theory implies is that, even in situations that are as simple as a double pendulum, feedback loops tend to lead to unpredictable behaviors in only a few cycles. So, simple input–output rules of the form {*Use treatment A and result B will occur.*} are not likely to work for situations involving student-teacher interactions, student-student interactions, teacher-treatment interactions, and student-treatment interactions – all functioning simultaneously.

No research methodology is "scientific" if it is based on assumptions that are inconsistent with those that are considered to be reasonable for the subjects and situations being investigated. So, a fundamental dilemma that mathematics education researchers face is that (quite often) they are trying to understand subjects that they (as a community) also are trying to change, design, or develop. This means that mathematics education researchers tend to be more like engineers and other "design scientists" than they are like "pure" scientists in fields such as physics or chemistry. In a completely "pure" science, a theory would tell which problems are priorities to solve; the theory also would determine the correctness of permissible solution processes; and the theory also would determine when the problem is solved. Whereas, in design sciences, problems arise in the "real world" (outside of any theory); solution processes usually need to integrate ideas and procedures drawn from a variety




**Theories of Learning Mathematics, Fig. 2** Stopping the two systems after 10 s

of disciplines (or textbook topic areas); and problems are not solved until the relevant real-life issue is resolved.

Why do realistically complex problems tend to require solutions which draw on more than a single theory? One reason is because "real-life" problems often involve partly conflicting constraints – such as high quality and low costs, low risk and high gain, simple and complete. Is a Jeep Cherokee a better buy than a Ford Taurus or a Toyota Prius? Answers depend on preferences of relevant decision makers. So, "one size fits all" is seldom a principle that decision makers will accept.

The central shortcoming of mathematics education research is not a lack of success in producing effective programs and materials. The central problem is lack of accumulation – coupled with the repeated recycling of previously discredited ways of thinking. And for accumulation to occur, it is important to notice that, in mature sciences, research communities tend to devote large portions of their time and energy to the development of tools to provide infrastructure for their own use. So, it is revealing that the mathematics education research community still does not have tools to document and assess the most important achievements that are expected of students, teachers, or programs.

To recognize why lack of accumulation has been such a problem in mathematics education, consider the following facts. If it were possible (*It isn't!*) to inspect the archives of all past curriculum innovation projects which have been supported by agencies such as the US *National Science Foundation*, then (beginning with early projects such as School Mathematics Study Group, The Madison Project, and MiniMast and continuing up to current times) inspectors of these archives would have no difficulty producing convincing evidence that important parts of most of these projects would be highly likely to be useful and effective today (under some conditions and for some students, some teachers, some schools, and some communities). On the other hand, other parts clearly would be missing or in need of significant revision. For example, most projects that focused on the development of innovative learning materials for children were not accompanied by adequate teacher development materials or implementation plans to help projects evolve from entry-level implementations (during the first year) to more complex and comprehensive implementations (during the Nth year). Furthermore, most of these projects did not provide assessment tools to document the achievements of higher-level achievements of students, teachers, or programs.

If mathematics education researchers pointed to one topic area where they believe theory development to be strongest, they'd likely point to either (a) early number concepts or (b) early algebraic reasoning (or rational numbers and proportional reasoning). Evidence of this theory development in learning is found in the literature related to Piaget-like cognitive structures (Steffe 1995; Steffe et al. 1996), cognitively guided instruction which focuses on task variables which are not at all like Piagetian cognitive structures, the focus on counting strategies and Vygotsky's socially mediated views of development, and focus on computer-based embodiments which are in some ways similar to those used by Zoltan Dienes (Sriraman 2008) but which also emphasize constructs similar to those emphasized by Steffe.

Yet, each of the preceding perspectives are based on significantly different (and in some ways incompatible) ways of thinking about mathematics concept development. One place where differences can be seen where the preceding perspectives differ has do with "learning trajectories" (or "learning progressions") through which development occurs. The notion of "learning trajectories" generally describes development (in both learning and problem-solving situations) as if it were like a point moving along a path. Yet, the following facts are well known:

1. It is easy to change the difficulty of a given task by several years by varying mathematically insignificant aspects of the task.
2. Research on models and modeling has shown that thinking is far more situated than traditional perspectives have suggested – because thinking tends to be organized around experience as much as it is organized around abstractions.
3. For a given concept, understandings develop along a variety of interacting dimensions: concrete-abstract, situated-decontextualized, specific-general, intuition-formalization, etc.
4. In each of the preceding dimensions, there exist "zones of proximal development" (ZPD) similar to those described by Vygotsky. Can these ZPDs be unpacked?
5. The development of "big ideas" interacts – so that understandings of any one of them depend partly on the development of others.

We conclude this encyclopedic entry with more questions than answers per se, with the hope of the community becoming interested in answering these fundamental questions in their quest for developing theories of mathematical learning.

- How do understandings of various "big ideas" interact?
- How does the development of "big ideas" interact with the development of "basic skills"?
- How does the development of "big ideas" interact with the ability to use these ideas in situations that are not pre-mathematized (outside of mathematics classrooms)?

## Cross-References

▶ History of Mathematics Teaching and Learning
▶ Policy Debates in Mathematics Education
▶ Zone of Proximal Development in Mathematics Education

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# Types of Technology in Mathematics Education

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## Keywords

Computers; Computer software; Communication technology; Handheld; Mobile; E-learning

## Terms and Definitions

Many of today's mathematics classrooms around the world are nowadays equipped with a variety of technologies. By using the term "technology," we mainly mean "new technology," as we refer to the "most prominent," recent, and "modern tool" in the teaching of mathematics that is labeled with terms "computers," "computer software," and "communication technology," according to Laborde and Sträßer (2010), p. 122. Another term "digital technology" which denotes a wide range of devices including a hardware (such as processor, memory, input–output, and peripheral devices) and software (applications of all kinds: technical, communicational, consuming, and educational) is used by Clark-Wilson, Oldknow, and Sutherland (2011). This is contrasted with yet another term Information and Communication Technology (ICT) widely used in a variety of educational contexts and describes the use of so-called "generic software" which means word processing, spreadsheets, along with presentational and communicational tools (such as e-mail and the Internet) (2011).

## Historical Background

Historically, technology and mathematics go alongside by mutually influencing each other's development (Moreno and Sriraman 2005). History does provide us with many technologies that enhance people to count (stones, pebbles, bones, fingers), to calculate (abacus, mechanic devoices, electronic devices), to measure (ruler, weights, calendar, clock), to construct (compass, ruler), and to record statistical data (cards with holes, spreadsheets) (Fig. 1).

As example of such devices, we can name the famous Ishango bone, an artifact of ingenious mind of our ancestors recently analyzed by Pletser and D. Huylebrouck (1999) who point at its possible function as one of the oldest known computational tool along with its other possible uses (calendar, number system, etc.). The invention of mechanical counting devices takes its origins from different kinds of abacus, such as Greek abax, meaning reckoning table covered with the dust or later version with disks moving along some lines (strings) (Kojima 1954). It is interesting that in some cultures, abacus was used till very recent times, as in Russia, in the everyday commerce to do calculations with moneys. Today, they may appear as educational support to enhance reasoning about quantities, such as Rekenrek (Blanke 2008).

**Types of Technology in Mathematics Education, Fig. 1** http://nrich.maths.org/6013

**Types of Technology in Mathematics Education, Fig. 2** http://www.sciencemuseum.org.uk/objects/mathematics/1927-912.aspx

Punch cards were invented by Hollerith, and his machine was used by the US Census Bureau to process data from 1890 till 1950s when it was replaced by computers (http://www.census.gov/history/www/innovations/technology/the_hollerith_tabulator.html) (Fig. 2).

## First Computers and Their Use in Education

Computers themselves can be seen as "mathematical devices," and their timeline goes back to abacus and is further marked by names of Leonardo da Vinci who conceived the first mechanical calculator (1500), followed by "Napier's bones" invented by Napier for multiplication (1600), based on the ancient numerical scheme known as the Arabian lattice; then comes the Pascaline, a mechanical calculator invented in 1642 by Pascal. Leibnitz (1673) and Babbage (1822) were among others who significantly contributed to the advancement in creation of automatic calculators which led, in the first half of the 1920s century, to the construction of the first computers, such as ENIAC (Electrical Numerical Integrator and Calculator), by Mauchly and Eckert, in 1946, mainly for military purposes. The second half of the twentieth century was marked by the rise of the IBM (International Business Machines); one of its models was used to prove the famous Four-Color Theorem (Appel and Haken 1976) (Fig. 3).

The time period after 1950 and till early 1980s was marked by as rather slow but sure penetration of mainframe and minicomputers in education, including mathematics education. With the main focus on accessibility of such devices for schools (question of costs and space), other questions arose by mathematics educators at that time regarding the purposes of its use and impact on learning. Zoet (1969) pointed at several dilemmas, namely, (1) about the capacity of computers to *process* data, like in business management to produce bills for millions of customers, on the one side, and to *compute* data, like in mathematical modelling where scientists need to do large amount of calculations in a short period of time; (2) about the time needed to master a particular part of technology (to solve mathematical problems) . . . which will soon be replaced with a new one; and (3) about the possibility of computer to assist a greater number of students to grasp principles of mathematics, as well as strengthen and broaden students' understanding, about whether mathematics learned by

**Types of Technology in Mathematics Education, Fig. 3** http://www-03.ibm.com/ibm/history/exhibits/mainframe/mainframe_PP3168.html

**Types of Technology in Mathematics Education, Fig. 4** http://el.media.mit.edu/logo-foundation/logo/turtle.html, repeat 3 [forward 50 right 60]

the students will be more functional, once they see how it is used in computers, or if small computers can be integrated into mathematics programs as the slide rule in the training of engineering students.

In 1970s–1980s, special languages, like FORTRAN, PASCAL, BASIC, were used as the first software, and their mastery was necessary to use computers effectively including mathematics calculations and modelling of mathematical processes and thus enhancing learning. One of such languages (LISP) was used to create the LOGO, a programming language designed by Papert (1980) specifically for educational purposes. According to Pimm and Johnston-Wilder (2005), a common starting point in creating LOGO programs was writing commands allowing for directing and controlling a "turtle" on the screen. This idea led to construction of specific mathematically rich learning environments called microworlds (Pimm and Johnston-Wilder (2005)) (Fig. 4).

In 1984, the NCTM (National Council of the Teachers of Mathematics) produced a yearbook entirely devoted to the topic on Computers in Mathematics Education (Hansen and Zweng 1984) portrays newest types of technologies called microcomputers as having endless list of applications available for mathematics teachers and learners which are becoming widely accessible for schools at low cost; it also adds graphics capabilities to support mostly two-dimensional representations (Fey and Heid 1984). Again this technology development interacts with pedagogical use as tutor, tool, and tutee (Fey and Heid 1984, referred to Taylor 1980) with questioning whether "traditional collection of mathematical skills and ideas needs" to be acquired by students to enable them "to operate intelligently in the computer-enhanced environment for scientific work," or one must have "new skills or understandings" to get prepared "for mathematical demands that lie in the twenty-first century" (Taylor 1980, p. 21). Regarding the format of integration of such technology in the process of teaching, educational institutions usually put computers in one classroom (computer lab) shared by several groups of students, or they can put a number of desktop computers (1–4) in a regular classroom, so teachers and students can work with them individually or in small groups.

On those computers, teachers could find general software, including spreadsheets (like Supercalc, Lotus, or Excel) that could be used in multiple teaching and learning purposes, for example, to conduct probabilistic experiments and simulations (Anand et al. 2012). 1980s and 1990s were also marked by widely spread use of educational games, on small floppy disks, and later multimedia on CD-ROMs and DVDs, helping even the very young students to learn basics about numbers and shapes and develop mathematical thinking while playing with patterns. Another kind of the software specifically designed for mathematics classroom based on a constructionist's ideas leads to the development of dynamic and interactive computer environments in geometry (dynamic geometry systems) and algebra (computer algebra systems). Different types of virtual manipulatives thus become available to teachers to make learning more visual, dynamic, and interactive (Moyer et al. 2002).

Computer networks – systems of interconnected computers and systems of their support called Intranet and Internet that emerge and spread out in 1990s and 2000s. The first (Intranet) allows to connect computers with a restraint number of people having access to it; often it is used within an organization, like school or school board, or university. The second (Internet) is open to a much wider audience, in many cases worldwide, although it can serve closed groups/communities built with different purposes. This technology, with the time becoming more rapid (high-speed), wireless, and handheld, enhances communication of people or machines with other people or machines to share information and resources in all areas including mathematics. As example of such kind of technology, we will analyze web 2.0 tools.

## E-learning: Web 2.0 Tools and Their Use in Mathematics

Solomon and Schrum (2007) use the year 2000 as a turning point in the development of a new Internet-based technology called Web 2.0. They begin their timeline with year 2000 when the number of web sites reached 20,000,000. The year 2001 was marked by the creation of Wikipedia, the first online encyclopedia written by everyone who wanted to contribute to the creation of the shared knowledge. In 2003, the site iTunes allowed creating and sharing musical fragments. In 2004, the Internet bookstore Amazon.com allowed buying books entirely online. In 2005, the video-sharing site Youtube.com appeared, allowing producing and sharing short video sequences. The authors state that by the year 2005, the Internet had grown more in 1 year than in all the years before 2000, reaching 1,000,000,000 sites by 2006.

The result of this tremendous growth of Internet-based environments and the educational resources generated by them is a transformation of e-learning itself. According to O'Hear (2006), the traditional approach to e-learning was based on the use of a Virtual Learning Environment (VLE) which tended to be structured around courses, timetables, and testing. That is an approach that is too often driven by the needs of the institution rather than the individual learner. In contrast, the approach used by e-learning 2.0 (a term introduced by Stephen Downes) is "small pieces, loosely joined," as it combines the use of discrete but complementary tools and web services – such as blogs, wikis, and other social software – to support the creation of ad hoc learning communities. Let us look at several features of these tools as we analyze a few examples of mathematical opportunities they create (adapted from Freiman 2008).

Wiki is an Internet tool allowing a collective writing of different texts as well as sharing a variety of information. Everybody can eventually be a contributor to the creation of a web site on a certain topic (or several topics, as it is in the case of the Wikipedia, www.wikipedia.org/).

*Podcasts* can be used to audio-share mathematical knowledge among a larger auditorium than one with people sitting in a traditional classroom. It can be used as a method of delivering mathematical lectures online as well as for the promotion of mathematics.

Video-casting opportunities are provided by multiple Internet sites, allowing the creation and sharing of video sequences produced by the users. For example, an article published in one local newspaper informs the readers about one university professor who put a 2-min video about a Mobius strip on the Youtube.com site.

The sequence was viewed by more than one million users within 2 weeks. The environment offers not only an opportunity to view the video but also to assess it (using a 5-star system) and to share it with others, as well as publish a comment.

Photo sharing is yet another form of creating and sharing knowledge, available on several dynamic sites with photo galleries like Flickr. Regrouped by categories that can be found by an easy-to-use search engine, the photos can be published and discussed by the members of a community, as for example, the community that discusses geometric beauty which numbers almost 5,000 members. Each photo is provided with a kind of ID card that documents useful information such as the date of its publication, the author's (or publisher's) username, as well as the list of all other categories to which the photo belongs, the date when the photo was taken, and how many other users added it to their albums.

Discussion forums allow building online communities that talk to each other by posting questions and giving answers. This collective work may enable a student who is struggling with mathematical homework to address other people and ask for help, as illustrated by the following example from the math forum site (mathforum.org). The message posted by one user says that "after having asked a teacher and having read a book," she "still had a feeling" that she needed more explanation, so she appealed to the whole virtual community asking for help. The discussion on some questions can take the form of multiple exchanges between members.

Blogs may provide multiple educational opportunities as they are built by means of easy-to-use software that removes the technical barriers to writing and publishing online. The "journal" format encourages students to keep a record of their thinking over time facilitating critical feedback by letting readers add comments – which could be from teachers, peers, or a wider audience. Students may use blogs for different purposes: to provide a personal space online, pose questions, publish work in progress, and link to and comment on other web sources.

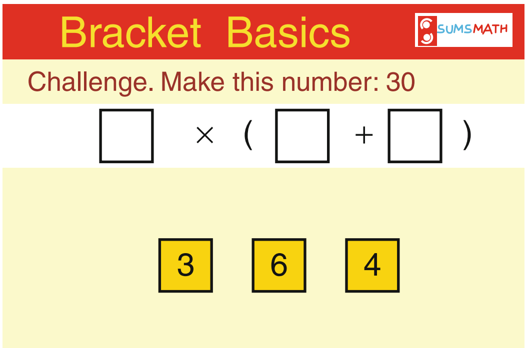
The learning model that can be extracted from our examples features three major educational trends related to the web 2.0 technology: knowledge building/co-constructing, knowledge sharing, and socialization by interaction with other people. Moreover, further development towards semantic web (web 3.0) technology has a potential to enhance self-learning, critical thinking, and collaborative and exploratory learning.

## M-Learning: Anytime, Anywhere with Laptops and Other Handheld Devices

Another recent trend is related to the rapid changes brought by so-called mobile technology that enhances anytime anywhere learning. Taking its roots from different types of calculators, it provides today's mathematics classrooms with several types of portable devices, such as laptop computers, iPads, iPhones, and other types of mobile technology (Fig. 5) (Jones et al. 2012).

According to Burrill et al. (2002), the first type of handheld technology mentioned as a part of the secondary school curriculum in 1986 was a Casio fx-7000G model. Even if the appropriate role of it in mathematics classroom was at that time (and still remains) debatable, it supported the creation of new visions for mathematics education while calling for broader access to deeper mathematics for all students (Burrill et al. 2002). Regarding the newest development of this type of technology, Burrill (2008) sees its potential to combine various learning environments like computer algebra systems (CAS) and Dynamic Geometry computer software, such as Dynamic Geometry Sketchpad or Cabri: "new technologies such as TI-Nspire bring together both of these environments in one handheld, providing the opportunity to create an even wider variety of dynamic linked representations, where a change in one representation is immediately and visibly reflected in another" (http://tsg.icme11.org/document/get/218).

Several laptop studies report about a variety of teaching and learning opportunities to use 1–1 portable technology for several subjects including mathematics. Freiman et al. (2011) developed and implemented problem-based learning (PBL)




**Types of Technology in Mathematics Education, Fig. 5** http://www.k12mathapps.com/

interdisciplinary scenarios (math, science, language arts) to measure and document students' actual learning process, particularly in terms of their ability to scientifically investigate authentic problems, to reason mathematically, and to communicate. In a rapidly changing world of technology and infinity of educational applications, mathematics teachers can now try to integrate newest technology, like iPads, in mathematics lessons. While only few research available, first pilot studies, like one reported by HMH (2010–2011, http://www.hmheducation.com/fuse/pdf/hmh-fuse-riverside-whitepaper.pdf) seems to have a positive impacts on students' performance. In this study, individual iPads were used along with the HMH Fuse: Agebra 1 programs. The application helped students use its multimedia components whenever and wherever they saw fit, regardless of Internet availability. In addition, students could take device home and "customize them," adding their own music, videos, and additional applications (Freiman et al. 2011).

Among other types of technologies to be mentioned are interactive whiteboards which, according to Jones (2004), might encourage more varied, creative, and seamless use of teaching materials; increase student' enjoyment and motivation; and facilitate their participation through the ability to interact with materials. While the whiteboards support and extend whole-class teaching in a more interactive way, haptic (in-touch) devices have a potential to enhance multimodal learning in 3-D spaces, on the individual base, or working in small groups, as the technology becomes less costly, more flexible in terms of usability, with better feedback options, allows for better merging with other mathematical learning environments, such as dynamic geometry (Güçler et al. 2012).

## Cross-References

- ▶ Information and Communication Technology (ICT) Affordances in Mathematics Education
- ▶ Learning Environments in Mathematics Education
- ▶ Learning Practices in Digital Environments
- ▶ Technology and Curricula in Mathematics Education
- ▶ Technology Design in Mathematics Education

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# U

## Urban Mathematics Education

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### Keywords

Culture; Diversity; Ethnicity; Race; "High-needs" schools; Socioeconomic class; Urban education; Urban schools

### Definition

*Urban Mathematics Education* (and/or *Urban Mathematics Education Research*) is characterized as a specific focus on the multilayered complexities as well as the challenges and promises of mathematics teaching and learning in high-density populated geographic areas. These "urban" areas more times than not contain greater human and cultural diversity in terms of "race," ethnicity, socioeconomic class, language, religion, disabilities/abilities, and sexual orientation and gender expression. Often times the phrase urban schools and, in turn, *urban mathematics education* are used as euphemistic proxies for (re)segregated schools and mathematics classrooms with high concentrations of poor and historical marginalized racial and/or ethnic student populations (e.g., in the United States, schools and classrooms with majority Black, Brown, and/or recent immigrant students [i.e., bilingual and multilingual students]).

### Suburban School/Urban School Binary

Over the past 40 years or so, a discursive binary between suburban school and urban school has emerged that privileges the suburban. This privileging has resulted in (re)segregated urban schools being further defined by euphemisms such as "hard-to-staff schools," "high-needs schools," and/or "low-performing schools" (Lipman 2011). Such euphemisms are used to gloss over challenges that too often plague urban schools such as ageing and inadequate infrastructures; dense and disconnected bureaucracies; uncertified or inadequately trained teachers; limiting and misdirected funds; and the ever-lingering damaging effects of race and racism, and xenophobia in general (Darling-Hammond 2010). These challenges, which typically have been found within what was commonly known as the urban "inner-city" school, are increasingly found in suburban and even rural schools (i.e., metropolitan suburban/rural sprawl) as schools in these geographic areas face similar challenges with the ever-changing racial, socioeconomic class, and citizenship status student demographic of suburban and rural communities and schools. These ever-changing demographics are motivated, in part, by gentrification of inner-city urban spaces (Lipman 2011).

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### Research in Urban Mathematics Education

Many researchers who work within the urban mathematics education domain deconstruct the euphemisms of urban schools as they make the social (Lerman 2000) and sociopolitical (Gutiérrez 2013) turns in mathematics education research. These researchers most often place an emphasis on contextualizing not only the mathematics classroom but also the concentric circles of school, district, community, and society at large in which the urban mathematics classroom is embedded (Weissglass 2002). Such contextualization makes possible a more complete analysis of the effects of the neoliberal and neoconservative agenda of urban (mathematics) education (Lipman 2011). Here, the mathematics teaching and learning dynamic of the classroom is not stripped of the sociocultural and sociopolitical power relations that exist within the multiplicity of interactions that occur in the mathematics classroom among teachers and students and the mathematics being taught and learned. Analyses of such power relations bring to the fore issues of equity and access, identity, and race, class, gender, language, and other sociocultural and sociopolitical discourses and practices that marginalize or silence groups of students, which, in turn, limit mathematics access, participation, and contribution of large groups of students.

Since the early to mid 2000s, a new trend in urban mathematics education research has emerged that highlights and examines the mathematics achievement and persistence of Black and Brown children within urban contexts and the effectiveness of urban teachers, schools, and districts (see, e.g., the edited volumes Leonard and Marin 2013; Martin 2009; Téllez et al. 2011). Much of this emerging research provides counter-stories or -narratives to the discourses of deficiency and ineffectiveness that too often frame urban mathematics students, teachers, and classrooms. For exemplars of urban mathematics education research, see the *Journal of Urban Mathematics Education* (*JUME*), a peer-reviewed, open-access, academic journal published twice a year: http://education.gsu.edu/JUME.

## Cross-References


- ▶ Bilingual/Multilingual Issues in Learning Mathematics
- ▶ Cultural Diversity in Mathematics Education
- ▶ Cultural Influences in Mathematics Education
- ▶ Equity and Access in Mathematics Education
- ▶ Immigrant Students in Mathematics Education
- ▶ Rural and Remote Mathematics Education
- ▶ Socioeconomic Class in Mathematics Education


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# V

## Values in Mathematics Education

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### Introduction

Values are a significant feature of education in any field, but it is only recently that values in mathematics education have been considered significant, or even recognized. This entry provides a historical perspective to the growing relevance of values in mathematics education. It also illustrates how different researchers have addressed different aspects of values depending on their theoretical and educational foci.

One focus has values being addressed as a characteristic of the person, which approaches values as a psychological construct. It builds on the research in mathematics education which explores values as related to learners' and teachers' attitudes, beliefs, and affect generally.

The other main research focus conceptualizes values as a sociocultural construct and is more concerned with the sociocultural context of mathematics education in which values are observed and negotiated. This approach builds on the relevant historical, cultural, and philosophical literatures at the intersection of mathematics and education.

While this entry describes generally the different meanings of values in mathematics education and how they seem to develop in both the individual and society, it finally introduces the challenging issues of *whether* desirable values can be developed in students through mathematics education and how values in mathematics education *should* be developed.

#### Values as Personal Constructs

Krathwohl et al. (1964) significant book on educational goals gives us a useful starting point. Their work was based on a behaviorist approach and was hierarchical in structure. Thus at their levels 3 and 4 (from 5), one finds the following categories of goals:

3. Valuing: 3.1 acceptance of a value, 3.2 preference for a value, and 3.3 commitment
4. Organization: 4.1 conceptualization of a value and 4.2 organization of a value system.

Of particular interest is their behaviorist background theory which gives us the distinction, and relationship, between values and valuing. From an educational viewpoint, this distinction is highly significant. "Valuing" is clearly a behavior but with no specification of what is to be valued. "Values" on the other hand represent what is to be valued, a totally different educational objective.

The research of Raths et al. (1987) offers a related perspective. They describe seven general criteria for calling something a value. Their criteria are (1) choosing freely, (2) choosing from alternatives, (3) choosing after thoughtful consideration of the consequences of each alternative, (4) prizing and cherishing; (5) affirming, (6) acting upon choices, and (7) repeating.

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They say "unless something satisfies all seven of the criteria, we do not call it a value, but rather a 'belief' or 'attitude' or something other than a value" (Raths et al. 1987, p. 199). They add "those processes collectively define valuing. Results of this valuing process are called values" (p. 201). Their emphasis on choices and choosing is also important in separating values from beliefs. One may hold several different beliefs but values are most likely to appear when the individual makes specific choices. This point is important for both research and practice.

Values in mathematics education have however generally been couched in terms of affect and attitudes. As a leading proponent of this research, McLeod (1989, 1992) separates beliefs, attitudes, and emotions, where beliefs can be about mathematics (e.g., mathematics is based on rules), about self (e.g., I am able to solve problems), about mathematics teaching (e.g., teaching is telling), and about the social context (e.g., learning is competitive). Attitudes can be exemplified by a dislike of geometric proof, the enjoyment of problem solving, or a preference for discovery learning. Emotions appear through, for example, joy (or frustration) in solving nonroutine problems or an aesthetic response to mathematics.

However he like others at that time made no reference to values, but one senses from his writing that he would see values as linking strongly with both beliefs and attitudes. Krathwohl et al. (1964) support this view: "Behaviour categorized at this level (3) is sufficiently consistent and stable to have taken on the characteristics of a belief or an attitude. The learner displays this behaviour with sufficient consistency in appropriate situations that he (sic) comes to be perceived as holding a value" (p. 180). So from this perspective, values grow out of beliefs and attitudes.

### Values as Sociocultural Constructs

The seminal work of Kroeber and Kluckhohn (1952) and Kluckhohn (1962) gives us an entrée into this other historical, and related, dimension of research on values. This is best summed up by the construct "cultural psychology," a branch of psychology which takes into consideration the culture of the context in which the learner (in this case) is operating. Lancy (1983) was an early researcher in this area, and he updated Piaget's work with his research from Papua New Guinea. He proposed that three stages were/are significant in a learner's development where cultural influence is paramount:

Stage 1, where genetic programming has its major influence and where socialization is the key focus of communication.

Stage 2, where enculturation takes over from socialization and, for example, where ethnomathematics becomes relevant.

Stage 3, which concerns the metacognitive level and where different cultural groups emphasize different theories of knowledge. These theories of knowledge represent the ideals and values lying behind the actual language and symbols developed by a cultural group. Thus in relation to the previous section, it is in Stages 2 and 3 that values are inculcated in the individual learners, and Stage 3 is where the value system is developed.

In the classic work by Kroeber and Kluckhohn (1952), they strongly support this general idea: "Values provide the only basis for the fully intelligible comprehension of culture because the actual organisation of all cultures is primarily in terms of their values" (p. 340). Moreover culture has been defined as an organized system of values which are transmitted to its members both formally and informally (McConatha and Schnell 1995, p. 81).

Thus from the perspective of mathematics education, the idea of mathematical thinking as a form of metacognition affected by the norms and values of the learner's society and culture is helpful. But where do these norms, values, and knowledge come from, and how are they framed in educational contexts?

Two points must be made here – firstly as Bishop (1988) has explained, it was the values which have been held by previous mathematicians which have shaped the field we know as mathematics today. Secondly, the research field of ethnomathematics has demonstrated that all cultures develop their own mathematical ideas and practices. This has not only generated a great deal of interesting evidence, but it has fundamentally changed many of our research ideas and constructs. The most significant influences have been in relation to:

- Human interactions. Ethnomathematics research concerns mathematical activities and practices in society, which take place outside school, and it thereby draws attention to the roles which people other than teachers and learners play in mathematics education.
- Values and beliefs. Ethnomathematics research makes us realize that any mathematical activity involves values, beliefs, and personal choices.
- Interactions between mathematics and languages. Ethnomathematics research makes us aware that languages act as the principal carriers of mathematical ideas and values in different cultures.
- Cultural roots. Ethnomathematics research is making us more aware of the cultural starting points and histories of mathematical development.

One example of an educational approach was derived from the cultural perspective of White (1959), an anthropologist interested in the ways cultures develop. Based on his research he argued that for all cultures to develop, they need cultural components which are technological, sentimental, societal, and ideological.

Translating this into mathematics, Bishop (1988) argued that the value dimensions could be formed of complementary pairs, using White's categories, producing six values: rationalism and objectivism (ideological), control and progress (sentimental), and openness and mystery (sociological). The technological component is given by the symbolic technology of mathematics. Using these categories research has explored teachers' values, students' values, and values in the mathematics curriculum and in teacher education.

### Educating Values and Developing Mathematics Education

One interesting fact is that there is little or no indications in the research literature above concerning the educational means of attaining any of the value goals and objectives outlined there, apart from the idea that values education should involve the existence of alternatives, choices and choosing, preferences, and consistency. Bishop et al. (2001) set out to investigate this in practice. The main conclusion was that values did not seem to mean much to the mathematics teachers in the study, while much harder still for them was the idea of trying to "teach" different values from the ones they normally "taught." A further study focused on understanding the values that the students were learning. The idea that values are revealed at choice points is only helpful when people have the opportunity to make valid and consistent choices. If one considers a "normal" mathematics classroom, however, students rarely have the opportunity to exercise any choices.

There are many connections between values in mathematics and in science (Bishop et al. 2006). Their study showed that useful research on values, and its associated data collection, should stay close to the experienced situation of the subjects, emphasizing as Raths et al. (1987) argued, that values are thoroughly personal attributes, and not easily developed within the social context of a classroom.

Not only are values personal attributes, they have a strong emotional characteristic, as McLeod (1992) also suggested. Future research can potentially increase our understanding of the relationship of values with the positive emotional side of mathematics learning.

## Cross-References

▶ Anthropological Approaches in Mathematics Education, French Perspectives
▶ Ethnomathematics
▶ Teacher Beliefs, Attitudes, and Self-Efficacy in Mathematics Education
▶ Students' Attitude in Mathematics Education

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## Visualization and Learning in Mathematics Education

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### Keywords

Signs; Symbols; Diagrams; Spatial aspect; Representation; Preferences of individual learners; Spatial abilities; Visual mental imagery; Inscriptions; Visual image; Ana-vis scale; Logic; Strength of mathematical processing; Type; Verbal-logical; Visual-pictorial; Analytic, geometric, and harmonic types; Reluctance to visualize; Pedagogy; Abstraction; Generalization; One-case concreteness; Prototype; Uncontrollable image; Compartmentalization; Dynamic imagery; Pattern imagery; Metaphor; Mnemonic advantages; Interactive dynamic geometry software; Gestures; Conversion processes; Registers; Connections; Idiosyncratic visual imagery; Reification; Computer technology; Overarching theory of visualization

### Definitions and Background

Visualization in mathematics learning is not new. Because mathematics involves the use of signs such as symbols and diagrams to represent abstract notions, there is a spatial aspect involved, that is, visualization is implicated in its representation. However, in contrast with the millennia in which mathematics has existed as a discipline, research on the use of visual thinking in learning mathematics is relatively new. Such research has been growing in volume and depth since the 1970s, initiated by Bishop (1973, 1980) and later Clements (1981, 1982), who investigated preferences of individual learners with regard to visualization in mathematics and how spatial abilities interacted with these preferences. Visualization has internal and external forms (Goldin 1992), which may be designated as visual mental imagery and inscriptions, respectively (Presmeg 2006). Presmeg defined a visual image as a mental sign depicting visual or spatial information and inscriptions as symbols, diagrams, information on computer screens, or any external representation with a visual component. Following Piaget and Inhelder's (1971) claim that visual imagery underlies the creation of a drawing or a spatial arrangement, Presmeg did not pursue the distinction between external and internal visual images.

Arcavi (2003, p. 217) blended definitions given by previous authors (Hershkowitz et al.

1989; Zimmermann and Cunningham 1991) to provide the following summary:

Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings.

### Preference for Visualization in Mathematics

Under the supervision of Ken Clements, early research on learners' preferences for using visualization in mathematics was carried out by Suwarsono (1982), who developed a *mathematical processing instrument* (MPI) for use with seventh graders in Australia. This instrument included word problems capable of solution by visual or by nonvisual mathematical means, and a questionnaire in which learners could identify the means they had used to solve the problems, yielding a score of *mathematical visuality* (MV). Presmeg (1985) followed Suwarsono's methodology in constructing his instrument, but designed her MPI for use with learners in grades 11–12 (parts A and B) and their mathematics teachers (parts B and C, more difficult), thus enabling comparison of the MV scores of teachers and students on part B, which was common. Nonparametric statistics revealed no significant difference between boys and girls in her study, but a significant difference between teachers and students: the learners needed more visual means than did their mathematics teachers. For most populations the preference for visuality (MV) scores follows a normal, Gaussian, frequency distribution. Factors that determine how a task will be approached include the following: the task itself, instructions to do the task in a certain way, individual preferences, and, finally, the culture of the mathematical learning environment including whether or not visualization is valued. At the far ends of the frequency distribution, some learners seldom resort to visualization, whereas there are others who *always* do so. The latter form part of a group of learners who are called *visualizers*.

### Theoretical Lenses

Early research on visualization in mathematics (e.g., Clements 1981, 1982) used a conceptual lens that opposed analysis and visualization, an "ana-vis" scale, on which individuals could be placed according to the preponderance of logical analysis or visualization in their mathematical thinking. However, Krutetskii (1976) argued, on the basis of his vast data pool, that without logical analysis there is no mathematics, whereas the use of what he termed "visual supports" is optional. Logic determines the strength of mathematical processing, whereas visualization (or its absence) determines the type. One might consider these two aspects of mathematical thinking on orthogonal axes: strength of logic on the x-axis and amount of visualization on the y-axis. Krutetskii (1976) worked with students who were considered "capable" in mathematics. On the basis of their problem solving in task-based interviews, he classified these students into groups according to the type of their thinking, i.e., according to whether verbal-logical or visual-pictorial thinking predominated (*analytical* and *geometric* types, respectively) or whether these aspects were in equilibrium (two types of harmonic thinkers – *abstract-harmonic* and *pictorial-harmonic*). These types would all lie in the right-hand quadrants of the orthogonal model, because of the ubiquitous strength of logic demonstrated by Krutetskii's learners. However, when Presmeg (1985) analyzed the mathematical achievements and type of thinking of grade 11 students according to her preference for visualization test, individuals could be classified in approximately equal numbers in all four of the quadrants. It is significant that not all students with strong spatial ability, who are capable of using visualization in their mathematical thinking, choose to do so. This aspect points to the interaction of visualization learning styles with other aspects of the classroom, as summarized in the next section.

### Interaction of Visualization Styles in Learning and Teaching

Dreyfus (1991) and Eisenberg (1994) suggested from their research that students are reluctant to visualize in mathematics. The evidence for their claim was largely from students learning college-level mathematics. However, this phenomenon could be the result of cultural environments in which visualization is not valued in mathematics. Presmeg's (1985 and later) frequency distribution graphs showed clearly that there is not a shortage of visualizers in mathematics. She explored the interactions between the teaching styles of 13 high school teachers and 54 visualizers in the mathematics classes of these teachers. It was noteworthy that there was a correlation of only 0.4 (Spearman's rho) between the teachers' mathematical visuality (MV) and teaching visuality (TV) scores. Several teachers realized that their students required more visual supports than they did and taught accordingly. The TV scores enabled the teachers to be distributed into a visual group, a middle group, and a nonvisual group. Visualizers in the classes of teachers in the nonvisual group attempted to follow the styles of their teachers, without visualization, and the result was lack of success, involving memorization without understanding. Surprisingly, visualizers with visual teachers also often experienced difficulty. It was the pedagogy of teachers in the middle group that was optimal for these visual learners. These teachers used and encouraged visual methods of working, but they *also* stressed that abstraction and generalization are important in mathematics.

## Difficulties and Affordances of Use of Visualization in Mathematics

Several research studies have emphasized that visualization needs to link with rigorous logic and analytical thought processes to be effective in mathematics (Arcavi 2003). Presmeg (1985, 1986) identified difficulties and strengths of mathematical visualization in data from task-based interviews with the 54 visualizers in her study. All the difficulties related in one way or another to the abstraction and generalization that are essential aspects of doing mathematics.

- The one-case concreteness of an image may be tied to irrelevant details or introduce false information.
- A prototype image may induce inflexible thinking.
- An uncontrollable image may persist, thereby preventing more fruitful avenues of thought.

Implicit in these difficulties is *compartmentalization*, whose damaging effect in learning mathematics has been noted by several authors (Duval 1999; Nardi et al. 2005; Presmeg 1992). There are two basic ways in which these difficulties can be overcome (Presmeg 1986, 1992). Firstly, a visual image or inscription of one concrete case can be the bearer of abstract information, that is, a sign for an abstract object. *Dynamic imagery* and *pattern imagery* are types of imagery that are useful in this regard. Secondly, *metaphor* can link the domain of abstract mathematical objects with visual imagery or inscriptions in a different domain. Visual images of all types have mnemonic advantages; pictures and spatial patterns are often memorable.

## Questions for Research on Visualization and Learning in Mathematics

Presmeg (2006, p. 227) put forward a list of 13 questions requiring further research, which she considered to be of significance for mathematics education. Many of these questions have received attention (e.g., Arcavi 2003; Nardi et al. 2005; Owens 1999; Presmeg 1992, 2008; Yerushalmy et al. 1999), but the list is still indicative of areas in which research is needed in order to increase knowledge of the role of visualization in effective learning of mathematics. Particularly in the computer age, the affordances of technology inevitably change the dynamics of the way in which mathematics is learned, including its visualization (Yerushalmy et al. 1999; Yu et al. 2009). Yu and colleagues found that the use of interactive dynamic geometry software in learning geometry at middle school level inverted the order of the levels of learning geometry established by van Hiele and van Hiele-Geldof in The Netherlands in the 1950s (Battista 2009).

1. What aspects of pedagogy are significant in promoting the strengths and obviating the

difficulties of use of visualization in learning mathematics?
2. What aspects of classroom cultures promote the active use of effective visual thinking in mathematics?
3. What aspects of the use of different types of imagery and visualization are effective in mathematical problem solving at various levels?
4. What are the roles of gestures in mathematical visualization?
5. What conversion processes are involved in moving flexibly amongst various mathematical registers, including those of a visual nature, thus combating the phenomenon of compartmentalization?
6. What is the role of metaphors in connecting different registers of mathematical inscriptions, including those of a visual nature?
7. How can teachers help learners to make connections between visual and symbolic inscriptions of the same mathematical notions?
8. How can teachers help learners to make connections between idiosyncratic visual imagery and inscriptions and conventional mathematical processes and notations?
9. How may the use of imagery and visual inscriptions facilitate or hinder the reification of processes as mathematical objects?
10. How may visualization be harnessed to promote mathematical abstraction and generalization?
11. How may the affect generated by personal imagery be harnessed by teachers to increase the enjoyment of learning and doing mathematics?
12. How do visual aspects of computer technology change the dynamics of the learning of mathematics?
13. What is the structure and what are the components of an overarching theory of visualization for mathematics education?

## Cross-References

► Abstraction in Mathematics Education
► Creativity in Mathematics Education
► Epistemological Obstacles in Mathematics Education
► Information and Communication Technology (ICT) Affordances in Mathematics Education
► Mathematics Teachers and Curricula
► Metaphors in Mathematics Education
► Problem Solving in Mathematics Education
► Semiotics in Mathematics Education
► Shape and Space – Geometry Teaching and Learning
► Visualization and Learning in Mathematics Education

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# W

## Wait Time in Mathematics Teaching

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### Keywords

Achievement; Cognitive level

### Definition

Research on the effects of the amount of time teachers wait for student responses after asking a question.

### Characteristics

Research in this area began in the 1960s in the science education field (Rowe and Hurd 1966) and asked the question of what connections there might be between the time teachers wait after asking a question and the cognitive level of students' responses. Tobin (1986) found, in a scientific study set in a whole-class mathematics instructional setting, that there were improvements in teachers' and students' discourse, including length of student responses, as well as higher mathematics achievement with increased wait time. More recently, the encouragement to teachers to wait longer than the typical 1s (though not more than 5, according to Tobin (1986), as advantages are lost) has gained further prominence by being seen as a key element of formative assessment (Black et al. 2002). Future research questions include research on gender- or other group-specific effects, the cognitive level of teachers' initial questions, and relationship to social and socio-mathematical norms.

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## Word Problems in Mathematics Education

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### Keywords

Affect; Algebra; Arithmetic; Cognitive psychology; Ethnomathematics; Metacognition;

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Modeling; Problem solving; Sociocultural theories; Word problems

## Definition and Function of Word Problems

Word problems are typically defined as verbal descriptions of problem situations wherein one or more questions are raised the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement (Verschaffel et al. 2000). As such they differ both from bare sums presented in written (e.g., 4 + 5 = ?; 5x + 2 = 22) or oral form (e.g., How much is 40 divided by 5?; What is the mean of the numbers 12, 17, 17, 18?), as well as from quantitative problems encountered in real life (e.g., Which type of loan should we take? Can I drive home from here without filling the tank?).

Importantly, the term "word problem" does not necessarily imply that every task that meets the above definition represents a true *problem*, in the cognitive-psychological sense of the word, for a given student, i.e., a task for which no routine method of solution is available and which therefore requires the activation of (meta) cognitive strategies (Schoenfeld 1992). Whether a word problem that a student encounters constitutes a genuine problem depends on his/her familiarity with the problem, his/her mastery of the various kinds of required knowledge and skills, the available tools, etc.

Word problems have always constituted an important part of school mathematics worldwide. Historically, their role in mathematics education dates back even to antiquity. One can find word problems already almost 4,000 years ago in Egyptian papyri. They also figure in, for instance, ancient Chinese and Indian manuscripts as well as in arithmetic textbooks from the early days of printing, such as the Treviso arithmetic of 1487, and they continue to fill current mathematics textbooks (Swetz 2009).

Despite this continuity across time and cultures, there has been little explicit discussion of why word problems should (continue to) be such a prominent part of the curriculum, and during the past decades writers have called for a reexamination of the rationale for this privileged position (see, e.g., Lave 1992). It can be inferred that word problems have been included to accomplish several goals, the most important one being to offer practice for everyday situations in which learners will need what they have learned in their arithmetic, geometry, or algebra lessons at school (the so-called application function). Other goals were and still are to motivate students to study mathematics, to train students to think creatively and to develop their problem-solving abilities, and to develop new mathematical concepts and skills.

## Characteristics

### Research Perspectives on Word Problem Solving

Word problems have already for a long time attracted the attention of researchers in psychology and (mathematics) education (see, e.g., Thorndike 1922). Before the emergence of the information-processing approach, research on word problems focused mainly on the effects on performance of various kinds of linguistic, computational, and/or presentational task features (e.g., number of words, grammatical complexity, presence of particular key words, number and nature of the required operations, nature and size of the given numbers) and subject features (e.g., age, gender, general intelligence, linguistic, and mathematical ability of the problem solver) (Goldin and McClintock 1984).

With the rise of the information-processing approach, researchers' attention shifted from learners' externally observable performance to the underlying cognitive schemes and thinking processes of students solving various kinds of word problems, and, accordingly, their research methods changed as well. Analyses of response accuracies were complemented with analyses of thinking aloud or retrospective protocols, individual interviews, reaction times, eye movements, and, most recently, neuropsychological measurements.

For instance, in the domain of one-step addition and subtraction word problems, a basic distinction emerged between three classes of problem situations: change problems (involving a change from an initial to a final state through the application of a transformation), combine problems (involving the combination of two discrete sets or splitting of one set into two discrete sets), and compare problems (involving the quantified comparison of two discrete sets of objects), each of which was further subdivided leading to a classification scheme of 14 problem types. Numerous cognitive-psychological studies with lower elementary school children provided evidence for the psychological validity of this classification scheme (Fuson 1992; Verschaffel et al. 2007).

Researchers also analyzed pupils' solution strategies in the domain of one-step addition and subtraction word problems (Carpenter and Moser 1984). These analyses first demonstrated that early in their development, children have a wide variety of successful material and verbal counting strategies, many of which are never taught explicitly and/or systematically at school. Gradually, these strategies develop into more formal mental solution strategies based on known and derived number facts. Second, it was found that the situational structure of a word problem significantly affects the nature of children's strategy choices. More specifically, children tended to solve each word problem with the type of strategy that corresponds most closely to its situation model. Similar findings have been found for the domain of multiplication and division word problems (Verschaffel et al. 2007).

Especially since the 1990s, insights from ethnomathematics and sociocultural theories have contributed to the insight that classical information-processing models are insufficient to grasp the full complexity of learners' word problem-solving processes. They need to be enriched with the idea that word problem solving is a human activity situated in the particular microcosm of a mathematics classroom (Lave 1992; Verschaffel et al. 2000), and that, therefore, students' word problem-solving behavior can only be understood by also seriously taking into account the tactics and the affects that they have built up along with their participation in the practice and culture of the mathematics classroom.

## Phases and Components of Competent Word Problem Solving

Currently the competent solution of a word problem is thought of as a complex multiphase process the "heart" of which is formed by (1) the construction of an internal model of the problem situation, reflecting an understanding of the elements and relations in the problem situation, and (2) the transformation of this situation model into a mathematical model of the elements and relations that are essential for the solution. These two steps are then followed by (3) working through the mathematical model to derive mathematical result(s), (4) interpreting the outcome of the computational work, (5) evaluating if the interpreted mathematical outcome is computationally correct and reasonable, and (6) communicating the obtained solution. This multiphase model is not considered to be purely sequential; rather, individuals can go back and forth through the different phases of the model (Blum and Niss 1991; Verschaffel et al. 2000).

Arguably, pupils' actual problem-solving processes do not always fit with this theoretical model. To the contrary, the process of actually solving word problems for many students is often along the lines of a "truncated" model, wherein the problem text immediately guides the mathematical model – the choice of an arithmetic operation, the selection of a geometric formula, or the composition of an algebraic expression – based on a quick and superficial analysis of the problem statement (e.g., by relying on key words in the text, such as the word "more" in the problem text automatically triggers an addition). The directly evoked mathematical operation, formula, or expression is then worked through, and the result of the calculation is found and given as the answer, typically without reference back to the problem text to verify whether the answer is meaningful in view of the original problem situation (Verschaffel et al. 2000). Concerning the competencies that are required to solve word problems, there is nowadays a rather broad consensus that they involve (Schoenfeld 1992; De Corte et al. 1996):

- A well-organized and flexibly accessible knowledge base involving the relevant conceptual knowledge (e.g., a schematic knowledge of the different problem types) and procedural knowledge (i.e., informal and formal solution strategies) that is relevant for solving word problems
- Heuristic methods, i.e., search strategies for problem analysis and transformation which increase the probability of finding a solution (e.g., making a drawing or a scheme) and metacognition, involving both metacognitive knowledge and metacognitive skills
- Positive task-related affects, involving positive beliefs, attitudes, and emotions, as well as meta-affect, involving knowledge about one's affects and skills for regulating one's affective processes

While there is evidence for the role of each of these aspects in students' word problem-solving processes and skills, it should be clear that they are strongly interrelated and interdependent.

## Solving Word Problems Versus Problems in the Real World

An issue that has received quite some attention during the past decades is the complex relation between word problems and reality. For a very long time, word problems have played their role as an unproblematic and transparent bridge between the world of mathematics and the real world. However, during the last 10–15 years, more and more researchers have questioned this role, partly on the basis of increasing empirical evidence of students' "suspension of sense-making" (Schoenfeld 1991) when doing school word problems and of aspects of the current practice and culture of word problem solving that seem directly responsible for this phenomenon (Verschaffel et al. 2000).

## Teaching Word Problem Solving

Besides ascertaining studies, researchers have also done numerous intervention studies – both design experiments and (quasi)experimental teaching experiments. While these intervention studies differ widely in terms of the age and mathematical background of the learners (from first graders up to university students) and the aspect(s) of word problem-solving expertise they are primarily aiming at (i.e., schematic knowledge, problem-solving skills, attitudes, and beliefs), common characteristics are:

- The use of varied, cognitively challenging, and/or realistic tasks, which lower the chance of developing superficial coping strategies (such as the key word strategy) and which involve the complexities of genuine mathematical application and modeling tasks (such as the necessity to seek and apply aspects of the real context to proceed, to discuss alternative models, to decide upon the required level of precision of the outcome).
- A variety of teaching methods and learner activities, including expert modeling of the strategic aspects of the problem-solving process, appropriate forms of scaffolding, small-group work, and whole-class discussions; typically, the focus is not on presenting and practicing well-established methods for solving well-defined types of problems, but rather on demonstrating, experiencing, articulating, and discussing what applied problem solving and modeling is all about.
- The creation of a classroom climate that is conducive to the development of a proper view of applied problem solving and mathematical modeling and of the accompanying skills and affects (Lesh and Doerr 2003; Verschaffel et al. 2000).

In most of these design experiments, (moderately) positive outcomes have been obtained in terms of performance, underlying (meta)cognitive processes, and affective aspects of learning. A final issue that has not yet elicited a lot of research, but that will become more important in the future, is whether word problems, which rely after all on an "old" vehicle for creating an applied problem situation (namely, printed text), will continue to keep their prominent position in the mathematics curriculum or whether they will be replaced or at least complemented by new and potentially more effective ways of bringing rich and real problems into the mathematics classroom, based on new information and communication technologies, such as video, computer graphics, and the Internet.

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# Z

## Zone of Proximal Development in Mathematics Education

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### Keywords

Vygotsky; Learning; Development; Expansion of agency; Collective agency; Activity theory

### Definition

The zone of proximal development is a category that emerged from the work of L. S. Vygotsky, the father of activity theory. Inspired by K. Marx, Vygotsky came to understand the specifically human characteristics in terms of society (Roth and Lee 2007). Explicitly referring to Marx, Vygotsky states that "any higher psychological function was external; this means it was social... *the relation between higher psychological functions was at one time a physical relation between people*" (Vygotsky 1989, p. 56). As a result of this perspective, our personalities are shaped by society: "*the psychological* nature of man is the totality of societal relations shifted to the inner sphere" (p. 59). Based on this understanding, he created a definition of the zone of proximal development that now has aphoristic qualities in educational circles. Thus, it denotes "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky 1978, p. 86).

### Zone of Proximal Development: An Example

The following example from a 3-D geometry lesson in a second grade class illustrates how the zone of proximal development tends to be used in education. On this day, the second graders each grabs a mystery object from a bag and places it with an existing group of objects or starts a new group. Connor has just completed placing his "mystery object" with a group of objects next to which one of the two teachers present has placed a label with the words "squares" and "cubes." The following fragment from the lesson begins when Mrs. Winter asks Connor what the group was about.

1. W: Em an' what did we say that group was about?
2. C: What do you mean like?
3. W: What was the– What did we put for the name of that group? What's written on the card?
4. C: Squares.
5. W: Square and...?
6. J: Cube.
7. W: Cube.

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At first, there is a counter-question rather than a reply: What does she mean? (turn 2). Mrs. Winter begins to rephrase: There is one abandoned question and then there are two full questions (turn 3). Now Connor replies providing one of the two words: Squares (turn 4). Mrs. Winter acknowledges his contribution by restating the word with a constative statement: Square (turn 5). She then says "and" with the rising intonation of a question. Jane says "cube" (turn 6), and Mrs. Winter acknowledges by repeating the word as she had done in the first instance (turn 7).

In the (dialogical) relation, Connor and Jane arrive at providing the sought-for answer because they do not do the entire task on their own. Here, they are part of a dialogical relation where the teacher takes one part of the task and the students the other. Now the task is spread across all participants. Later, once they are able to state the name and properties of the group without being asked, the children are said to have internalized. But it is evident that this description does not entirely match the situation. For the children to take their part in the relation, they already have to mobilize their understanding so that they can take their position in the question answer game that produces the result. The fact is, as Vygotsky's other way of framing says much more clearly, that the higher psychological functions exist in and as external relations between people. Thus, "the relation of psychological functions is genetically linked to real relations between people: regulation of the word, verbalized behavior = power–submission" (Vygotsky 1989, p. 57). In the exchange, under the tutelage of the teacher, they learn to provide the right words; she regulates the production of the words and regulated verbal behavior. As a result of the exchange, when they no longer need the external relation to name and characterize the group of cubes, Connor and Jane are in a position to individualize the social relation – they develop.

In the way the zone of proximal development is defined, there is an asymmetrical relation between those who know (teachers, more advanced peers) and those who do not (students). Researchers have drawn on the asymmetry between learner and the social other, because it orients to "the ways in which more capable participants structure interactions so that novices (children) can participate in activities that they are not themselves capable of" and to the fact that "with repeated practice, children gradually increase their relative responsibility until they can manage the adult role" (Cole 1984, p. 155). It thereby leads us to think about the learning process through the lens of the teacher who, because she/he is responsible for structuring the learning situation, becomes the "real subject in/of the child's learning" (Holzkamp 1993, p. 418).

## Recent Critical Reworking of the Notion

Recent work in mathematics education shows that the relations between teachers and students, such as Mrs. Winter and Connor and Jane, are much more symmetrical (Roth and Radford 2010). This is so because each has to understand the other for the episode to unfold as it does. For example, Connor already has to understand that Mrs. Winter is asking him a question, and he has to understand that he has trouble with her question. Similarly, Mrs. Winter has to understand that Connor is asking her to restate the question. Thus, the relation is more symmetrical than researchers have led on in the past. For example, precisely because her first question (turn 1) was not intelligible, Mrs. Winter has to rephrase it. She gives it several tries and eventually finds one that allows the children to provide first one and then the other expected word. That is, it is precisely in such relations that teachers such as Mrs. Winter become better and better at asking appropriate questions. The zone of proximal development, therefore, works both ways. Connor and Jane learn to talk about, name, and characterize mathematical objects; and Mrs. Winter develops as a teacher by learning to ask age-appropriate questions in a unit of 3-D geometry (it is her first time to teach such a unit at that grade level).

From a systemic perspective on cultural-historical activity theory, the concept of a zone of proximal development can be reformulated as the "distance between the present everyday actions of the individuals and the historically new form of the societal activity that can be collectively generated as a solution to the double bind potentially embedded in the everyday actions" (Engeström 1987, p. 174). We may exemplify the core idea in this revised definition in the following way. If Connor and Jane had been in a class based on discovery learning, they would have been left on their own to make mathematical discoveries. They would have arrived at certain results, which, in all likelihood, would have been less advanced than the results that they contribute to producing in the presence of the teacher. Because there are now more people working together, but with clearly different role in the division of labor, a new form of activity has emerged. This new form of societal activity gives rise to higher-level actions on the part of the children then in the hypothetical discovery learning context; it also gives the teacher new opportunities to learn to teach.

A third way of defining the zone of proximal development takes the perspective of individuals who are integral and irreducible parts of society (Holzkamp 1993). Individuals can expand their individual agency and control over life conditions by contributing to collective agency and collective control over conditions. In mathematics classrooms, this means that students engage in collaborations with others, because they increase their individual agency and task control when they contribute to the expansion of collective agency and control by active participation. Thus, Connor and Jane already participate with the teacher; and it is because of their participation that their agency expands. If one or the other had said to the teacher, "I want to do this on my own" or "I don't need help," then they would have actively rejected contributing to the collective agency and control and, perhaps, never arrived at the point where they did.

This final definition allows us more easily than the other two to conceptualize the important distinction between learning and development, which, in the Vygotskian framing, are part of one and the same process. In the relation, the students expand their agency and control over the mathematical task conditions: They learn. But when they no longer need the relation with the teacher or peers, they have reached the new developmental level. Learning and development, very different concepts in the constructivist paradigm of J. Piaget, are now two different sides of the same movement (Roth and Radford 2011). Similarly, by working *with* the children, in the interest of allowing them to learn, the teacher expands her own agency and control over the conditions: She develops.

## Summary

In summary, the zone of proximal development is a powerful category for understanding learning that arises when people enter relations with others. Aphoristically we may state: *What these relations are today will be psychological functions of the participants tomorrow*.

## Cross-References

▶ Activity Theory in Mathematics Education
▶ Concept Development in Mathematics Education
▶ Dialogic Teaching and Learning in Mathematics Education
▶ Learner-Centered Teaching in Mathematics Education
▶ Scaffolding in Mathematics Education

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